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APPLICATION OF THE TAYLOR DIFFERENTIAL TRANSFORMATION IN THE CALCULUS OF VARIATIONS

Abstract. The paper presents the method of solving some problems belonging to the area of the calculus of variations, that is the problems of searching for the selected types of functionals which can be transformed to some, nonlinear in general, ordinary differential equations or systems of such equations. The obtained equations are solved on the basis of the Taylor differential transformation.

1. Selected problems of the calculus of variations

One of the problems belonging to the area of the calculus of variations is the task of searching for the extremum of functional

$$J_1[y(x)] = \int_a^b F_1(x, y(x), y'(x))dx \tag{1}$$

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in the class of continuous functions y with the continuous first derivative in the interval $\langle a, b\rangle$, that is for functions $y \in D\langle a, b\rangle$ satisfying the conditions

$$y(a) = A, \quad y(b) = B, \tag{2}$$

where a, b, A, $B \in \mathbb{R}$. The function F_1 is the continuous function possessing the continuous partial derivatives of the first and second order in set $\Phi = \langle a, b\rangle \times \mathbb{R}^2$, that is $F_1 \in C_2(\Phi)$ and $\Phi = \{(x, y, y')\colon a \le x \le b,\ y, y' - \text{any values}\}$.

Geometrically, the problem consists in finding the curve connecting two given points and realizing the extremum of the functional.

The condition, under which the functional (1), defined in the set of functions $y \in D\langle a, b\rangle$ satisfying conditions (2), takes the local extremum for a given function y is the fulfillment of the Euler equation [1, 5, 7, 9] by function F_1:

$$\frac{\partial F_1}{\partial y} - \frac{d}{dx}\left(\frac{\partial F_1}{\partial y'}\right) = 0 \tag{3}$$

by preserving conditions (2).

In general case the Euler equation (3) is the second order differential equation, mostly nonlinear, which can be written in the form

$$\frac{\partial F_1}{\partial y} - \frac{\partial}{\partial y'}\left(\frac{\partial F_1}{\partial x}\right) - \frac{\partial}{\partial y'}\left(\frac{\partial F_1}{\partial y}\right) y' - \frac{\partial}{\partial y'}\left(\frac{\partial F_1}{\partial y'}\right) y'' = 0.$$

Solving this equation may lead to the problems in determining its general integral as well as, due to the boundary conditions, in finding its particular integral.

Equation (3) expresses the necessary condition for the existence of extremum of the functional and its solutions are called the extremals or the stationary functions. In many problems the fulfillment of the necessary condition suffices in practice for solving the problem, that is the found extremal realizes the sought local extremum (which simultaneously the global extremum) and it results from the geometrical or physical sense of the investigated problem.

Similar problem is the problem of searching for the extremum of functional

$$J_2[y(x)] = \int_a^b F_2(x, y(x), y'(x), \ldots, y^{(n)}(x))dx \tag{4}$$

in the class of continuous functions y with the continuous derivatives till the n-th order in the interval $\langle a, b\rangle$, that is for the functions $y \in D_n\langle a, b\rangle$ satisfying the boundary conditions

$$y^{(k)}(a) = A_k, \quad y^{(k)}(b) = B_k, \quad k = 0, 1, \ldots, n-1, \tag{5}$$

where a, b, A_k, $B_k \in \mathbb{R}$, whereas the function F_2 is of class $C_{n+1}(\Phi)$ for $\Phi = \langle a, b\rangle \times \mathbb{R}^{n+1}$.

In this case, the necessary condition, under which the functional (4), defined in the set of functions $y \in D_n\langle a, b\rangle$ satisfying conditions (5), takes the local extremum for a given function y, is the fulfillment of the Euler-Poisson equation [1,5,7,9] by function F_2:

$$\frac{\partial F_2}{\partial y} - \frac{d}{dx}\left(\frac{\partial F_2}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F_2}{\partial y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F_2}{\partial y^{(n)}}\right) = 0 \tag{6}$$

with conditions (5).

Equation (6) is the differential equation of order $2n$ and, similarly like in the previous case, satisfying this equation is only the necessary condition for the existence of extremal of functional (4), however in many cases it suffices to find the extremal realizing the global extremum.

Another important problem in the calculus of variations is the problem of searching for the extremum of functional depending on the multi-variable functions. In particular the problem concerns the following functionals

$$J_3[y_1(x), y_2(x), \ldots, y_n(x)] = \int_a^b F_3(x, y_1(x), \ldots, y_n(x), y_1'(x), \ldots, y_n'(x))dx \tag{7}$$

in the class of functions $y_i \in D\langle a, b\rangle$, $i = 1, 2, \ldots, n$, satisfying the conditions

$$y_i(a) = A_i, \quad y_i(b) = B_i, \quad i = 1, 2, \ldots, n, \tag{8}$$

where a, b, A_i, $B_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, whereas the function F_3 is of class $C_{2n+1}(\Phi)$ for $\Phi = \langle a, b\rangle \times \mathbb{R}^{2n+1}$.

In this case the necessary condition for taking the local extremum by functional (7),defined in the set of functions $y_i \in D\langle a, b\rangle$, $i = 1, 2, \ldots, n$, satisfying conditions (8), for the given functions y_i, $i = 1, 2, \ldots, n$, is the fulfillment by these functions of the Euler-Lagrange system of equations of the form [1, 5, 7, 9]:

$$\frac{\partial F_3}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F_3}{\partial y_i'}\right) = 0, \quad i = 1, 2, \ldots, n \tag{9}$$

with boundary conditions (8).

The system of equations (9) is in general composed of n nonlinear ordinary differential equations of the second order with the unknown functions y_i, $i = 1, 2, \ldots, n$ and, although satisfying these equations is only the necessary condition for the existence of extremum of function (7), in practice it usually suffices to find the stationary functions realizing the global extremum.

The formulated above problems show that the search for the extremum of the selected class of functionals reduces to the solution of the boundary problems for mostly nonlinear ordinary differential equations of the second order or systems of such equations, relatively to the solution of the boundary problems for ordinary differential equations of order $2n$, $n \geq 2$. The discussed problems not always possess the solutions and even if the solutions exist, they may be not unique. However, one may prove that the discussed problems have the unique solutions under certain assumptions. In particular, it concerns the group of tasks reflected in the geometrical, physical or technical problems. In this paper we assume that the conditions ensuring the existence and the uniqueness of solutions of the investigated problems are satisfied and these solutions will be sought by applying the Taylor transformation combined with the "shooting" method frequently applied to solve the boundary value problems for ordinary differential equations [8]. The "shooting" method reduces the boundary value problem to finding the initial conditions that give a root, which may be interpreted as "shooting" the solutions in different directions until we find the solution possessing the desired boundary value. This method is implemented in *Mathematica* as "Shooting", however in this elaboration we used the implementations originally programmed by us.

2. The Taylor transformation

Let us assume that we consider only such functions of real variable x defined in some region $X \subset \mathbb{R}$, that can be expanded into the Taylor series in some neighbourhood of point $\alpha \in X$. These functions will be called the originals or the transformable functions and will be denoted by the small letters of Latin alphabet, for example f, y, u, v, w, and so on. Thus, if a function y is the original then the following equality is satisfied

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(\alpha)}{k!}(x-\alpha)^k, \tag{10}$$

where $\alpha \in X$ is a point, in the neighbourhood of which function y is expanded into the Taylor series.

Each original y corresponds to the function Y_α of nonnegative integer arguments $k = 0, 1, 2, \ldots$, according to formula

$$Y_\alpha(k) = \frac{y^{(k)}(\alpha)}{k!}, \quad k = 0, 1, 2, \ldots \tag{11}$$

Function Y_α will be called the image of function y, T_α-function of function y or the transform of function y and the transformation itself will be called the Taylor transformation or the Taylor transform.

An obvious fact is that by having the T_α-function Y_α one can find, according to formulas (10) and (11), the corresponding original in the form of its expansion into the Taylor series, that is

$$y(x) = \sum_{k=0}^{\infty} Y_\alpha(k)(x-\alpha)^k, \quad x, \alpha \in X. \tag{12}$$

Transformation (11), associating the original with its image, will be called the direct transformation. Whereas transformation (12), associating the image with the corresponding original, will be named the inverse transformation. The connection between these two transformations will be denoted with the following symbols

$$Y_\alpha(k) = \mathcal{T}[y(x); k, \alpha], \tag{13}$$

for the direct transformation and

$$y(x) = \mathcal{T}^{-1}[Y_\alpha(k); x], \tag{14}$$

for the inverse transformation, where $\mathcal{T}$ and $\mathcal{T}^{-1}$ are the symbols of the given transformation.

According to the taken notational convention, for example for the function $y(x) = e^x$ and $\alpha = 0$ we have as follows

$$Y_0(k) = \mathcal{T}[y(x); k, \alpha] = \mathcal{T}[e^x; k, \alpha] = \mathcal{T}\left[\sum_{k=0}^{\infty} \frac{x^k}{k!}; k, \alpha\right] = \frac{1}{k!},$$

where $k = 0, 1, 2, \ldots$ Whereas in case of the inverse transformation for the above function we obtain

$$y(x) = \mathcal{T}^{-1}[Y_0(k); x] = \mathcal{T}^{-1}\left[\frac{1}{k!}; x\right] = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

The Taylor transformation possesses a number of properties, thanks to which the usage of this tool, supported by the computational platforms giving the possibility of symbolic computations like, for example, *Mathematica*, is quite simple [2]. In particular, the basic properties are very useful, namely the following ones [3]:

$$\mathcal{T}[c \cdot u(x); k, \alpha] = c \cdot U_\alpha(k), \tag{15}$$

$$\mathcal{T}[u(x) \pm w(x); k, \alpha] = U_\alpha(k) \pm W_\alpha(k), \tag{16}$$

$$\mathcal{T}[u(x) \cdot w(x); k, \alpha] = \sum_{r=0}^{k} U_\alpha(r) W_\alpha(k-r), \tag{17}$$

$$\mathcal{T}[u'(x); k, \alpha] = (k+1) U_\alpha(k+1), \tag{18}$$

$$\mathcal{T}[u^{(m)}(x); k, \alpha] = \frac{(k+m)!}{k!} U_\alpha(k+m), \tag{19}$$

$$\mathcal{T}[u'(x) \cdot w'(x); k, \alpha] = \sum_{r=0}^{k} (r+1)(k-r+1) U_\alpha(r+1) W_\alpha(k-r+1), \tag{20}$$

where $\alpha, c \in \mathbb{R}$, $k = 0, 1, 2, \ldots$, whereas U_α and W_α denote the respective images of functions u and w.

3. Method of solution for the discussed variational problems

As we have already discussed, the problem of searching for the extremum of functionals, considered in this paper, can be reduced to the solution of boundary problems for the ordinary differential equations of the second order, or systems of such equations, or boundary problems for the ordinary differential equations of order $2n$, $n > 1$. However, even if we know that the solution of such associated boundary problem exists and is unique, determination of this solution, basing on the classical methods dedicated to the boundary problems for ordinary differential equations, is very often impossible. In these cases we may use the non-classical methods, like the method based on the Taylor transformation which allows to find the solution in the form of the power series or, in the worst case, in the form of the partial sum of such series.

Thus, considering the described problem of the calculus of variations, we seek the stationary functions of functional J with the aid of Taylor transformation of the form

$$y(x) = \sum_{k=0}^{\infty} Y_\alpha(k)(x-\alpha)^k, \quad x, \alpha \in \langle a, b \rangle, \tag{21}$$

where Y_α is the $\mathcal{T}_\alpha$-function of function y.

However, in most cases we will just search for the sub-stationary function y_m based on m initial values of $\mathcal{T}_\alpha$-function of function Y_α, that is

$$y(x) \approx y_m(x) = \sum_{k=0}^{m} Y_\alpha(k)(x-\alpha)^k, \quad x, \alpha \in \langle a, b \rangle, \tag{22}$$

which, for the properly chosen m, approximates sufficiently well the stationary function y.

Below we present, by examples, how to apply the discussed method in practise.

4. Examples of calculations

4.1. Functional J_1

Example 1. *We determine the stationary function of functional*

$$J[y(x)] = \int_0^1 \left[(y' + x)^2 + (y + x)^2 + x^2\right] dx \tag{23}$$

satisfying the conditions

$$y(0) = 1, \quad y(1) = 0. \tag{24}$$

Problem, formulated in this way, according to the facts presented in Section 1 can be reduced to the solution of the boundary problem given below

$$\begin{cases} \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \\ y(0) = 1, \quad y(1) = 0, \end{cases} \tag{25}$$

where function F is of the form

$$F(x, y, y') = (y' + x)^2 + (y + x)^2 + x^2. \tag{26}$$

It follows from (25) and (26) that determination of the stationary function y of functional (23) under conditions (24) consists finally in solving the ordinary differential equation of the second order

$$y'' - y - x + 1 = 0, \tag{27}$$

with boundary conditions

$$y(0) = 1, \tag{28}$$

$$y(1) = 0. \tag{29}$$

If we assume that the $\mathcal{T}_\alpha$-function of the sought stationary function $y(x)$ is function $Y_0(k)$, it means $\alpha = 0$, then formula (21) implies that

$$y(x) = \sum_{k=0}^{\infty} Y_0(k) x^k. \tag{30}$$

Whereas, on the ground of properties (15)–(19) of the Taylor transformation, we get from differential equation (27) the following relations

$$(k+1)(k+2)Y_0(k+2) - Y_0(k) - \delta(k-1) + \delta(k-0) = 0, \quad k = 0, 1, 2, \ldots, \tag{31}$$

where

$$\delta(k - l) = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

Simultaneously, condition (28) implies that

$$Y_0(0) = 1. \tag{32}$$

Analyzing the obtained relations (31) and (32) we conclude that we are not able to determine the value $Y_0(1)$ on their ground. So we assume additionally that

$$Y_0(1) = s, \quad s \in \mathbb{R}. \tag{33}$$

Taken assumption (33), together with relations (31) and (32), enables to determine the successive values $Y_0(k)$, $k = 2, 3, \ldots$, namely we have

$$Y_0(k) = 0, \quad k = 2l, \quad l = 1, 2, 3, \ldots,$$

and

$$Y_0(k) = \frac{s+1}{k!}, \quad k = 2l+1, \quad l = 1, 2, 3, \ldots$$

Considering the above result, according to formula (30) we have

$$y(x) = 1 + sx + \sum_{l=1}^{\infty} \frac{s+1}{(2l+1)!} x^{2l+1}. \tag{34}$$

We need to calculate yet the value of parameter s. To do that we use condition (29), which implies that parameter s must satisfy the equation

$$1 + \sum_{l=1}^{\infty} \frac{1}{(2l+1)!} + s \left(1 + \sum_{l=1}^{\infty} \frac{1}{(2l+1)!} \right) = 0. \tag{35}$$

It results from this equation that $s = -1$ which, by substituting the value of parameter s to relation (34), leads to the conclusion that the stationary function of functional (23) is the function

$$y = 1 - x, \quad x \in \langle 0, 1 \rangle. \tag{36}$$

Example 2. *We determine the extremal of functional*

$$J[y(x)] = \int_0^1 \frac{1 + (y(x))^2}{(y'(x))^2} dx, \tag{37}$$

satisfying the conditions

$$y(0) = 0, \quad y(1) = e. \tag{38}$$

Formulated above problem reduces to the solution of the boundary problem

$$\begin{cases} \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \\ y(0) = 0, \quad y(1) = e, \end{cases} \tag{39}$$

where function F takes the form

$$F(x, y, y') = \frac{1 + y^2}{(y')^2}. \tag{40}$$

It results from (39) and (40) that determination of the extremal of functional (37) under conditions (38) consists in solving the equation

$$y'' + y''y^2 - y(y')^2 = 0, \tag{41}$$

with boundary conditions

$$y(0) = 0, \tag{42}$$

$$y(1) = e. \tag{43}$$

If we assume that the function $y(x)$ is expected in the form of its expansion into the Taylor series at point $\alpha = 0$, then, in view of the Taylor transformation, we have

$$y(x) = \sum_{k=0}^{\infty} Y_0(k)x^k, \tag{44}$$

where Y_0 is the $\mathcal{T}_\alpha$-function of function y.

From boundary condition (42) and differential equation (41), on the ground of properties (15)–(20) of the Taylor transformation, it follows that

$$Y_0(0) = 0 \tag{45}$$

and that for $k = 0, 1, 2, \ldots$:

$$\begin{aligned} &(k+1)(k+2)Y_0(k+2)+ \\ &+ \sum_{l=0}^{k} \sum_{m=0}^{k-l} (k+1-m-l)(k+2-m-l)Y_0(k+2-m-l)Y_0(l)+ \\ &- \sum_{l=0}^{k} \sum_{m=0}^{k-l} (m+1)(k+1-m-l)Y_0(m+1)Y_0(k+1-m-l)Y_0(l) = 0. \end{aligned} \tag{46}$$

Analyzing the obtained relations (45) and (46) we see that we cannot determine the value $Y_0(1)$ on their basis. Let us assume then, for a moment, that

$$Y_0(1) = s, \quad s \in \mathbb{R}. \tag{47}$$

Taken assumption and the relations (45) and (46) give now the possibility to determine the successive values $Y_0(k)$, $k = 2, 3, \ldots$, so we get

$$\begin{array}{llll}
Y_0(2) = 0, & Y_0(3) = \dfrac{s^3}{6}, & Y_0(4) = 0, & Y_0(5) = \dfrac{s^5}{120}, \\
Y_0(6) = 0, & Y_0(7) = \dfrac{s^7}{5040}, & Y_0(8) = 0, & Y_0(9) = \dfrac{s^9}{362880}, \\
Y_0(10) = 0 & Y_0(11) = \dfrac{s^{11}}{39916800}, & Y_0(12) = 0, & \ldots
\end{array}$$

Since we cannot find any pattern while calculating the odd values $Y_0(k)$, we take at the moment, as the solution of problem (41)–(43), the approximate solution y_7, depending still on parameter s, based on the eight-element partial sum of the series from relation (44), that is

$$y(x) \approx y_7(x; s) = \sum_{k=0}^{7} Y_0(k)x^k = sx + \frac{s^3}{6}x^3 + \frac{s^5}{120}x^5 + \frac{s^7}{5040}x^7. \tag{48}$$

We have to determine the value of parameter s. For this purpose we use condition (43), according to which parameter s must fulfill the equation

$$s + \frac{s^3}{6} + \frac{s^5}{120} + \frac{s^7}{5040} = e, \tag{49}$$

and the sole real root of this equation is $s = 1.72552$. It means that the assumed approximate solution has the form

$$y_7(x) = 1.72552x + 0.856259x^3 + 0.127471x^5 + 0.00903651x^7. \tag{50}$$

If we take twelve initial values of the $\mathcal{T}_\alpha$-function Y of function y, then we obtain the solution of the form

$$y_{11}(x; s) = sx + \frac{s^3}{6}x^3 + \frac{s^5}{120}x^5 + \frac{s^7}{7!}x^7 + \frac{s^9}{9!}x^9 + \frac{s^{11}}{11!}x^{11}, \tag{51}$$

and the value of parameter s is equal in this case to $s = 1.72538$.

Actually, the investigated problem possesses the exact solution possible to be found od the ground of series

$$y(x;s) = \sum_{i=1}^{\infty} \frac{(sx)^{2i-1}}{(2i-1)!} x^{2i-1} = \sinh(sx)$$

and condition (43), from which we obtain the value $s = \operatorname{arcsinh} e$, and finally we get

$$y(x) = \sinh(\operatorname{arcsinh}(e)x). \tag{52}$$

Figure 1 shows the plots of both approximate solutions y_7 and y_{11}, as well as the plot of exact solution y. Whereas Figure 2 presents the plots of absolute errors

$$\Delta y_m = |y - y_m|, \quad m = 7, 11, \tag{53}$$

arising when we substitute the exact solution with the respective approximate solutions.

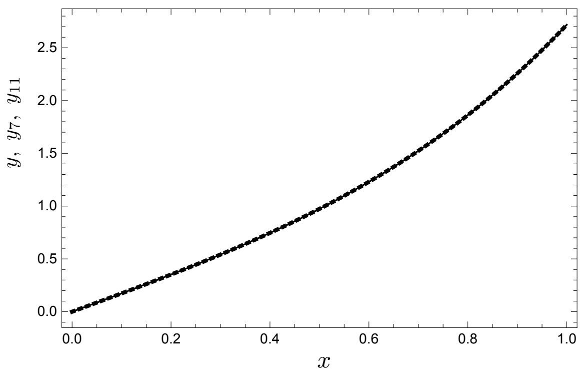

Fig. 1. Plots of the solutions from Example 2: exact solution y – solid line and approximate solutions: y_7 – dashed line and y_{11} – dot line

4.2. Functional J_2

Example 3. *We find the stationary function of functional*

$$J[y(x)] = \int_0^2 \left[(y' - x - y'')^2 + (y + y'y'' - y'')x^2 + (y'' - x)^2 \right] dx \tag{54}$$

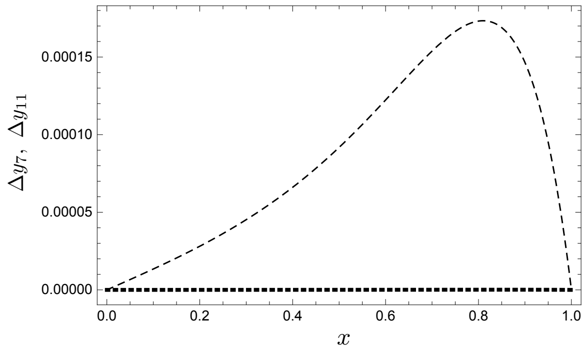

Fig. 2. Plots of the absolute errors of solutions: Δy_7 – dashed line and Δy_{11} – dot line

satisfying the conditions

$$y(0) = 2, \quad y'(0) = 1, \quad y(2) = 1, \quad y'(2) = 2. \tag{55}$$

By taking $F(x, y, y', y'') = (y' - x - y'')^2 + (y + y'y'' - y'')x^2 + (y'' - x)^2$ and by using equation (6), we get the differential equation

$$4y^{(4)} + 2(x-1)y'' + 2y' + x^2 = 0, \tag{56}$$

which, under conditions (55), will be solved with the aid of the discussed method. Similarly as before, we easily notice that $Y_0(0) = 2$ and $Y_0(1) = 1$, additionally we assume that $Y_0(2) = s$ and $Y_0(3) = t$, $s, t \in \mathbb{R}$. On the basis of the respective properties of the Taylor transformation, the equation (56) can be transformed to the form

$$\begin{aligned} &4(k+1)(k+2)(k+3)(k+4)Y_0(k+4)+ \\ &\qquad + 2\left(\begin{cases} 0, & k=0, \\ k(k+1)Y_0(k+1), & k \geq 1 \end{cases} - (k+1)(k+2)Y_0(k+2)\right) + \\ &\qquad + 2(k+1)Y_0(k+1) + \delta(k-2) = 0, \quad k = 0, 1, 2, \ldots \end{aligned} \tag{57}$$

From equation (57) for $k = 0$ we obtain $Y_0(4) = \frac{2Y_0(2) - Y_0(1)}{48}$, whereas for the other $k \geq 1$ we get the following recurrence relation

$$Y_0(k+4) = \frac{2(k+1)(k+2)Y_0(k+2) - 2(k+1)^2 Y_0(k+1) - \delta(k-2)}{4(k+1)(k+2)(k+3)(k+4)}. \tag{58}$$

The successive terms $Y_0(k)$, $k \geq 2$, depend obviously on the unknown parameters s and t. We calculate these parameters by using the third and fourth condition from among conditions (55). Thus, if we take as the final solution the approximate solution

$$y_7(x; s, t) = \sum_{k=0}^{7} Y_0(k)x^k = 2 + x + sx^2 + tx^3 + \frac{1}{48}(2s-1)x^4 + \frac{1}{120}x^5(3t-2s) + \\ + \frac{x^6(2s-36t-1)}{2880} + \frac{x^7(-6s+3t+2)}{10080},$$

then, by solving the appropriate system of equations with respect to variables s and t, we receive $s = -2.61339$ and $t = 1.00138$, and hence

$$y_7(x) = 2 + x - 2.613x^2 + 1.001x^3 - 0.13x^4 + 0.069x^5 - 0.015x^6 + 0.002x^7. \quad (59)$$

In Figure 3 there are displayed the approximate solutions y_7 (dashed line), y_8 (dot line) and y_{10} (dash–dot line) together with the numerical solution (considered as the exact solution) obtained with the aid of instruction `NDSolve` available in the computational platform *Mathematica*. Whereas in Figure 4 there are presented the absolute errors of these approximate solutions (denoted respectively in the same way as in Figure 3).

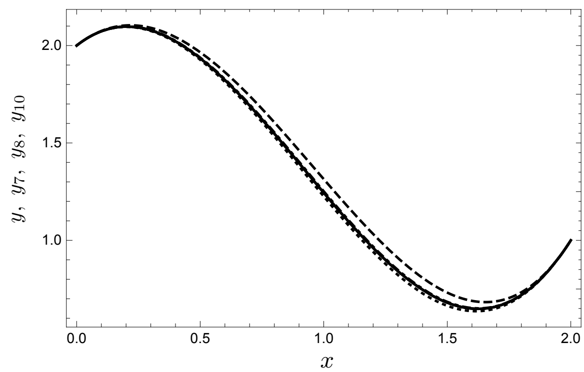

Fig. 3. Plots of the solutions from Example 3: exact solution y – solid line and approximate solutions: y_7 – dashed line, y_8 – dot line and y_{10} – dash–dot line

Example 4. *We determine the stationary function of functional*

$$J[y(x)] = \int_0^{\frac{\pi}{2}} \left[(f(x) - y'')(x + y'') + (y')^3 - 2xyy'' \right] dx, \quad (60)$$

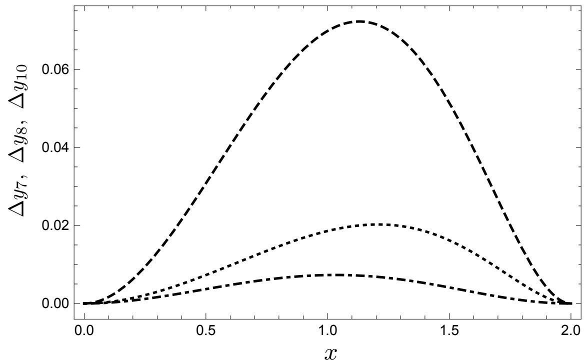

Fig. 4. Plots of the absolute errors of solutions from Example 3

satisfying the conditions

$$y(0) = 1, \quad y'(0) = -2, \quad y\left(\frac{\pi}{2}\right) = 1, \quad y'\left(\frac{\pi}{2}\right) = 2, \tag{61}$$

where $f(x) = \frac{3}{2}\sin 4x - 4x\sin 2x + 8\sin 2x - 2\cos 2x$.

By taking $F(x, y, y', y'') = (f(x) - y'')(x + y'') + (y')^3 - 2xyy'$ and by applying equation (6), we obtain the differential equation

$$y^{(4)} + 2xy'' + y'(3y'' + 2) + g(x) = 0, \tag{62}$$

where $g(x) = 16\sin 2x + 12\sin 4x + 4\cos 2x - 8x\sin 2x$. We solve the above equation, with conditions (61), by using the introduced method. Similarly like in previous examples we notice that $Y_0(0) = 1$ and $Y_0(1) = -2$, additionally we take that $Y_0(2) = s$ and $Y_0(3) = t$, $s, t \in \mathbb{R}$. In view of the respective properties of the Taylor transformation the equation (62) transforms to the form

$$\begin{aligned}(k+1)(k+2)(k+3)(k+4)Y_0(k+4) + 2(k+1)^2 Y_0(k+1) +& \\ + 3\sum_{j=0}^{k}(j+1)(k-j+1)(k-j+2)Y_0(j+1)Y_0(k-j+2) +& \\ + G_0(k) = 0, \quad k = 0, 1, 2, \ldots&\end{aligned} \tag{63}$$

From equation (63) for $k = 0$ we get $Y_0(4) = \frac{Y_0(2)}{2}$, whereas for the other $k \geq 1$ we obtain the recurrence relation, from which we determine the values of $Y_0(k+4)$ for each $k \geq 1$. The successive terms $Y_0(k)$, $k \geq 2$, depend certainly on the unknown parameters s and t. We calculate values of these parameters by using the third and fourth condition from among conditions (61). So, if we take the approximate solution y_8 as our final solution, then, after solving the appropriate system of equations with respect to variables s and t, the chosen approximate solution takes the following form (for $s = 0.05403$ and $t = 1.28693$):

$$\begin{aligned} y_8(x) = 1 - 2x + 0.054x^2 + 1.287x^3 + 0.027x^4 - 0.284x^5 + \\ - 0.003x^6 + 0.029x^7 - 0.002x^8. \end{aligned} \tag{64}$$

Figure 5 presents the approximate solutions y_8 (dot line) and y_{10} (dash–dot line), as well as the exact solution (solid line – one can easily verify that the exact solution is the function $y = 1 - \sin 2x$). The next Figure 6 shows the absolute errors of these approximate solutions (denoted respectively in the same way as in Figure 5).

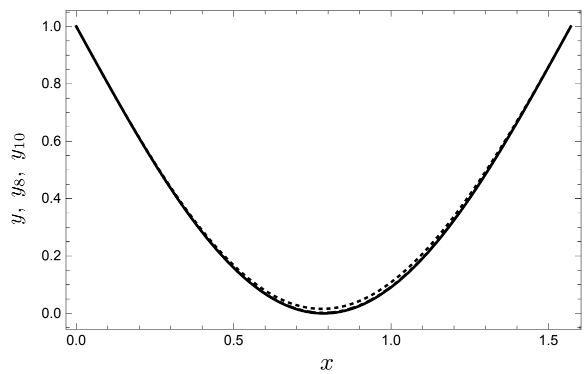

Fig. 5. Plots of the solutions from Example 4: exact solution y – solid line and approximate solutions: y_8 – dot line and y_{10} – dash–dot line

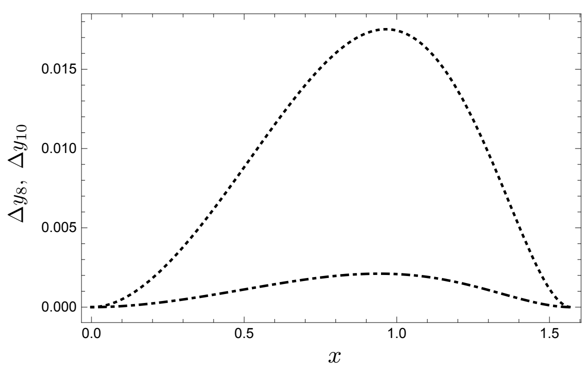

Fig. 6. Plots of the absolute errors of solutions from Example 4

4.3. Functional J_3

Example 5. *We determine the stationary function of functional*

$$J[y(x)] = \int_0^1 \left[y_2'(y_1 y_2 - y_2') - 8(x-1)^2 (y_2 - y_1') \right] dx, \tag{65}$$

fulfilling conditions

$$y_1(0) = 1, \quad y_1(1) = 0, \quad y_2(0) = 4, \quad y_2(1) = 0. \tag{66}$$

By taking $F(x, y_1, y_2, y_1', y_2') = y_2'(y_1 y_2 - y_2') - 8(x-1)^2(y_2 - y_1')$ and by using equations (9), we obtain the system of differential equations

$$\begin{cases} y_2 y_2' - 16(x-1) = 0, \\ 2y_2'' - y_1' y_2 - 8(x-1)^2 = 0, \end{cases} \tag{67}$$

which, under conditions (66), will be solved by means of the introduced method. Just like before, we have easily $Y1_0(0) = 1$ and $Y2_0(0) = 4$, moreover we take that $Y1_0(1) = s$, $s \in \mathbb{R}$. On the ground of the respective properties of the Taylor transformation, the system of equations (67) can be transformed to the form

$$\begin{cases} \sum_{j=0}^{k} (k-j+1) Y2_0(j) Y2_0(k-j+1) - 16\left(\delta(k-1) - \delta(k-0)\right) = 0, \\ 2(k+1)(k+2) Y2_0(k+2) - 8\left(\delta(k-2) - 2\delta(k-1) + \delta(k-0)\right) + \\ \qquad - \sum_{j=0}^{k} (j+1) Y1_0(j+1) Y2_0(k-j) = 0, \quad k = 0, 1, 2, \ldots \end{cases} \tag{68}$$

From the first equation of system (68) for $k = 0$ we get $Y2_0(1) = -4$, and for $k = 1$ we have $Y1_0(1) = -2$. From the second equation of this system for $k = 0$ we get $Y2_0(2) = 0$ and, returning to the first equation, for $k = 1$ we obtain $Y1_0(2) = -1$. Considering in this order the successive values of k we conclude that the other values $Y1_0(k)$, $k \geq 3$, and $Y2_0(k)$, $k \geq 2$, are qual to zero. It means that the solution received thanks to the Taylor transformation is of the form

$$\begin{cases} y_1(x) = (x-1)^2, \\ y_2(x) = -4(x-1), \end{cases} \tag{69}$$

for $x \in \langle 0, 1 \rangle$, and one can easily check that this is the exact solution of system (67).

5. Conclusion

The paper discusses the possibility of using the Taylor transformation for solving some selected types of problems belonging to the area of the calculus of variations. Presented examples show that the introduced method is effective in solving the problems of considered kind and, what is more, the additional advantage of this approach is the simplicity of its application.

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