ZESZYTY
NAUKOWE
POLITECHNIKI
ŚLĄSKIEJ

MARIAN BŁACHUTA

CONTRIBUTIONS TO THE THEORY OF DISCRETE-TIME CONTROL FOR CONTINUOUS-TIME SYSTEMS

AUTOMATYKA
Z. 127

GLIWICE
1999

POLITECHNIKA ŚLĄSKA

ZESZYTY NAUKOWE

Nr 1416

MARIAN BŁACHUTA

CONTRIBUTIONS TO THE THEORY OF DISCRETE-TIME CONTROL FOR CONTINUOUS-TIME SYSTEMS

GLIWICE 1999

OPINIODAWCY

Dr hab. inż. Jan Tadeusz Duda
Prof. dr hab. Jakub Gutenbaum

Wydano za zgodą
Rektora Politechniki Śląskiej

PL ISSN 0434-0760

Wydawnictwo Politechniki Śląskiej
ul. Akademicka 5, 44-100 Gliwice
tel./fax 237-13-81

Nakład 110+83 egz. Ark. wyd. 15. Ark. druk. 11,875. Papier offset. kl. III 70x100 80 g
Oddano i podpisano do druku 25.05.1999 r. Druk ukończono w czerwcu 1999 r.
Zam. 15/99

Fotokopie, druk i oprawę wykonano w UKiP sc, *J&D. Gębka*,
Gliwice, ul. Pszczyńska 44, tel./fax 231-87-09

Preface

This work is about sampled data control systems, i.e. systems which consist of a continuous-time plant to be controlled, a sampling device, and a digital controller driving the plant through a hold device.

In spite of the continuous-time character of real physical processes, totally discrete-time descriptions were traditionally chosen as a sole basis of operation. As a result, sampled data control systems were treated as discrete-time systems in the literature, and only modeled the phenomena at sampling instants.

Unfortunately, when implementing certain advanced discrete-time algorithms it appeared that continuous-time output was plagued by unacceptable inter-sample ripple or even non-stability, particularly at high sampling rates. These phenomena denied the intuitive feeling that the system behavior should approach that of a continuous-time system when the sampling rate increased.

Due to the fast development of digital controllers, sampled data control systems have become a much studied topic during the last decade, and many approaches have been developed to overcome the limitations of the purely discrete time theory. Their common disadvantage is that they are mathematically complicated.

The route taken here is to get more within the existing framework, and to assure a proper intersample behavior. To do that the sources of problems within the discrete-time methods have been identified, and methods and models immune against these problems have been developed. A general rule was to keep in mind that systems work in continuous time, and to require discrete-time models to keep track with the ultimate continuous-time factors.

This work summarizes the authors experience gained during a long period. It is the authors pleasure to deeply thank several people and institutions who, directly or indirectly, contributed to this work.

A significant body of the results was obtained during the author's stays at the Institute for Advanced Studies, Vienna, with Prof. Manfred Deistler and the University of Birmingham with Prof. Mieczysław Brdyś, holding grants from the Austrian Federal Ministry of Science and Research and the European Community, respectively.

Most of the work summarized here was done within several projects conducted in the Institute of Automation, Silesian Technical University, Gliwice, headed by Prof. Ryszard Gessing, under financial support from the Polish Ministry of Education and the Polish Committee for Scientific Research (KBN). The final version was written within a project supported by the KBN grant no. 8T11A 006 14.

Partial results were presented and discussed during the seminars chaired respectively by Prof. Ryszard Gessing and Prof. Antoni Niederliński, both at the Silesian Technical University, Gliwice; Prof. Alexander Weinmann at Vienna University of Technology; Prof. Manfred Deistler in the Institute for Advanced Studies, Vienna; Prof. Tadeusz Kaczorek at Warsaw University of Technology; Prof. Peter Roberts at the City University, London; and Prof. Józef Korbicz at the Technical University of Zielona Góra. The author thanks them for providing this opportunity.

It is the author's great pleasure to express a very special thank to Prof. Bozenna Pasik-Duncan, Vice President of the IEEE Control Systems Society, for persistent encouragement to scientific activity within CSS, and the CSS Board of Governors for providing financial support to attend CSS flagship conferences. Similar acknowledgments are due to Mrs. Sabina Tiffert, the proprietor of the International Haulage Enterprise ETOR II, for sponsoring the author's travel to the USA to attend the 36th CDC, and to Mr. Czesław Sobierajski, MP, for arranging for the financial assistance. Without the support from those people this work would have never achieved the present quality.

Finally, the author wants to thank his family, who tolerated him spending long hours at the office and allowed time to travel across the world.

Contents

Spis treści

1. Introduction

Due to the advent of high speed low cost computational tools virtually every advanced control system depends upon sampling, modulation and digital signal processing.

Sampling and modulation, however, cause information loss and as a result the performance of sampled-data control systems is usually poorer than that of continuous-time ones, particularly at small sampling rates. It is therefore reasonable to expect that increasing the sampling rate should result in the continuous-time performance recovery. Unfortunately, this may not be the case for certain classical designs when 'ringing' of the control signal leading to an unacceptable intersample ripple of the output is observed.

The classical purely discrete-time approach to sampled-data systems requires the continuous-time plant to be discretized prior to controller design. This approach gave rise to the so called discrete-time control systems theory defining the control task and performance at discrete-time sampling instants. Although most frequently met in textbooks it overlooks the intersample behavior and is vulnerable to fail at high sampling rates.

Design methods yielding controllers which have good properties for a wide range of sampling periods and recover the continuous-time performance at high sampling rates are highly desirable. The present work aims at contributing towards achieving this target.

There are two main sources of problems when using the discrete-time approach to a lumped parameters system: (1) whatever the value of the relative degree of the continuous-time plant, the relative degree of the discretized system equals generically to 1, and (2) excessive zeros produced by the discretization process are either unstable or badly damped.

Zeros belong to fundamental characteristics of linear time-invariant systems. However, while the mapping between the discrete-time poles and their continuous-time counterparts is very simple, this is not the case with zeros, for which no general closed form expressions exist. Therefore an extensive study of zeros of pulse transfer functions is performed in two next chapters.

In particular, in Chapter 2 the famous Åström-Hagander-Sternby theorem on limiting zeros of the pulse transfer function of a system with the zero-order hold (ZOH) is extended by determining the accuracy of the asymptotic results for both the discretization and the intrinsic zeros when the sampling interval is small. Closed form formulae are derived that express both the degree of the principal term of the Taylor expansion of the difference between the true zeros and asymptotic ones as a function of the relative degree of the underlying continuous-time system, as well as the value of the corresponding coefficient itself. Certain known results on asymptotic zeros are shown to be particular cases of the result presented.

Similar approach is used in Chapter 3 to the analysis of limiting zeros of systems with the first-order hold (FOH). In particular, the Hagiwara-Yuasa-Araki theorems on limiting zeros of the pulse transfer function of sampled-data system with first-order hold are extended by stating that limiting intrinsic zeros can be expressed as exponential functions of continuous-time zeros, and by determining the accuracy of the asymptotic results for both the discretization and the intrinsic zeros when the sampling interval is small.

Most of the designs depend upon the relative degree of the continuous-time system, either explicitly or implicitly. Unfortunately, this most important design parameter becomes hidden once discretization is done and is not taken into account by purely discrete-time design procedures. An associated effect is that discretization zeros often appear in the characteristic polynomial of the closed loop system, which leads to badly damped control producing intersample ripple. Another possible effect is an impulsive behavior of the control signal when the sampling period becomes small.

Negative effects of unstable discretization zeros and of the reduction of the relative degree can be circumvented by using approximate pulse transfer functions discussed in Chapter 4, where a systematic approach to a class of approximations to the pulse transfer function of a system consisting of a zero-order hold and a linear continuous-time plant is presented. It is based on the asymptotic results on zeros developed in Chapter 2, and on the bilinear transformation. Superiority of the approximations considered over a δ-operator based truncated approximation of Goodwin, Leal, Mayne, & Middleton (1984) is shown. Since the number of intrinsic parameters does not change in the discretization process, model matching control, robust control and identification are suggested as possible areas of application. The results are illustrated by an example.

It is interesting to note that although the theory of discrete-time modeling of sampled systems seems to be well developed certain important issue remained to be revisited. In Chapter 5 discrete-time modeling is addressed when both a continuous-time plant and a discrete-time controller have a feedthrough. It is pointed out that in this case discrete-time models which can be found in most references and program packages should not be used in the closed-loop context. A new state-space model appropriate for the closed-loop modeling, and formulae for calculating the related discrete-time pulse transfer functions are derived. Intersample phenomena are studied and the feasibility of that model to describe systems with parasiting dynamics is emphasized. Examples from the literature illustrate the relevance of the issue.

The so called hybrid approach which performs direct design taking the intersample behavior into account is another remedy against bad intersample behavior. It has been receiving increasing recognition for the last years but its main disadvantage is a great mathematical and numerical load. Two simple approaches to the synthesis of a discrete-time model reference controller for a continuous-time system are presented and compared in Chapter 6. A model reference control task is defined in such manner that the output is required to fulfill a predefined differential or difference equation, or to be close to its solution while the overall closed-loop system is stable. The first, purely discrete approach, bases on the discrete-time model of a dynamic system and on a discrete quadratic infinite horizon performance index while the second is based on the continuous-time integral performance index. When the sampling time tends to zero the control variable in the former problem does not converge to its continuous time prototype whereas in the latter does. The relative order of the continuous-time plant itself and a proper relationship between the model and plant relative orders are detected to be crucial to avoid the impulsive control signal behavior at high sampling rates.

Control systems usually work in the presence of external disturbances best modeled by stochastic continuous-time processes.

In Chapter 7, which starts the second part of this work, models of sampling continuous-time processes are discussed. As a result of sampling, discrete second-order random processes described by linear time-invariant state-space models with a random vector driving input are obtained. Equivalent representations with the number of noise inputs reduced to one are found. In contrast to the innovations approach these representations have time-invariant parameters. The relationship with ARMA models is presented and the Representations Theorem is generalized to a class of nonstationary processes. The issue of identification of continuous-time models is discussed.

The reduced models obtained in Chapter 7 form a basis for definition of stochastic discrete-time control problems usually handled by LQG or predictive control philosophy. In Chapter 8 a unified approach to the MV, LQG and GPC control problems based on the input-output and state-space representations of Box-Jenkins models is presented. Its two main advantages are: an integral action of the controller attained with a realistic stationary model of the disturbance, and a reduction of the computational complexity. Moreover, it will be shown that Chandrasekhar equations improve the computational efficiency for receding-horizon control problems as compared to the use of Riccati equations. The approach is also shown to be an efficient design method for the optimal infinite horizon control systems.

Bearing in mind that the output of the controlled system is continuous-time, intersample output characteristics are of primary importance to asses the control performance. Chapter 9 deals with discrete-time control of continuous-time systems driven by ZOH with pulse amplitude modulation and disturbed by a stationary Gaussian process with a rational spectral density. The algorithms considered have the form of a linear feedback from the Kalman filter. We concentrate on inter-sample mean and variance of the input and output to characterize the performance of the continuous-time system with discrete feedback. A methodology of calculation of these functions is developed. Some results of the related works in the area are generalized and extended.

Part I

Deterministic Systems

2. Zeros of Systems with Zero-Order Hold

Zeros, along with poles, are fundamental characteristics of linear time-invariant systems. While the mapping between the discrete-time poles and their continuous-time counterparts is very simple, this is not the case with zeros, for which no general closed form expressions exist. Therefore, it is desirable to have formulae that relate all discrete-time zeros with the continuous-time ones, at least approximately.

The famous Åström-Hagander-Sternby theorem on limiting zeros of the pulse transfer function is extended by determining the accuracy of the asymptotic results for both the discretization and the intrinsic zeros when the sampling interval is small. Closed form formulae are derived that express the degree of the principal term of Taylor expansion of the difference between the true zeros and asymptotic ones as a function of the relative degree of the underlying continuous-time system, and the value of the corresponding coefficient itself. Certain known results on asymptotic zeros are shown to be particular cases of the result presented. [1]

2.1 Introduction

As far as limiting zeros at high sampling rates are concerned only a limited set of particular results has been known to date.

Perhaps the first attempt to study zeros was that by Lindorff (1965), who conjectured that the continuous-time zeros map to discrete-time ones approximately exponentially.

This is also stated in the ÅHS theorem of Åström, Hagander & Sternby (1984), which describes the asymptotic behavior of the discrete-time zeros for small h as functions of their continuous-time counterparts, and of the relative order of the system being discretized. Due to this theorem, a part of zeros called intrinsic (Hagiwara, 1996; Hagiwara, Yuasa & Araki, 1993) go to 1 while the remaining discretization (Hagiwara et al., 1993) zeros, which are due to sampling and modulation, go towards zeros of certain polynomial called Euler (Frobenius, 1910; Sobolev, 1977), normal (Kowalczuk, 1983) or reciprocal

[1] The chapter is based on (Błachuta, 1997*f*) and (Błachuta, 1998*d*)

(Hagiwara et al., 1993; Jury, 1964) polynomial completely determined by the value of the relative order of the continuous-time system.

The correspondence between the intrinsic zeros and continuous-time zeros was clarified in a more precise manner in the HYA theorem by Hagiwara et al. (1993).

A study on intrinsic zeros based on the state-space description introduced by Hayakawa, Hosoe & Ito (1983) has been presented in the recent paper of Hagiwara (1996) with the outcome that the Taylor expansion of the zero coincides with that predicted by the ÅHS theorem at least up to the second order term, in general, and up to the third order term if the relative degree of the continuous-time system is greater than or equal to two. Although being the hitherto most advanced extension to the ÅHS theorem, the above result of Hagiwara (1996) is limited to single intrinsic zeros and its extension to higher order coefficients of the Taylor expansion in an explicit compact form does not seem to be simple within that framework.

Due to the dead-beat and MV pole-zero canceling control algorithms, see (Åström & Wittenmark, 1997; Clarke, 1984) and references therein, a great deal of work has been devoted to determine conditions for stable zeros, e.g. (Åström et al., 1984; Fu & Dumont, 1989; Hagander, 1993; Hagiwara, 1996; Hara, Katori & Kondo, 1989; Ishitobi, 1992), and (Ishitobi, 1993).

This problem has become much less important in the purely discrete-time LQR context (Åström & Wittenmark, 1997; Chen & Francis, 1995) where unstable zeros do not influence the closed loop stability but, as shown in Chapter 6 [2], discretization zeros can still lead to intersample ripple caused by controller 'ringing' if there is no control costing in the performance index. Finally, with the advent of hybrid methods, (Chen & Francis, 1995) and references therein, the stability problem of discretization zeros has become completely irrelevant (Błachuta, 1997*b*) for the contemporary $\mathcal{H}_2$-norm (Chen & Francis, 1995) and LQR (Błachuta, 1997*b*) optimal sampled-data control systems.

The aim of the chapter is to find how close limiting zeros are to actual intrinsic and discretization ones, irrespective of whether they are stable or not. The approach used could be referred to as an extension of that of (Åström et al., 1984). The results will be applied in Chapter 4 [3] to investigate the accuracy of certain approximate pulse transfer functions that base on limiting zeros or their Padé approximation.

The chapter is organized as follows. The formulation of the problem, the fundamental lemmas, and the ÅHS theorem along with its alternative proof are presented in the preliminary section 2.2. The main result is presented in section 2.3 and then the rapprochement of some results of (Hagiwara et al., 1993) and (Hagiwara, 1996) with our result is shown in section 2.4. The proofs of lemmas and theorems are collected in Appendix A.1 and conclusion is drawn in section 2.5.

[2] presented first in (Błachuta, 1997*b*)
[3] see also (Błachuta, 1997*d*)

2.2 Preliminaries

2.2.1 Pulse transfer functions

Rational strictly proper continuous-time transfer functions $G(s)$ with the relative order $k = n - m > 0$ are considered of the form:

$$G(s) = \frac{\sum_{j=0}^{m} \beta_j s^j}{\sum_{i=0}^{n} \alpha_i s^i} = g_k \frac{\prod_{i=1}^{m} (s - \sigma_i)}{\prod_{i=1}^{n} (s - \pi_i)}. \tag{2.1}$$

Assume that $\alpha_n \neq 0$, $\beta_m \neq 0$, and $G(s)$ is of type $l \geq 0$, i.e. $\alpha_0 = \ldots \alpha_{l-1} = 0$ and $\alpha_l \neq 0$. Moreover, a unity gain, i.e. $\alpha_l = 1$ and $\beta_0 = 1$ will be assumed for simplicity. Then

$$g_k = \frac{\beta_m}{\alpha_n} = \frac{\prod_{i=1}^{n-l} (-\pi_i)}{\prod_{i=1}^{m} (-\sigma_i)} \tag{2.2}$$

is the k-th Markov parameter of (2.1).

Let $H(z)$ be the pulse transfer function of a series connection of a zero-order hold and a continuous-time system with the transfer function $G(s)$, and let h be the sampling period. Then the general form of $H(z)$ is

$$H(z) = (1 - z^{-1}) \mathcal{Z}\{\frac{G(s)}{s}\}. \tag{2.3}$$

$H(z)$ has $n - 1$ zeros for almost every h, so that

$$H(z) = \frac{\sum_{j=0}^{n-1} b_j z^j}{\sum_{i=0}^{n} a_i z^i} = b_{n-1} \frac{\prod_{i=1}^{n-1} (z - z_i)}{\prod_{i=1}^{n} (z - p_i)} \tag{2.4}$$

with

$$p_i = e^{\pi_i h} \tag{2.5}$$

and $a_n = 1$. A link between a continuous-time transfer function $G(s)$ and its discrete-time counterpart $H(z)$ is defined, e.g. (Ackermann, 1993), by the Poisson formula:

$$H(e^{sh}) = \frac{1 - e^{-sh}}{h} \sum_{l=-\infty}^{\infty} \frac{G(s + jl\omega_s)}{s + jl\omega_s}, \ \omega_s = \frac{2\pi}{h}. \tag{2.6}$$

Let us divide equation (2.6) into two parts:

$$H(e^{sh}) = G_h(s) + \Delta_h(s), \tag{2.7}$$

where

$$G_h(s) = \frac{1-e^{-sh}}{sh} G(s) \tag{2.8}$$

is the $l = 0$ term. From (2.6)-(2.8) we have $\lim_{s\to 0} \Delta_h(s) = 0$, which means that the steady-state properties of $H(e^{sh})$ are the same as those of $G_h(s)$ and only transients are affected by $\Delta_h(s)$. For sh small enough $\Delta_h(s)$ is supposed to be small. This is specified in Lemma 2.2.1 of subsection 2.2.4, which plays a crucial role in further argument.

2.2.2 HYA Theorem

The following theorem due to Hagiwara et al. (1993) clarifies the correspondence between the zeros of $G(s)$ and the intrinsic zeros of $H(z)$.

Theorem 2.2.1 (Hagiwara, Yuasa & Araki, 1993). *Let σ_i be a zero of $G(s)$ with multiplicity μ. Suppose that S is a simply-connected bounded domain which includes σ_i inside and has no other zeros of $G(s)$ inside nor on its boundary. Then, there exist some h_S such that for every h satisfying $0 < h < h_S$, $H(z)$ has μ zeros inside the domain*

$$e^{Sh} := e^{sh}|_{s\in S}. \tag{2.9}$$

Corollary 2.2.1. *Denote z_i, $i = 1, 2 \ldots m$ the intrinsic zeros of $H(z)$, which due to the HYA theorem are related to the zeros σ_i of $G(s)$, while $\zeta_i = z_{m+i}$, $i = 1, 2, \ldots k-1$ denote the discretization zeros. As a result of Theorem 2.2.1, for h small enough, $H(z)$ admits the following factorization:*

$$H(z) = b_{n-1} \frac{E_k(z) \prod_{i=1}^{m} (z - z_i)}{\prod_{i=1}^{n} (z - p_i)} \tag{2.10}$$

with a polynomial $E_k(z)$:

$$E_k(z) = \prod_{i=1}^{k-1} (z - \zeta_i). \tag{2.11}$$

2.2.3 Euler polynomials

The polynomials $\mathcal{E}_k(z)$, called Euler (Frobenius, 1910; Sobolev, 1977), normal (Kowalczuk, 1983) or reciprocal (Hagiwara et al., 1993; Jury, 1964), defined as

$$\mathcal{E}_k(z) = \epsilon_1^k z^{k-1} + \epsilon_2^k z^{k-2} \ldots + \epsilon_k^k, \tag{2.12}$$

with

$$\epsilon_i^k = \sum_{j=1}^{i} (-1)^{i-j} \binom{k+1}{i-j} j^k, \; i = 1 \ldots k, \tag{2.13}$$

or recursively, (Frobenius, 1910; Weller, Moran, Ninness & Pollington, 1997b),

$$\mathcal{E}_1(z) = 1,$$
$$\mathcal{E}_{l+1}(z) = (1 + lz)\mathcal{E}_l(z) + z(1-z)\frac{d\mathcal{E}_l(z)}{dz}, \; l = 1, 2, \ldots \tag{2.14}$$

play an important role in the study of limiting discretization zeros. Due to (Frobenius, 1910; Sobolev, 1977) their roots ζ_i' are real, simple and negative for any k. If ζ_i' is a root of $\mathcal{E}_k(z)$ then $1/\zeta_i'$ is also a root. Thus $\mathcal{E}_k(-1) = 0$ for even k. The zeros of polynomials having progressively higher degree are interlaced on the negative real axis. Moreover, the coefficients ϵ_i^k are symmetric positive integers for all $i = 1 \ldots k$, i.e. $\epsilon_i^k = \epsilon_{k-i+1}^k$ for $i = 1 \ldots q(k)$, where $q(k) = \frac{1}{2}(k-1)$ for k odd and $q(k) = \frac{k}{2} - 1$ for k even, and

$$\mathcal{E}_k(1) = \sum_{i=1}^{k} \epsilon_i^k = k!. \tag{2.15}$$

$\mathcal{E}_i(z)$ for $i = 1 \ldots 5$ are listed below:

$$\begin{aligned} \mathcal{E}_1(z) &= 1 \\ \mathcal{E}_2(z) &= z + 1 \\ \mathcal{E}_3(z) &= z^2 + 4z + 1 \\ \mathcal{E}_4(z) &= z^3 + 11z^2 + 11z + 1 \\ \mathcal{E}_5(z) &= z^4 + 26z^3 + 66z^2 + 26z + 1. \end{aligned} \tag{2.16}$$

The Euler polynomials characterize the pulse transfer function of an integrator of arbitrary order. Their feasibility to approximate any pulse transfer function for small h is specified in Lemma 2.2.3.

2.2.4 Fundamental lemmas

Lemma 2.2.1. *For any finite $s \in \mathbb{C}$*

$$\Delta_h(s) = \phi_\alpha(k, s) h^{k+\alpha} + o(h^{k+\alpha}), \tag{2.17}$$

where $\alpha = 1$ for k odd, $\alpha = 2$ for k even, and

$$\phi_1(k, s) = -\frac{B_{k+1}}{(k+1)!} g_k s \tag{2.18}$$

$$\phi_2(k, s) = -\frac{B_{k+2}}{(k+2)!} g_k s[(k+1)s + \sum_{i=1}^{m} \sigma_i - \sum_{i=1}^{n} \pi_i], \tag{2.19}$$

where B_k are the Bernoulli numbers.

Remark 2.2.1. Bernoulli numbers obey the following recursive formula:

$$B_0 = 1,\ 1 + \binom{k}{1} B_1 + \binom{k}{2} B_2 \cdots + \binom{k}{k-1} B_{k-1} = 0,\ k = 2, 3, \ldots \tag{2.20}$$

There is $B_{2l+1} = 0$ for $l > 0$, $B_{2l} > 0$ for l odd and $B_{2l} < 0$ for l even (Edwards, 1974; Titchmarsh, 1986). The first nonzero values of B_k are: $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42} \cdots$

Remark 2.2.2. From Lemma 2.2.1 it follows that for finite s

$$\lim_{h \to 0} \Delta_h(s) = 0, \tag{2.21}$$

which in accordance with (2.7) implies $H(e^{sh}) \to G(s)$ as $h \to 0$.

Lemma 2.2.2. *For $s \neq 0$, $G(s)$ can be presented as follows*

$$G(s) = \frac{g_k}{s^k} + \frac{g_{k+1}}{s^{k+1}} + o(\frac{1}{s^{k+1}}), \tag{2.22}$$

where

$$g_{k+1} = g_k(\frac{\beta_{m-1}}{\beta_m} - \frac{\alpha_{n-1}}{\alpha_n}) = g_k\left(\sum_{i=1}^{n} \pi_i - \sum_{i=1}^{m} \sigma_i\right). \tag{2.23}$$

Remark 2.2.3. From (2.23) it is seen that $g_{k+1} = 0$ can be the case. Then a better resolution of (2.22) can be attained by taking the first nonzero higher order Markov parameter into account.

Lemma 2.2.3. *For any finite $z \in \mathbb{C}$, $z \neq 1$, $H(z)$ admits the following expansion:*

$$H(z) = c_k(z)h^k + c_{k+1}(z)h^{k+1} + \Delta H(z), \tag{2.24}$$

where

$$c_i(z) = \frac{g_i}{i!}\frac{\mathcal{E}_i(z)}{(z-1)^i},\ i = k, k+1,\ \Delta H(z) = o(h^{k+1}) \tag{2.25}$$

and g_i is the i-th Markov parameter of $G(s)$.

2.2.5 The ÅHS Theorem

The following famous theorem of Åström et al. (1984), adapted to the notation used here, gives a limiting relationship between the continuous-time and discrete-time zeros.

Theorem 2.2.2 (Åström, Hagander & Sternby, 1984). *Let $G(s)$ in (2.1) be a continuous-time transfer function and $H(z)$ in (2.10) be the corresponding pulse transfer function of a series connection of a zero-order hold and a continuous-time system. Then, as the sampling period $h \to 0$,*

(i) m zeros z_i of $H(z)$ go to 1 as $e^{\sigma_i h}$, and

(ii) the remaining $k-1$ zeros ζ_i of $E_k(z)$ go to the zeros ζ_i' of the Euler polynomial $\mathcal{E}_k(z)$.

Proof. From (2.24)-(2.25) it results that:

$$\lim_{h \to 0} h^{m-n} H(z) = \frac{g_k}{\mathcal{E}_k(1)} \frac{\mathcal{E}_k(z)(z-1)^m}{(z-1)^n}, \tag{2.26}$$

which proves item (*ii*) of the theorem. Due to (2.8), $G_h(\sigma_i) = 0$ for any h, and $H(e^{\sigma_i h}) = \Delta_h(\sigma_i)$. According to (2.21), $\Delta_h(e^{\sigma_i h}) \to 0$ as $h \to 0$. For any other $s \neq \sigma_i$, $H(e^{sh}) \to G(s) \neq 0$. Hence $z_i' = e^{\sigma_i h}$ is an asymptotic value of the zero z_i. This, together with (2.26), proves item (*i*). □

Remark 2.2.4. Note that since the assertion of Remark 2.2.2 lacks in (Åström et al., 1984), the second part of item (*i*) has not been proved there; also compare (Hagiwara et al., 1993) for what is recognized as the ÅHS Theorem. Another proof of Theorem 2.2.2 can be found in (Gessing, 1993).

2.3 The Main Result

Both the ÅHS and HYA theorems lack any estimate of how close $z_i'(h)$ is to $z_i(h)$. Theorem 2.3.1 addresses this issue giving more insight into the characterization of pulse transfer functions at high sampling rates.

Theorem 2.3.1. *Let σ_j denotes a zero of $G(s)$ with multiplicity μ, and π_i, $i = 1 \ldots n$, denote poles. Then under assumptions of Theorem 2.2.2:*

(i)

$$b_{n-1} = \frac{g_k}{k!} h^k + o(h^k) \tag{2.27}$$

(ii) the intrinsic zeros $z_{j+i}(h)$ of $H(z)$ obey:

$$\prod_{i=0}^{\mu-1} (z_{j+i} - e^{\sigma_j h}) = (-1)^{\mu-1} \Theta_j^{\alpha} h^{k+\mu+\alpha} + o(h^{k+\mu+\alpha}) \tag{2.28}$$

with $\alpha = 1$ for k odd and $\alpha = 2$ for k even, and *(iii) for the remaining $k-1$ discretization zeros $\zeta_j(h)$ there is:*

$$\zeta_j(h) = \zeta_j' + \Omega_j^0 h + o(h), \tag{2.29}$$

where $\mathcal{E}_k(\zeta_j') = 0$ for $j = 1, 2 \ldots k-1$, and

$$\Omega_j^0 = \frac{\sum_{i=1}^{m} \sigma_i - \sum_{i=1}^{n} \pi_i}{k+1} \frac{\mathcal{E}_{k+1}(\zeta_j')}{(\zeta_j' - 1)\prod_{\substack{i=1\\ i\neq j}}^{k-1}(\zeta_j' - \zeta_i')} \tag{2.30}$$

$$\Theta_j^1 = \frac{B_{k+1}}{(k+1)!} g_k \frac{\mu! \sigma_j}{G^{(\mu)}(\sigma_j)} \tag{2.31}$$

$$\Theta_j^2 = \frac{B_{k+2}}{(k+2)!} g_k \frac{\mu! \sigma_j}{G^{(\mu)}(\sigma_j)} [(k+1)\sigma_j + \sum_{i=1}^{m} \sigma_i - \sum_{i=1}^{n} \pi_i]. \tag{2.32}$$

Remark 2.3.1. Denote $\mathcal{J} = \{j, j+1, \cdots j+\mu-1\}$ a set of integers indicating μ multiple zeros, $\sigma_j = \sigma_{j+1} \ldots = \sigma_{j+\mu-1}$. Then there is

$$G^{(\mu)}(\sigma_j) = (d/ds)^\mu G(s)|_{s=\sigma_j} = g_k \frac{\mu! \prod_{\substack{i=1\\ i\notin \mathcal{J}}}^{m} (\sigma_j - \sigma_i)}{\prod_{i=1}^{n} (\sigma_j - \pi_i)}. \tag{2.33}$$

Corollary 2.3.1. *Single intrinsic zeros $z_j(h)$ of $H(z)$ obey:*

$$z_j(h) = e^{\sigma_j h} + \Theta_j^{\alpha} h^{k+\alpha+1} + o(h^{k+\alpha+1}) \tag{2.34}$$

for $j = 1, 2 \ldots, m$, with $\alpha = 1$ for k odd and $\alpha = 2$ for k even, and

$$\Theta_j^1 = \frac{B_{k+1}}{(k+1)!} g_k \frac{\sigma_j}{G'(\sigma_j)} \tag{2.35}$$

$$\Theta_j^2 = \frac{B_{k+2}}{(k+2)!} g_k \frac{\sigma_j}{G'(\sigma_j)} [(k+1)\sigma_j + \sum_{i=1}^{m} \sigma_i - \sum_{i=1}^{n} \pi_i]. \tag{2.36}$$

2.4 Correspondence with Known Results

(a) Theorem 1 of (Hagiwara, 1996) stating that for a single intrinsic zero $z_j(h)$:

$$z_j(h) = 1 + \frac{\sigma_j}{1!} h + \frac{\sigma_j^2}{2!} h^2 + \left(\frac{\sigma_j^3}{3!} + g_1 \frac{\sigma_j}{12 G'(\sigma_j)} \right) h^3 + O(h^4) \tag{2.37}$$

follows directly from (2.34), which in the particular case of $k = 1$ reads:

$$z_j(h) = e^{\sigma_j h} + g_1 \frac{\sigma_j}{12 G'(\sigma_j)} h^3 + o(h^3). \tag{2.38}$$

(b) Let $k = 2$. Then from (2.30) it results that

$$\zeta(h) = -1 + \frac{1}{3} (\sum_{i=1}^{m} \sigma_i - \sum_{i=1}^{n} \pi_i) h + o(h). \tag{2.39}$$

Theorem 5 of (Hagiwara et al., 1993), which asserts that $H(z)$ has a limiting discretization zero $\zeta = -1$ for $k = 2$, where the direction of approach is from the inside of the unit disc if $\sum_{i=1}^{n} \pi_i < \sum_{i=1}^{m} \sigma_i$, results directly from (2.39). It is clear that (2.39) provides more insight into the limiting behavior of zero than the above stability criterion of (Hagiwara et al., 1993).

(c) It is known (Hagiwara, 1996) that the pulse transfer function $H(z)$ for

$$G(s) = \frac{s - \gamma}{(s-p)(s-q)(s-2\gamma)}, \ \gamma = \frac{p+q}{2} \tag{2.40}$$

has an intrinsic zero $z_1 = e^{\gamma h}$ and a discretization zero $\zeta_1 = -e^{\gamma h}$ so that it must be $\Omega_1^0 = -\gamma$ and $\Theta_1^2 = 0$. From (2.30) or (2.39) and (2.32) it is easy to check that this is indeed the case.

(d) A sampled-data system with the following plant:

$$G(s) = \frac{(as+1)(bs+1)}{(s+1)^3} \tag{2.41}$$

having a triple pole ($\pi_1 = \pi_2 = \pi_3 = -1$) and - depending on time constants a and b - none, one ($\sigma_1 = -1/a$) or two ($\sigma_1 = -1/a$, $\sigma_2 = -1/b$) finite zeros is considered.

We then have

$$H(z) = \frac{b_2 z^2 + b_1 z + b_0}{(z - e^{-h})^3} = b_2 \frac{(z - z_1)(z - z_2)}{(z - e^{-h})^3}. \tag{2.42}$$

Depending on the relative order of $G(s)$, one gets 3 different cases studied in detail in what follows.

The case $a = 0$, $b = 0$: The relative order of (2.41) equals to 3 and (2.42) takes the form:

$$H(z) = b_2 \frac{(z - \zeta_1)(z - \zeta_2)}{(z - e^{-h})^3}, \tag{2.43}$$

where, according to Theorem 2.3.1:

$$\zeta_1 = \zeta_1' + \Omega_1^0 h + o_1(h), \ \zeta_2 = \zeta_2' + \Omega_2^0 h + o_2(h) \tag{2.44}$$

with:

$$\zeta_1' = -(2-\sqrt{3}),\ \Omega_1^0 = -\frac{3}{4}(2-\sqrt{3}) \tag{2.45}$$

$$\zeta_2' = -(2+\sqrt{3}),\ \Omega_2^0 = -\frac{3}{4}(2+\sqrt{3}). \tag{2.46}$$

The case $a \neq 0$, $b = 0$: The relative order of (2.41) equals to 2 and (2.42) takes the form:

$$H(z) = b_2\frac{(z-\zeta_1)(z-z_1)}{(z-e^{-h})^3}, \tag{2.47}$$

where, according to Theorem 2.3.1:

$$\zeta_1 = -1 + \Omega_1^0 h + o_1(h),\ z_1 = e^{-h/a} + \Omega_1^2 h^5 + o_2(h^5) \tag{2.48}$$

with:

$$\Omega_1^0 = \frac{1-3a}{3a},\ \Omega_1^2 = \frac{1}{720}\frac{(1-a)^3(3a-4)}{a^5}. \tag{2.49}$$

The case $a \neq 0$, $b \neq 0$: The relative order of (2.41) equals to 1. Two subcases are to be considered.

(a) different continuous-time zeros $(a \neq b)$

The system in (2.42) has the following zeros:

$$z_1 = e^{-h/a} + \Omega_1^1 h^3 + o_1(h^3),\ z_2 = e^{-h/b} + \Omega_2^1 h^3 + o_2(h^3) \tag{2.50}$$

with:

$$\Omega_1^1 = -\frac{1}{12}\frac{b(1-a)^3}{a^3(a-b)},\ \Omega_2^1 = \frac{1}{12}\frac{a(1-b)^3}{b^3(a-b)}. \tag{2.51}$$

(b) double continuous-time zero $(a = b)$

We have:

$$(z_1 - z_1')(z_2 - z_1') = \Theta_1^1 h^4 + o(h^4) \tag{2.52}$$

with:

$$z_1' = e^{-h/a},\ \Theta_1^1 = -\frac{1}{12}\frac{(1-a)^3}{a^3}. \tag{2.53}$$

Analytic expression for $H(z)$ has been found in (Błachuta, 1997*f*) giving the following values for the numerator coefficients of $H(z)$ in (2.42):

$$b_2 = 1 - [1 + \alpha_1 h + \alpha_2 h^2]e^{-h} \tag{2.54}$$

$$b_1 = [-2 + \alpha_1 h + \alpha_2 h^2]e^{-h} + [2 + \alpha_1 h - \alpha_2 h^2]e^{-2h} \tag{2.55}$$

$$b_0 = [1 - \alpha_1 h + \alpha_2 h^2]e^{-2h} - e^{-3h} \tag{2.56}$$

with

$$\alpha_1 = 1 - ab,\ \alpha_2 = \frac{1 + ab - a - b}{2}. \tag{2.57}$$

Both zeros, z_1 and z_2, can be found analytically, and the derivatives, $\gamma_i = b_2^{(i)}(0)$, of $b_2(h)$ with respect to h at $h = 0$ are: $\gamma_1 = ab$, $\gamma_2 = a + b - 3ab$, $\gamma_3 = 1 - 3a - 3b + 6ab$, $\gamma_4 = -3 + 6a + 6b - 10ab$. This yields: $b_2 = h^3/6 + o(h^3)$ if $a = b = 0$, $b_2 = ah^2/2 + o(h^2)$ if $b = 0$, and $b_2 = abh + o(h)$ when both a and b are nonzero. Formulae (2.29) and (2.34)-(2.36) were checked on this example for $k = 1, 2, 3$ based on exact zeros, the rule of de l'Hospital and symbolic computations.

2.5 Conclusion

A theorem has been proved that, for small sampling periods, characterizes the accuracy of all limiting zeros of the pulse transfer function of a system composed of a zero-order hold followed by a continuous-time plant.

The main result has a form of a correction to the asymptotic result of Åström et al. (1984) in the form of a power term of h, whose degree depends on the relative order of the continuous-time counterpart and its contribution is expressed in terms of Bernoulli numbers and the poles and zeros of the continuous-time transfer function.

The discussion is based on two fundamental lemmas. The first lemma yields two terms of the Taylor series expansion of the pulse transfer function around $h = 0$ and the second characterizes the magnitude of the difference between the exact pulse transfer function and the principal term of its Poisson representation as a function of h.

Similar methods can be applied to study limiting zeros for pulse transfer functions of systems with a first-order hold. This will be done in chapter 3.

One of possible applications of the result is investigation of the accuracy of approximate pulse-transfer functions. This issue will be discussed in chapter 4.

3. Zeros of Systems with First-Order Hold

The Hagiwara-Yuasa-Araki theorems on limiting zeros of the pulse transfer function of sampled-data systems with first-order holds are extended by stating that limiting intrinsic zeros can be expressed as exponential functions of continuous-time zeros, and by determining the accuracy of the asymptotic results for both the discretization and the intrinsic zeros when the sampling interval is small. Closed form formulae are derived that express both the degree of the principal term of Taylor expansion of the difference between the true zeros and the limiting ones as a function of the relative degree of the underlying continuous-time system and the value of the corresponding coefficient itself.

3.1 Introduction

The chapter [1] is concerned with the zeros of sampled-data systems resulting from continuous-time systems preceded by a first-order hold (FOH) and followed by a sampler. The main motivation for FOH is reduction of intersample ripple, particularly in the steady state for a ramp-wise reference when the continuous-time plant is of Type 0.

As far as limiting zeros of sampled-data systems with zero-order hold (ZOH) at high sampling rates are concerned, quite a large number of results are known to date (Åström et al., 1984; Błachuta, 1997*f*; Hagiwara, 1996; Hagiwara et al., 1993; Lindorff, 1965).

The main reference in the area of interest is (Hagiwara et al., 1993), where stress is put on stability of limiting discretization zeros and (Weller, Moran, Ninness & Pollington, 1997*a*; Weller et al., 1997*b*), where some conjectures stated in (Hagiwara et al., 1993) are proved.

The aim of this chapter is to extend the methodology of chapter 2 to systems with FOH in order to show that intrinsic zeros are related to continuous-time zeros approximately exponentially, and to determine the accuracy of the asymptotic formulae of both intrinsic and discretization zeros at high sampling rates.

[1]The material of this chapter is based on (Błachuta, 1997*f*) and (Błachuta, 1998*d*)

The results can be applied to investigate the accuracy of certain approximate pulse transfer functions that base on limiting zeros or their Padé approximation as it will be done in Chapter 4.

The chapter is organized as follows. Known results on zeros are briefly surveyed in section 3.2. Lemmas necessary for proofs of new theorems are collected in section 3.3 while the theorems themselves are formulated in section 3.4. Proofs of lemmas and the main theorem are collected in Appendix A.2. Conclusions are drawn in section 3.5.

3.2 Survey of Known Results

Rational strictly proper continuous-time transfer functions $G(s)$ with the relative order $k = n - m > 0$ are considered of the form:

$$G(s) = \frac{\sum_{j=0}^{m} \beta_j s^j}{\sum_{i=0}^{n} \alpha_i s^i} = g_k \frac{\prod_{i=1}^{m} (s - \sigma_i)}{\prod_{i=1}^{n} (s - \pi_i)}. \tag{3.1}$$

Assume that $\alpha_n \neq 0$, $\beta_m \neq 0$, and $G(s)$ is of Type l, $l \geq 0$, i.e. $\alpha_0 = \dots \alpha_{l-1} = 0$ and $\alpha_l \neq 0$. Moreover, a unity gain, i.e. $\alpha_l = 1$ and $\beta_0 = 1$ will be assumed for simplicity. Then

$$g_k = \frac{\beta_m}{\alpha_n} = \frac{\prod_{i=1}^{n-l} (-\pi_i)}{\prod_{i=1}^{m} (-\sigma_i)} \tag{3.2}$$

is the k-th Markov parameter of (3.1).

3.2.1 Pulse Transfer Functions

Let $H(z)$ be the pulse transfer function of a series connection of a first-order hold and a continuous-time system with the transfer function $G(s)$, and let h be the sampling period. Then, according to Jury (1958) the general form of $H(z)$ is

$$H(z) = (1 - z^{-1})^2 \mathcal{Z}\{\frac{1 + sh}{s^2 h} G(s)\}. \tag{3.3}$$

For almost every sampling period h the pulse transfer function $H(z)$ has n zeros. As a result

$$H(z) = \frac{\sum_{j=0}^{n} b_j z^j}{z \sum_{i=0}^{n} a_i z^i} = b_n \frac{\prod_{i=1}^{n} (z - z_i)}{z \prod_{i=1}^{n} (z - p_i)} \tag{3.4}$$

with $p_i = e^{\pi_i h}$.

3.2.2 Relevant Polynomials

Polynomials

$$\mathcal{F}_k(z) = \sum_{i=0}^{k} \phi_i^k z^{k-i}$$

defined in terms of Euler polynomials (2.12)-(2.14) as

$$\mathcal{F}_k(z) = \mathcal{E}_{k+1}(z) + (k+1)(z-1)\mathcal{E}_k(z) \tag{3.5}$$

play an important role in the study of limiting discretization zeros. Equivalent nonrecursive definitions of $\mathcal{E}_k(z)$, which are called Euler (Frobenius, 1910), normal (Kowalczuk, 1983) or reciprocal (Hagiwara et al., 1993) polynomials, can be found in (Åström et al., 1984; Błachuta, 1997f; Hagiwara et al., 1993; Jury, 1964) and (Kowalczuk, 1983).

The following is known about $\mathcal{E}_k(z)$ and $\mathcal{F}_k(z)$:

(a) All roots ξ_i of $\mathcal{E}_k(z)$ are single and negative real for any k, i.e. $\xi_1 < \dots < \xi_{k-1} < 0$. Furthermore, the roots of $\mathcal{E}_k(z)$ interlace the roots η_i of $\mathcal{E}_{k+1}(z)$ on the negative real axis, i.e. $\eta_1 < \xi_1 < \eta_2 < \xi_2 < \cdots < \xi_{k-1} < \eta_k < 0$.

(b) All roots ζ_i of $\mathcal{F}_k(z)$ are single and real for any k, i.e. $\zeta_1 < \dots < \zeta_k$. Furthermore, the ith smallest root of $\mathcal{F}_k(z)$ lies between the ith smallest root of $\mathcal{E}_k(z)$ and the ith smallest root of $\mathcal{E}_{k+1}(z)$, i.e.

$$\eta_1 < \zeta_1 < \xi_1 \cdots < \eta_{k-1} < \zeta_{k-1} < \xi_{k-1} < \eta_k < 0 < \zeta_k < 1.$$

(c) The largest root of $\mathcal{F}_k(z)$ approaches $z = 1/e$ as $k \to \infty$, where e is the base of natural logarithm.

(d) $\mathcal{E}_k(z)$ are symmetrical, i.e. $\epsilon_i^k = \epsilon_{k-i+1}^k$ for $i = 1, \dots, k$ and the roots ξ_i of are pair-wise reciprocal, i.e. $\xi_i \xi_{k-i} = 1$ for $i = 1, \dots, k-1$.

(e) For even k, $z = -1$ is a root of $\mathcal{E}_k(z)$.

(f) For $k = 2$, $\mathcal{E}_k(z)$ has a root on the unit disk and for $k \geq 3$ outside the closed unit disk.

(g) For $k \geq 2$, $\mathcal{F}_k(z)$ has a root outside the closed unit disk.

(h) $\mathcal{E}_k(1) = k!$, $\mathcal{F}_k(1) = (k+1)!$.

(i) $\phi_0^k = k + 2$.

Items (a)-(c) were conjectured in (Hagiwara et al., 1993) based on numerical evidence for k up to 50. However, item (a), which due to (Hagiwara et al., 1993) implies (b), appeared to be already known (Frobenius, 1910). Item (c) was proved in (Weller et al., 1997b) based on the theory of Sobolev (1977).

3.2.3 Theorems on Zeros of Systems with FOH

The following two theorems of (Hagiwara et al., 1993), adapted to the notation used here, characterize limiting zeros of $H(z)$ and, for small h, the correspondence between finite continuous-time and discrete-time zeros.

Theorem 3.2.1.

$$\lim_{h\to 0} h^{m-n} H(z) = \frac{g_k}{\mathcal{F}_k(1)} \frac{\mathcal{F}_k(z)(z-1)^m}{z(z-1)^n}. \tag{3.6}$$

This theorem suggests that the m limiting zeros approaching $z = 1$ correspond to the continuous-time zeros and that the remaining k zeros approaching the roots of $\mathcal{F}_k(z)$ are newly generated by discretization. The former zeros are called *intrinsic zeros* while the latter *discretization zeros.*

For h finite but small enough, the correspondence between the intrinsic zeros and continuous-time zeros is characterized by Theorem 3.2.2.

Theorem 3.2.2. *Let σ_i be a zero of $G(s)$ with multiplicity μ. Suppose that $\mathcal{S}$ is a simply-connected bounded domain which includes σ_i inside and has no other zeros of $G(s)$ inside nor on its boundary. Then, there exist some $h_{\mathcal{S}}$ such that for every h satisfying $0 < h < h_{\mathcal{S}}$, $H(z)$ has μ zeros inside the domain*

$$e^{\mathcal{S}h} := e^{sh}|_{s\in\mathcal{S}}. \tag{3.7}$$

As a result, if σ_i is a stable (respectively unstable) zero of $G(s)$, then the corresponding limiting zero of $H(z)$ is also stable (respectively unstable).

3.3 Fundamental Lemmas

A link between a continuous-time transfer function $G(s)$ and its discrete-time counterpart $H(z)$ is defined (Jury, 1964) by the formula:

$$H(e^{sh}) = \frac{(1-e^{-sh})^2}{h^2} \sum_{l=-\infty}^{\infty} \frac{1+(s+jl\omega_s)h}{(s+jl\omega_s)^2} G(s+jl\omega_s), \tag{3.8}$$

where $\omega_s = 2\pi/h$.

Let us divide equation (3.8) into two parts:

$$H(e^{sh}) = G_h(s) + \Delta_h(s), \tag{3.9}$$

where

$$G_h(s) = \frac{(1+sh)(1-e^{-sh})^2}{s^2h^2} G(s) \tag{3.10}$$

is the $l = 0$ term. From (3.8)-(3.10) we have $\lim_{s\to 0} \Delta_h(s) = 0$, which means that the steady-state properties of $H(e^{sh})$ are the same as those of $G_h(s)$ and only transients are affected by $\Delta_h(s)$. For sh small enough $\Delta_h(s)$ is supposed to be small. This is specified in Lemma 3.3.1, which plays a crucial role in further argument.

Proofs of lemmas to follow are collected in Appendix A.1.

Lemma 3.3.1. For any finite $s \in \mathbb{C}$

$$\Delta_h(s) = \phi_\alpha(k,s)h^{k+2} + o(h^{k+2}), \tag{3.11}$$

where $\alpha = 1$ for k odd, $\alpha = 2$ for k even,

$$\phi_1(k,s) = -\frac{B_{k+1}}{(k+1)!} g_k s^2$$

$$\phi_2(k,s) = \frac{B_{k+2}}{(k+2)!} g_k s^2 \tag{3.12}$$

and B_{k+1}, B_{k+2} are the Bernoulli numbers.

Lemma 3.3.2. *For any finite $z \in \mathbb{C}$, $z \neq 0$ and $z \neq 1$, $H(z)$ admits the following expansion:*

$$H(z) = c_k(z)h^k + c_{k+1}(z)h^{k+1} + o(h^{k+1}), \tag{3.13}$$

where

$$c_i(z) = \frac{g_i}{\mathcal{F}_i(1)} \frac{\mathcal{F}_i(z)}{z(z-1)^i}, \quad i = k, k+1 \tag{3.14}$$

and g_i is the i-th Markov parameter of $G(s)$.

Remark 3.3.1. Lemma 3.3.2 provides an immediate proof of Theorem 3.2.1 as an alternative to that in (Hagiwara et al., 1993).

3.4 New Results

Similarly to the ZOH case (Hagiwara, 1996), a direct consequence of Theorem 3.2.2 is that for h small enough $z_i'(h) = e^{\sigma_i h}$ is an approximation of the zero $z_i(h)$ which corresponds to the continuous-time zero σ_i. The following theorem expresses the fact that $z_i'(h)$ is also a limiting zero.

Theorem 3.4.1. *Let σ_j denotes a finite zero of $G(s)$. Then $z_i'(h) = e^{s_i h}$ is a limiting intrinsic zero of $H(z)$, i.e. $\lim_{h\to 0} H[z_i'(h)] = 0$, if and only if $s_i = \sigma_i$.*

Proof of Theorem 3.4.1 From (3.9)-(3.11) it results that

$$\lim_{h\to 0} H(e^{s_i h}) = G(s_i). \tag{3.15}$$

□

Theorem 3.4.1 allows to say that intrinsic zeros approach 1 as $e^{\sigma_i h}$. However, both Theorem 3.2.2 and Theorem 3.4.1 lack any estimate of how close $z_i'(h)$ is to $z_i(h)$ for small h. Theorem 3.4.2 addresses this issue giving more insight into the characterization of discrete-time zeros at high sampling rates.

Theorem 3.4.2. *Let σ_j denotes a finite zero of $G(s)$ with multiplicity μ, and π_i, $i = 1\dots n$, denote poles. Then:*

(i)

$$b_n = \frac{k+2}{(k+1)!} g_k h^k + o(h^k) \tag{3.16}$$

(ii) the intrinsic zeros $z_{j+i}(h)$ of $H(z)$ obey:

$$\prod_{i=0}^{\mu-1} (z_{j+i} - e^{\sigma_j h}) = (-1)^{\mu-1}\Theta_j^\alpha h^{k+\mu+2} + o(h^{k+\mu+2}) \tag{3.17}$$

with $\alpha = 1$ for k odd and $\alpha = 2$ for k even, and

(iii) for the remaining k discretization zeros $\zeta_j(h)$ there is:

$$\zeta_j(h) = \zeta_j' + \Omega_j^0 h + o(h), \tag{3.18}$$

where $\mathcal{F}_k(\zeta_j') = 0$ for $j = 1, 2\dots k$, and

$$\Omega_j^0 = \frac{\sum_{i=1}^m \sigma_i - \sum_{i=1}^n \pi_i}{(k+2)^2} \frac{\mathcal{F}_{k+1}(\zeta_j')}{(\zeta_j' - 1)\prod_{\substack{i=1\\ i\neq j}}^k (\zeta_j' - \zeta_i')} \tag{3.19}$$

$$\Theta_j^1 = \frac{B_{k+1}}{(k+1)!} g_k \frac{\mu!\sigma_j^2}{G^{(\mu)}(\sigma_j)} \tag{3.20}$$

$$\Theta_j^2 = \frac{B_{k+2}}{(k+2)!} g_k \frac{\mu!\sigma_j^2}{G^{(\mu)}(\sigma_j)}. \tag{3.21}$$

Remark 3.4.1. Denote $\mathcal{J} = \{j, j+1, \cdots j+\mu-1\}$ a set of integers indicating μ multiple zeros, $\sigma_j = \sigma_{j+1}\dots = \sigma_{j+\mu-1}$. Then there is

$$G^{(\mu)}(\sigma_j) = (\frac{d}{ds})^\mu G(s)|_{s=\sigma_j} = g_k \frac{\mu! \prod_{\substack{i=1\\ i\notin\mathcal{J}}}^m (\sigma_j - \sigma_i)}{\prod_{i=1}^n (\sigma_j - \pi_i)}. \tag{3.22}$$

Corollary 3.4.1. *Single intrinsic zeros $z_j(h)$ of $H(z)$ obey:*

$$z_j(h) = e^{\sigma_j h} + \Theta_j^\alpha h^{k+3} + o(h^{k+3}) \tag{3.23}$$

for $j = 1, 2\dots, m$, with $\alpha = 1$ for k odd and $\alpha = 2$ for k even, and

$$\Theta_j^1 = \frac{B_{k+1}}{(k+1)!} g_k \frac{\sigma_j^2}{G'(\sigma_j)}, \quad \Theta_j^2 = \frac{B_{k+2}}{(k+2)!} g_k \frac{\sigma_j^2}{G'(\sigma_j)}. \tag{3.24}$$

3.5 Conclusion

Two theorems concerning zeros of sampled data systems with a first order hold at high sampling rates have been proved. The first shows that the limiting intrinsic discrete-time zeros are determined by exponential mappings of continuous-time zeros. The second characterizes the accuracy of all limiting zeros including the discretization ones.

The proofs are based on two fundamental lemmas. The first characterizes the magnitude of the difference between the exact pulse transfer function and the principal term of its infinite series representation as a function of h and the second yields two terms of the Taylor series expansion of the pulse transfer function around $h = 0$.

Similarly to the ZOH case, the main result has the form of a correcting power term in h added to the asymptotic zero, whose degree depends on the relative order of the continuous-time counterpart and its contribution is expressed in terms of Bernoulli numbers and parameters of the continuous-time transfer function.

One of possible areas of application is investigation of the accuracy of approximate pulse-transfer functions, similarly as it will be done in Chapter 4 for systems with ZOH.

4. Approximate Pulse Transfer Functions

In this chapter [1] a systematic approach to a class of approximations to the pulse transfer function of a system consisting of a zero-order hold and a linear continuous-time plant is presented. It is based on the asymptotic result of Åström, Hagander & Sternby (1984) on zeros of sampled systems at high sampling rates, and on the bilinear transformation. Since the number of intrinsic parameters does not change in the discretization process, model matching control, robust control and identification are suggested as possible areas of application. Superiority of the approximations considered over a δ-operator based truncated approximation of Goodwin, Leal, Mayne & Middleton (1986) is shown. The results are illustrated by an example.

4.1 Introduction

The exact pulse transfer function of a sampled-data system consisting of a zero-order hold (ZOH) and a continuous-time plant is easily calculated numerically or symbolically. However, an important feature is that discrete-time parameters are complicated functions of continuous-time parameters. This particularly concerns the numerator of the pulse transfer function which depends on all continuous-time parameters.

Approximate pulse transfer functions are proposed that base on the notion of *limiting zeros* (Åström et al., 1984; Błachuta, 1997*f*; Hagiwara et al., 1993). Using the bilinear transformation into the w variable domain, three further approximations are also determined which not only require much less computations but also offer additional structural advantages. Since they are related more directly to the continuous-time parameters than the exact ones, they not only contribute to better understanding of discrete-time parameters but are also useful for system identification and control. Moreover, our approximations are shown to be superior to the *truncated* approximate transfer function of (Goodwin et al., 1986) obtained using δ-operator (Middleton & Goodwin, 1990).

Standard discrete-time identification methods assume that the parameters to be estimated are independent. As the number of discrete-time parameters is usually greater

[1] The chapter is based on (Błachuta, 1997*d*) and (Błachuta, 1998*a*)

than that of the underlying continuous-time system, the discrete-time model is over-parametrized and the result of identification can be incorrect. This problem is easily solved by using approximate transfer functions. Complicated relationships between the parameters of continuous-time and discretized system make it difficult (Ackermann & Hu, 1991) to map the uncertainty between both domains. For certain approximations, relationships between the parameters are linear. This greatly simplifies the robustness analysis of sampled-data systems with uncertain physical parameters. Discretization zeros lying close or outside the unit circle lead to a 'ringing' or diverging control signal in sampled-data *exact model matching* control systems. Due to structural properties of approximate pulse transfer functions the problem of *approximate model matching* with perfect intersample behavior is successfully solved within our framework.

The chapter is organized as follows. Section 4.2 formalizes the problem. In section 4.3, an asymptotic pulse transfer function that results from the Åström-Hagander-Sternby theorem is presented. Based on a bilinear transformation, further approximations are derived in section 4.4. Accuracy of the approximations considered is studied in section 4.5. Links with digital approximations methods are shown in section 4.6. Computational issues of the approximations based on bilinear transformation are presented in section 4.7. The application of approximate pulse transfer functions to the model matching control, robust control and to identification of both discrete- and continuous-time models is proposed in section 4.8. Remarks on the δ-operator approach are presented in section 4.9. Theoretical considerations are supported by a numerical example in section 4.10, and conclusions are drawn in section 4.11.

4.2 Problem Formalization

Let $G(s)$ be a rational continuous-time transfer function of Type l, $l \geq 0$:

$$G(s) = \frac{\beta(s)}{s^l \alpha(s)} = \frac{\sum_{j=0}^{m} \beta_j s^j}{s^l \sum_{i=0}^{n-l} \alpha_i s^i} \tag{4.1}$$

with $\deg \alpha(s) = n - l$, $\deg \beta(s) = m$, and the relative order

$$k = n - m > 0, \tag{4.2}$$

where it is assumed for simplicity that $\beta_0/\alpha_0 = 1$.

$G(s)$ can also be expressed in the time constant and pole-zero forms:

$$G(s) = \frac{\prod_{i=1}^{m}(s\tau_i + 1)}{s^l \prod_{i=1}^{n-l}(sT_i + 1)} = g_k \frac{\prod_{i=1}^{m}(s - \sigma_i)}{s^l \prod_{j=1}^{n-l}(s - \pi_j)}, \tag{4.3}$$

where $g_k = \prod_{i=1}^{m} \tau_i / \prod_{i=1}^{n-l} T_i$, $\sigma_i = -\tau_i^{-1}$, $\pi_i = -T_i^{-1}$.

The pulse transfer function $H(z)$ for a system consisting of a ZOH and a continuous-time plant with a transfer function $G(s)$ has the form:

$$H(z) = \frac{B(z)}{(z-1)^l A(z)}, \tag{4.4}$$

where $\deg A(z) = n - l$, $\deg B(z) = n - 1$ for almost every h, and $B(1)/A(1) = h^l$.

The following class of functions is studied:

$$\mathcal{H}(z) = \frac{\mathcal{B}(z)}{(z-1)^l \mathcal{A}(z)} = \frac{\mathcal{P}(z)\mathcal{B}_0(z)}{(z-1)^l \mathcal{A}(z)}, \tag{4.5}$$

where the coefficients of $\mathcal{B}_0(z)$ only depend on the coefficients of $\beta(s)$, $\mathcal{P}(z)$ is a matching polynomial which only depends on k and does not depend either on h or on the parameters of $G(s)$, $\deg \mathcal{A}(z) = n - l$, $\deg \mathcal{B}(z) = n - 1$, $\deg \mathcal{B}_0(z) = m$, $\deg \mathcal{P}(z) = k - 1$. $\mathcal{H}(z)$ can also be presented in the factorized form:

$$\mathcal{H}(z) = b_{n-1} \frac{\mathcal{P}(z) \prod_{i=1}^{m}(z - \bar{z}_i)}{(z-1)^l \prod_{i=1}^{n-l}(z - \bar{p}_i)}, \tag{4.6}$$

where

$$b_{n-1} = \frac{c_m^j}{\mathcal{P}(1)}, \; c_m^j = h^l \frac{\prod_{i=1}^{n-l}(1 - \bar{p}_i)}{\prod_{i=1}^{m}(1 - \bar{z}_i)}, \tag{4.7}$$

$\bar{p}_i = \varphi(\pi_i)$, $\bar{z}_j = \varphi(\sigma_j)$, $i = 1 \ldots n-l$, $j = 1 \ldots m$, and both $\varphi(z)$ and $\mathcal{P}(z)$ differ depending on the particular approximation chosen.

It is easy to check that for $n > m$, $\mathcal{H}(z)$ and $H(z)$ have the same steady-state gain:

$$\lim_{z \to 1}\left(\frac{z-1}{h}\right)^l \mathcal{H}(z) = \lim_{z \to 1}\left(\frac{z-1}{h}\right)^l H(z) = \lim_{s \to 0} s^l G(s) \tag{4.8}$$

irrespective of the value of h.

It should be stressed that the strength of $\mathcal{H}(z)$ consists in its structure, where the appropriate choice of the matching polynomial $\mathcal{P}(z)$ admits $\mathcal{H}(z)$ to be characterized by the same number of parameters as $G(s)$ while retaining the vital properties of $H(z)$.

4.3 Approximation Based on the Å-H-S Theorem

Based on the asymptotic result of Åström et al. (1984), an approximation $H_0(z)$ of the pulse transfer function $H(z)$ is proposed in the form of:

$$H_0(z) = b^0_{n-1}\frac{\mathcal{E}_k(z)\prod_{i=1}^{m}(z-z_i^0)}{(z-1)^l\prod_{i=1}^{n-l}(z-p_i^0)}, \tag{4.9}$$

where

$$b^0_{n-1} = \frac{c^0_m}{\mathcal{E}_k(1)}, \quad c^0_m = h^l\frac{\prod_{i=1}^{n-l}(1-p_i^0)}{\prod_{i=1}^{m}(1-z_i^0)} \tag{4.10}$$

$$p_i^0 = e^{\pi_i h}, \quad z_i^0 = e^{\sigma_i h}. \tag{4.11}$$

The polynomial $\mathcal{E}_k(z)$, called Euler (Frobenius, 1910) or reciprocal (Hagiwara et al., 1993) polynomial, is defined as:

$$\mathcal{E}_k(z) = \epsilon_1^k z^{k-1} + \epsilon_2^k z^{k-2} + \ldots + \epsilon_k^k \tag{4.12}$$

with the coefficients ϵ_i^k:

$$\epsilon_i^k = \sum_{j=1}^{i}(-1)^{i-j} j^k \binom{k+1}{i-j}, \quad i = 1\ldots k. \tag{4.13}$$

According to (Frobenius, 1910), $\mathcal{E}_k(z)$ can be expressed as follows

$$\mathcal{E}_k(z) = \begin{cases} \prod_{i=1}^{q(k)}(z-\zeta_i)(z-\zeta_i^{-1}), & k \text{ - odd} \\ (z+1)\prod_{i=1}^{q(k)}(z-\zeta_i)(z-\zeta_i^{-1}), & k \text{ - even.} \end{cases} \tag{4.14}$$

where $q(k) = (k-1)/2$ for k odd and $q(k) = k/2 - 1$ for k even, and ζ_i are single, real and negative.

4.4 Further Approximations

An important feature of the w-plane domain,

$$w = \frac{2}{h}\frac{z-1}{z+1}, \tag{4.15}$$

is that the intrinsic zeros map to the positions close to the original continuous-time ones while the discretization zeros map to positions on the negative real axis far away from the origin. This suggests further approximations of $H_0(z)$. Define

$$G_0(w) = H_0(z)\mid_{z=\frac{2+wh}{2-wh}}. \tag{4.16}$$

From (4.14) and (4.9) one gets

$$G_0(w) = \phi_0(w)\frac{\prod_{i=1}^{m}(w\tau_i^0+1)}{w^l\prod_{i=1}^{n-l}(wT_i^0+1)}, \tag{4.17}$$

$$\phi_0(w) = \phi_1(w)\phi_2(w), \quad \phi_1(w) = 1 - w\frac{h}{2}, \quad \phi_2(w) = \prod_{i=1}^{q(k)}[1-(\omega_i^0 w)^2] \tag{4.18}$$

and

$$T_i^0 = \frac{h}{2}\frac{1+p_i^0}{1-p_i^0}, \quad \tau_i^0 = \frac{h}{2}\frac{1+z_i^0}{1-z_i^0}, \quad \omega_i^0 = \frac{h}{2}\frac{1+\zeta_i}{1-\zeta_i}, \tag{4.19}$$

where i changes in the limits specified in (4.17). $\phi_0(w)$ can be interpreted as a representation of the ZOH. The form of (4.18) results from the fact that the zeros of $\mathcal{E}_k(z)$ occur in reciprocal pairs, ζ_i and ζ_i^{-1}, which implies that the time constants of $\phi_2(w)$ also occur in symmetrical pairs, ω_i^0 and $-\omega_i^0$. From (4.10), (4.11) and (4.19) it results that

$$T_i^0 = T_i + O(h^2), \quad \tau_i^0 = \tau_i + O(h^2), \quad \omega_i^0 = O(h). \tag{4.20}$$

The poles and m zeros of $G_0(w)$ converge to their continuous time counterparts, i.e. $T_i^0 \to T_i$, $\tau_i^0 \to \tau_i$, and the remaining zeros go to infinity, i.e. $\omega_i \to 0$, as $h \to 0$. As a result $G_0(w) \to G(w)$ for finite w. From equations (4.17)-(4.19) it is seen that the effect of using original time constants instead of T_i^0 and τ_i^0, and that of omitting discretization time constants ω_i are relatively small. As a result, the following further approximations are proposed:

$$G_1(w) = \phi_0(w)\frac{\prod_{i=1}^{m}(w\tau_i+1)}{w^l\prod_{i=1}^{n-l}(wT_i+1)} \tag{4.21}$$

$$G_2(w) = \phi_1(w)\frac{\prod_{i=1}^{m}(w\tau_i^0+1)}{w^l\prod_{i=1}^{n-l}(wT_i^0+1)} \tag{4.22}$$

$$G_3(w) = \phi_1(w)\frac{\prod_{i=1}^{m}(w\tau_i+1)}{w^l\prod_{i=1}^{n-l}(wT_i+1)}. \tag{4.23}$$

Transforming $G_1(w)$, $G_2(w)$ and $G_3(w)$ back to the z-plane yields:

$$H_1(z) = b^1_{n-1}\frac{\mathcal{E}_k(z)\prod_{i=1}^{m}(z-z_i^*)}{(z-1)^l\prod_{i=1}^{n-l}(z-p_i^*)} \tag{4.24}$$

$$H_2(z) = b_{n-1}^2 \frac{\mathcal{N}_k(z) \prod_{i=1}^{m} (z - z_i^0)}{(z-1)^l \prod_{i=1}^{n-l} (z - p_i^0)} \tag{4.25}$$

$$H_3(z) = b_{n-1}^3 \frac{\mathcal{N}_k(z) \prod_{i=1}^{m} (z - z_i^*)}{(z-1)^l \prod_{i=1}^{n-l} (z - p_i^*)}, \tag{4.26}$$

where z_i^0, p_i^0 and c_m^0 are defined in (4.10)-(4.11), and

$$z_i^* = \frac{2+\sigma_i h}{2-\sigma_i h},\ p_i^* = \frac{2+\pi_i h}{2-\pi_i h},\ c_m^* = h^l \frac{\prod_{i=1}^{n-l} (1 - p_i^*)}{\prod_{i=1}^{m} (1 - z_i^*)} \tag{4.27}$$

$$b_{n-1}^1 = \frac{c_m^*}{\mathcal{E}_k(1)},\ b_{n-1}^2 = \frac{c_m^0}{\mathcal{N}_k(1)},\ b_{n-1}^3 = \frac{c_m^*}{\mathcal{N}_k(1)}. \tag{4.28}$$

Newton binomials $\mathcal{N}_k(z) = (z+1)^{k-1}$ can be expanded to

$$\mathcal{N}_k(z) = \nu_1^k z^{k-1} + \nu_2^k z^{k-2} + \ldots + \nu_k^k \tag{4.29}$$

with

$$\nu_i^k = \frac{(k-1)!}{i!(k-i-1)!} \tag{4.30}$$

An important feature of $H_1(z)$ and $H_3(z)$ is that they can be obtained directly from (4.1) by using the bilinear transformation without the need of calculating poles and zeros. This issue is further discussed in sections 4.6 and 4.7. The polynomials $\mathcal{E}_k(z)$ and $\mathcal{N}_k(z)$ differ for $k \geq 3$. As a result, the numerators of $H_2(z)$ and $H_3(z)$ do not converge to the numerator of $H(z)$ as $h \to 0$ when $k \geq 3$. Nevertheless, $\lim_{h \to 0} H_i(e^{sh}) = G(s)$ for finite s and all $i = 0, 1, 2, 3$.

4.5 Accuracy of Approximations

The accuracy of poles, zeros and frequency plots is the ultimate factor to asses the applicability of approximate pulse transfer function for control purposes. It has been shown in Chapter 2 that for single intrinsic zeros $z_i(h)$, $i = 1, 2 \ldots m$ and discretization zeros $z_{m+j}(h)$, $j = 1, 2 \ldots k-1$ there is:

$$z_i(h) - z_i^0(h) = O(h^\kappa),\ z_{m+j}(h) - \zeta_j = O(h), \tag{4.31}$$

where $\kappa = k+2$ for k odd , $\kappa = k+3$ for k even. Since $\kappa \geq 3$ for k odd and $\kappa \geq 5$ for k even, the accuracy of the asymptotic approximations of intrinsic zeros is quite high. This contrasts with the accuracy of limiting discretization zeros.

Consider $r_0(z) = H_0(z)/H(z)$. As a result of (4.31), for $k \geq 2$ and finite ω there is:

$$r_0(e^{j\omega h}) = 1 + O(h) + jO(h^2). \tag{4.32}$$

The relative accuracy of further approximations can be studied based on the relative values:

$$r_1(z) = \frac{H_1(z)}{H_0(z)} = \frac{H_3(z)}{H_2(z)},\ r_2(z) = \frac{H_2(z)}{H_0(z)} = \frac{H_3(z)}{H_1(z)}. \tag{4.33}$$

If we take into account that $p_i^0 - p_i^* = O(h^3)$ and $z_i^0 - z_i^* = O(h^3)$ then for finite ω

$$r_1(e^{j\omega h}) = 1 + jO(h^2). \tag{4.34}$$

From (4.14) and (4.24)-(4.26) there is

$$r_2(e^{j\omega h}) = \frac{[2\cos(\omega h/2)]^{2q(k)}}{\prod_{i=1}^{q(k)} [2\cos(\omega h) + \gamma_i]} \frac{\mathcal{E}_k(1)}{\mathcal{N}_k(1)}, \tag{4.35}$$

where $\gamma_i = -(\zeta_i + \zeta_i^{-1})$. This leads to:

$$r_2(e^{j\omega h}) = 1 + O(\omega h). \tag{4.36}$$

As a result, as long as ω belongs to the range of frequencies that are important for control design and h is chosen according to the standard guidelines, both the Nyquist and Bode plots of approximate pulse transfer functions $H_i(z)$, $i = 0, 1, 2, 3$ are close to the exact ones, and for $k \geq 2$ the accuracy of further approximations is of the same order as that of $H_0(z)$. Replacing $\mathcal{E}_k(z)$ by $\mathcal{N}_k(z)$ only affects the relative magnitude of frequency plots for higher values of ω and thus initial values of time responses.

4.6 Links with Digital Approximation Methods

Our approximations refer to sampled-data systems with a ZOH. In contrast to this, Tustin transformation (Tustin, 1947) defining $H_T(z)$:

$$H_T(z) = G(s)\,\big|_{s=\frac{2}{h}\frac{z-1}{z+1}} \tag{4.37}$$

and the so called matched pole-zero method (MPZ) (Franklin, Powell & Workman, 1990) defining $H_{MPZ}(z)$ are established techniques for discrete approximation of continuous-time systems (Kowalczuk, 1993) performed e.g. to compute the time response of a continuous-time system to a continuous-time excitation digitally. Since $H_T(z)$ does not account for a hold and the Nyquist plots of $H_T(z)$ and $G(s)$ overlap for any h, it should be stressed that unlike (Isermann, 1989), (Janiszowski, 1993) and other references $H_T(z)$

must not be regarded as an approximation of the transfer function $H(z)$, see also (Gessing, 1995) for further interpretations. This is illuminated by examples in section 4.10. Similar remarks apply to $H_{MPZ}(z)$. The relationships:

$$H_T(z) = \frac{z+1}{2} H_3(z), \; H_{MPZ}(z) = \frac{z+1}{2} H_2(z) \tag{4.38}$$

along with the discussion on $H_2(z)$ and $H_3(z)$ provide an additional insight into the Tustin and MPZ digitization methods.

$H_T(z)$ was used in (Sinha, 1972; Wymer, 1972) for identification of $G(s)$ based on sampled input and output data from a purely continuous-time system. A natural extension of this idea to a ZOH driven sampled system leads to the use of $H_i(z)$. This issue is further discussed in section 4.8.

4.7 Computation of $H_1(z)$ and $H_3(z)$

Denote

$$G^*(\lambda) = G(\frac{2}{h}\lambda) = h^{n-m} \frac{\beta^*(\lambda)}{\lambda^l \alpha^*(\lambda)}, \tag{4.39}$$

with $\beta_i^* = \beta_i(\frac{1}{2}h)^{m-i}$, $\alpha_i^* = \alpha_i(\frac{1}{2}h)^{n-l-i}$ and

$$\lambda = (z-1)/(z+1).$$

Then upon (4.37)-(4.38) $H_1(z)$ and $H_3(z)$ become:

$$H_1(z) = \frac{h^{n-m}}{\mathcal{E}_k(1)} \frac{\mathcal{E}_k(z) C^*(z)}{(z-1)^l A^*(z)} \tag{4.40}$$

$$H_3(z) = \frac{h^{n-m}}{\mathcal{N}_k(1)} \frac{\mathcal{N}_k(z) C^*(z)}{(z-1)^l A^*(z)}, \tag{4.41}$$

where:

$$C^*(z) = \sum_{i=0}^{m} c_i^* z^i = \sum_{i=0}^{m} \beta_i^*(z-1)^i (z+1)^{m-i} \tag{4.42}$$

$$A^*(z) = \sum_{i=0}^{n-l} a_i^* z^i = \sum_{i=0}^{n-l} \alpha_i^*(z-1)^i (z+1)^{n-l-i}. \tag{4.43}$$

Coefficients of $\mathcal{E}_k(z)$ and $\mathcal{N}_k(z)$ are determined respectively by (4.13) and (4.30). Algorithms to perform calculations in (4.42)-(4.43) can be summarized as follows. Assume that

$$q(\lambda) = \sum_{i=0}^{n} q_i \lambda^i$$

and

$$p(z) = \sum_{i=0}^{n} p_i z^i$$

are n-th degree polynomials related by

$$q(\lambda)\,|_{\lambda=\frac{z-1}{z+1}} = \frac{p(z)}{(z+1)^n}, \tag{4.44}$$

where

$$p(z) = \sum_{i=0}^{n} p_i z^i = \sum_{i=0}^{n} q_i (z-1)^i (z+1)^{n-i}, \tag{4.45}$$

and denote $\boldsymbol{p} = [p_n, \ldots, p_0]'$ and $\boldsymbol{q} = [q_n, \ldots, q_0]'$. The problem of interest is: given $\boldsymbol{q}$ find $\boldsymbol{p}$. Efficient algorithms for a slightly different transformation $\lambda' = (z+1)/(z-1)$ are presented in (Bose, 1983; Bush & Fielder, 1973; Davies, 1974; Ismail & Vakilazadian, 1989; Jury & Chan, 1973; Power, 1967; Power, 1968; Scott, 1994). Four main approaches were developed. The first and perhaps the most illustrative one, started by Power (Power, 1967; Power, 1968) and continued in (Bose, 1983; Bush & Fielder, 1973; Jury & Chan, 1973) is devoted to the so called $\boldsymbol{Q_n}$-matrix, such that $\boldsymbol{p} = \boldsymbol{Q_n q}$. The mechanization of the transformation (4.44)-(4.45) is then as follows: a) fill the first row of the $(n+1) \times (n+1)$ matrix $\boldsymbol{Q_n}$ with $1's$ and b) the last column with the binomial coefficients ν_i^{n+1} of (4.30), and c) calculate the remaining entries from the known upper row and right column elements by using a recurrence relation:

$$q_{i+1,j} = q_{i+1,j+1} - q_{i,j+1} - q_{i,j}, \;\; n \geq i, j \geq 1. \tag{4.46}$$

The first three $\boldsymbol{Q_n}$-matrices are:

$$\boldsymbol{Q}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \; \boldsymbol{Q}_2 = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, \; \boldsymbol{Q}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \\ 3 & -1 & -1 & 3 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \tag{4.47}$$

The second approach (Malvar, 1985; Parthasarathy & Jayasimha, 1984) is based on the subdivision of the bilinear transformation into a sequence of elementary transformations on $q(s)$. Since $s = -2/(z+1) + 1$, the algorithm below performs the transformation (4.44)-(4.45) in the order indicated by the arrow:

$$q(s) \to q(z+1) \to q(1/z+1) \to q(-2/z+1) \to q((-2/(z+1)+1) \to p(z). \tag{4.48}$$

The operations in (4.48) involve only a) scaling the magnitude of zeros, b) replacing the zeros by their reciprocals, and c) shifting the zeros by real constants. The first two operations are trivial and for the third the so called synthetic division is used, whose algorithm is presented in Fig.4.1. Ismail & Vakilazadian (1989) presented another approach based

```
r(1)=q(1)
for j=1 to n+1
    k=n+2-j
    for i=2 to k    r(i)=q(i)-r(i-1)
    for i=2 to k    q(i)=r(i)
```

Fig. 4.1. Calculation of $r(z) = q(z+1)$ by the synthetic division; high order coefficients first

on the theory of continued fractions. Perhaps the most efficient is the recent algorithm of Scott (1994), which bases on a recurrent formula for calculating successive derivatives of certain polynomial. The algorithms in (Davies, 1974; Scott, 1994; Ismail & Vakilazadian, 1989) can also be applied to the general bilinear transformation $\lambda = (az+b)/(cz+d)$. The most general algorithm for arbitrary polynomial transformation is presented in (Heinen & Siddique, 1988).

4.8 Areas of Application

Model matching control, robust control and identification are the areas where benefits are gained when using certain $H_i(z)$.

4.8.1 Model matching control

Given a plant having a minimum-phase transfer function $G(s)$ with relative order k, a stable proper 1 DOF continuous-time controller

$$K(s) = \frac{1}{G(s)} \frac{T(s)}{1 - T(s)} \tag{4.49}$$

can be found such that the transfer function of a stable unity feedback closed-loop system equals to $T(s)$ whose relative order is equal to k (Wolovich, 1994). Suppose that a discrete-time controller

$$K(z) = \frac{1}{H(z)} \frac{T(z)}{1 - T(z)} \tag{4.50}$$

of (Isermann, 1989) is applied with $T(z)$ being a step invariant transform of $T(s)$ so that at sampling instants the output of the closed loop system matches that of the continuous-time one assuming a step-wise set-point. Unfortunately, for $k \geq 2$ discretization zeros may lead to controller 'ringing', unacceptable output ripple or even to unstable control. Then the problem of approximate model matching with excellent intersample behavior is successfully solved within our framework by choosing

$$K_i(z) = \frac{1}{H_i(z)} \frac{T_i(z)}{1 - T_i(z)}, \tag{4.51}$$

where $H_i(z)$ and $T_i(z)$ are the discrete-time counterparts of $G(s)$ and $T(s)$ for any $i = 0, 1, 2, 3$. The point is that $H_i(z)$ and $T_i(z)$ share the same matching polynomial, $\mathcal{E}_k(z)$ or $\mathcal{N}_k(z)$, which cancels out from the controller transfer function. Similar remarks apply to 2 DOF controllers (Ichikawa, 1985; Wolovich, 1994).

4.8.2 System identification

Denote $\boldsymbol{\theta}_c = [\alpha_1, \ldots, \alpha_{n-l}, \beta_0, \ldots, \beta_m]'$ a $\partial\boldsymbol{\theta}_c = n + m - l + 1$ parameter vector of the continuous-time transfer function $G(s)$ of eq. (4.1) and $\boldsymbol{\theta}_d = [a_1, \ldots, a_{n-l}, b_0, \ldots, b_{n-1}]$ a $\partial\boldsymbol{\theta}_d = 2n - l$ parameter vector of the pulse transfer function $H(z)$ of eq. (4.4), where $\partial\boldsymbol{\theta}_d \geq \partial\boldsymbol{\theta}_c$. These two vectors are related by a non-linear relationship $\boldsymbol{\theta}_d = \boldsymbol{f}(\boldsymbol{\theta}_c)$. Given sampled measurements of the output of a continuous-time system driven by a ZOH, there are two ways to identify $\boldsymbol{\theta}_c$ and $\boldsymbol{\theta}_d$:

(a) Determine $\hat{\boldsymbol{\theta}}_d$ by means of any standard, e.g. (Isermann, 1989), parameter estimation procedure. Then $\hat{\boldsymbol{\theta}}_c = \arg\min ||\hat{\boldsymbol{\theta}}_d - \boldsymbol{f}(\boldsymbol{\theta}_c)||$.

(b) Apply a nonlinear estimation procedure, e.g. (Maine & Iliff, 1981), to find $\hat{\boldsymbol{\theta}}_c$ directly from data. Then $\hat{\boldsymbol{\theta}}_d$ is calculated from $\hat{\boldsymbol{\theta}}_d = \boldsymbol{f}(\hat{\boldsymbol{\theta}}_c)$.

Unfortunately, if $m < n-1$ then usually $||\hat{\boldsymbol{\theta}}_d - \boldsymbol{f}(\hat{\boldsymbol{\theta}}_c)|| \neq 0$ in *(a)*, e.g. (Söderström, 1991), while the procedure in *(b)* is both complex and time-consuming. This issue is greatly simplified by using the concept of approximate pulse transfer functions. Denote q a forward shift operator, and filtered variables $x(i) = (q-1)^l y(i)$, and $v(i) = \mathcal{P}(q)u(i)$. Then the following equation with a possible stochastic disturbance $e(i)$ gives rise to estimation of $n + m - l + 1$ unknown parameters $\boldsymbol{\theta}_d^0$ of the polynomials $\mathcal{A}(z)$ and $\mathcal{B}_0(z)$:

$$\mathcal{A}(q)x(i) = \mathcal{B}_0(q)v(i) + e(i). \tag{4.52}$$

The estimates $\hat{\mathcal{A}}(z)$ and $\hat{\mathcal{B}}_0(z)$ obtained this way can either be used as a final result or they can be transformed to the continuous-time domain e.g. by the relationship $\hat{\boldsymbol{\theta}}_c = \boldsymbol{Q}_n^{-1}\hat{\boldsymbol{\theta}}_d^0$, or eventually $\hat{\boldsymbol{\theta}}_c$ may serve as a starting point to a nonlinear estimation procedure in *(b)*. Statistical properties of estimators can be analyzed similarly as it was done in (Wymer, 1972). It should be stressed that the methods of continuous-time system identification that base on orthogonal functions or Poisson moment functionals described in (Unbehauen & Rao, 1987) and (Sinha & Rao, 1991) also identify the continuous-time parameters directly but they require either continuous-time signals or their dense samples, and the identified system is not driven by a hold.

4.8.3 Robust control

For the relationships between $\boldsymbol{\theta}_c$ and $\boldsymbol{\theta}_d$ are linear in the case of $H_1(z)$ and $H_3(z)$, they are particularly suited to map the uncertainty of parameters, e.g. those obtained as estimated confidence intervals, from the continuous-time domain to the discrete-time domain and vice versa, compare (Janiszowski, 1993). This greatly simplifies the robustness analysis of uncertain systems (Ackermann & Hu, 1991).

4.9 Remarks on the Truncated Approximation

The δ-operator, defined as

$$\delta x_i = \frac{x_{i+1} - x_i)}{h}, \tag{4.53}$$

has been used in (Goodwin et al., 1986) to arrive at an approximate transfer function that provides a rapprochement between continuous-time and discrete-time transfer functions in such sense that they have the same relative orders and thus the same numbers of zeros. These features were then used for identification and model reference control. Denote $\gamma = (z-1)/h$ a complex variable related to the δ-operator. Then

$$G^\delta(\gamma) = H(z)\mid_{z=\gamma h+1} = \frac{\beta_\epsilon^\delta(\gamma) + \beta_r^\delta(\gamma)}{\gamma^l \alpha^\delta(\gamma)} \tag{4.54}$$

with

$$\alpha^\delta(\gamma) = \gamma^n + \alpha_{n-1}^\delta \gamma^{n-1} + \ldots + \alpha_0^\delta \tag{4.55}$$

$$\beta_\epsilon^\delta(\gamma) = \epsilon_{n-1}\gamma^{n-1} + \ldots + \epsilon_{m+1}\gamma^{m+1} \tag{4.56}$$

$$\beta_r^\delta(\gamma) = \beta_m^\delta \gamma^m + \ldots + \beta_0^\delta. \tag{4.57}$$

Since there is $\alpha_i^\delta \to \alpha_i$, $\epsilon_i^\delta \to 0$ and $\beta_i^\delta \to \beta_i$ as $h \to 0$, where α_i and β_i are the coefficients of $\alpha(s)$ and $\beta(s)$ of $G(s)$ of equation (4.1), this gave rise in (Goodwin et al., 1986) to the *truncated* approximation $G_r^\delta(\gamma)$ of $G^\delta(\gamma)$:

$$G_r^\delta(\gamma) = \frac{\beta_r^\delta(\gamma)}{\gamma^l \alpha^\delta(\gamma)} \tag{4.58}$$

with the polynomial $\beta_\epsilon^\delta(\gamma)$ simply ignored. Reference (León de la Barra, 1997) suggests that this procedure leads to the undesired phase lag, which is explained as the effect of neglecting the discretization zeros of $G^\delta(\gamma)$, whose direct analysis in the δ-operator domain can be found in (Tesfaye & Tomizuka, 1995; Weller, 1998).

For it is difficult to study the accuracy of $G_r^\delta(\gamma)$, consider the mapping $G_0^\delta(\gamma) = H_0(z)\mid_{z=\gamma h+1}$:

$$G_0^\delta(\gamma) = \frac{\mathcal{E}_k^\delta(\gamma h)}{\mathcal{E}_k^\delta(0)} \frac{\prod_{i=1}^{m}(\gamma\tau_i^\delta + 1)}{\gamma^l \prod_{i=1}^{n-l}(\gamma T_i^\delta + 1)} \tag{4.59}$$

with

$$\frac{\mathcal{E}_k^\delta(\gamma h)}{\mathcal{E}_k(1)} = \begin{cases} \prod_{i=1}^{q(k)}(\omega_i\gamma + 1)(\theta_i\gamma + 1) \\ (\gamma h + 2)\prod_{i=1}^{q(k)}(\omega_i\gamma + 1)(\theta_i\gamma + 1), \end{cases} \tag{4.60}$$

where the first row is for odd k and the second for even k, and

$$T_i^\delta = \frac{h}{1 - p_i^0},\ \tau_i^\delta = \frac{h}{1 - z_i^0},\ \omega_i = \frac{h}{1 - \zeta_i},\ \theta_i = -\frac{h\zeta_i}{1 - \zeta_i}, \tag{4.61}$$

where i changes in the limits specified in (4.59). It is easy to check that

$$T_i^\delta = T_i + O(h),\ \tau_i^\delta = \tau_i + O(h),\ \omega_i = O(h),\ \theta_i = O(h). \tag{4.62}$$

Although the convergence of the time constants in (4.62) is slower than of those in (4.20), a natural and tractable approximation $G_1^\delta(\gamma)$ closely related to that of (4.58) and fulfilling the aformentioned conditions of rapprochement is obtained for h small by neglecting the discretization zeros, i.e. replacing $\mathcal{E}_k^\delta(\gamma h)$ by $\mathcal{E}_k^\delta(0)$. $H_1^\delta(z)$, the z-plane counterpart of $G_1^\delta(\gamma)$ and $K_1^\delta(w)$, its w-plane counterpart, have then the forms:

$$H_1^\delta(z) = c_m^0 \frac{\prod_{i=1}^{m}(z - z_i^0)}{(z-1)^l \prod_{i=1}^{n-l}(z - p_i^0)} \tag{4.63}$$

$$K_1^\delta(w) = (1 - w\frac{h}{2})^k \frac{\prod_{i=1}^{m}(w\tau_i^0 + 1)}{w^l \prod_{i=1}^{n-l}(wT_i^0 + 1)} \tag{4.64}$$

with c_m^0 defined in (4.10). The matching polynomials $\mathcal{E}_k(z)$ or $\mathcal{N}_k(z)$ disappear in $H_1^\delta(z)$, which results in an excessive phase lag for $k > 1$ and thus in poorer accuracy of $H_1^\delta(z)$. This is easily seen in $K_1^\delta(w)$ and illustrated in the next section.

4.10 Example

A sampled-data system is considered consisting of a ZOH and a continuous-time part with the transfer function $G(s)$,

$$G(s) = K\frac{(as+1)(bs+1)}{(s+1)^3}. \tag{4.65}$$

Nyquist plots depicted in Fig.4.2 - Fig.4.4 for the range of frequencies important for control design show that for $h = 0.4$ there is little to choose between $H(z)$ (heavy solid line) and any of $H_i(z)$, $i = 0, 1, 2, 3$ (light lines) as opposite to the Nyquist plot of $H_1^\delta(z)$ (heavy dotted line), which except for $k = 1$ deviates strongly from that of $H(z)$. The same applies for any k to the Nyquist plots of $G(s)$ and $H_T(z)$ (heavy dash-dotted line).

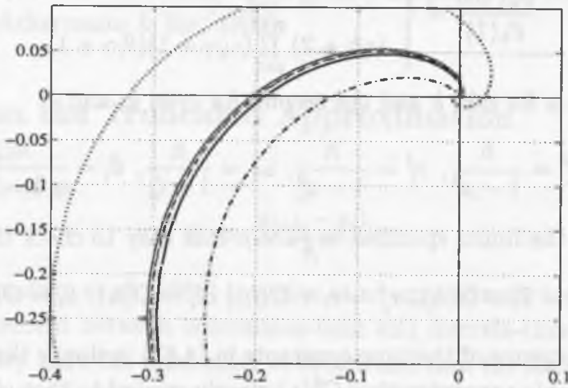

Fig. 4.2. Nyquist plots for $K = 1, a = 0, b = 0$ $(k = 3)$

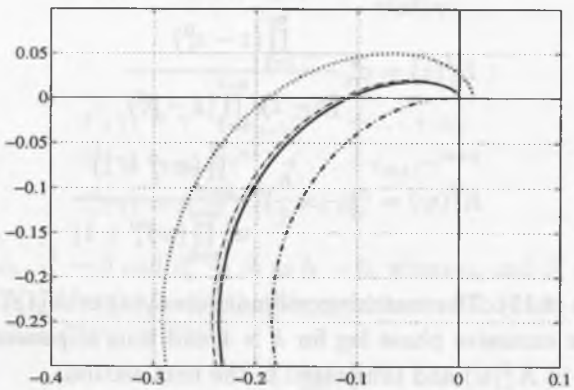

Fig. 4.3. Nyquist plots for $K = 1, a = 0.3, b = 0$ $(k = 2)$

For $h \leq 0.2$ the Nyquist plots of the true pulse transfer function and those of the approximations $H_i(z)$, $i = 0, 1, 2, 3$ are hard to distinguish. In Fig.4.5 - Fig.4.6 output and control signals from the exact (based on $H(z)$) and approximate (based on $H_3(z) = H_1(z)$) sampled-data model matching control systems are displayed for $h = 0.4$ (heavy lines) and compared with those of a continuous-time control system (light lines) with

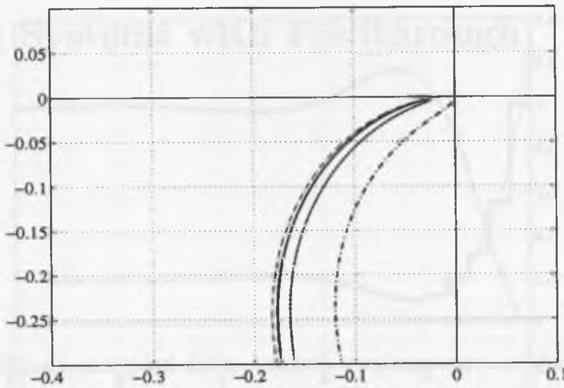

Fig. 4.4. Nyquist plots for $K = 1, a = 0.3, b = 0.25$ $(k = 1)$

$K = 5, a = 0.3, b = 0$ and

$$T(s) = \frac{2}{s^2 + \sqrt{2}s + 2}.$$

For $a > 1/3$ the exact model matching control system becomes stable, which is predicted by discretization zero stability results of (Błachuta, 1997f, Hagiwara et al., 1993). The results for $H_2(z) = H_0(z)$ are similar to those of Fig.4.6. The truncated approximation (4.58) leads to an unstable system when $h = 0.4$.

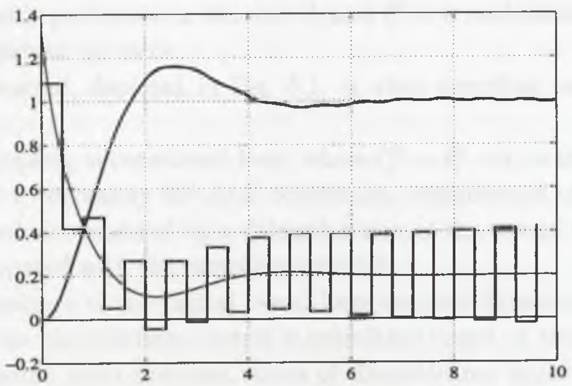

Fig. 4.5. Exact model matching: $K = 5, a = 0.3, b = 0, H(z)$

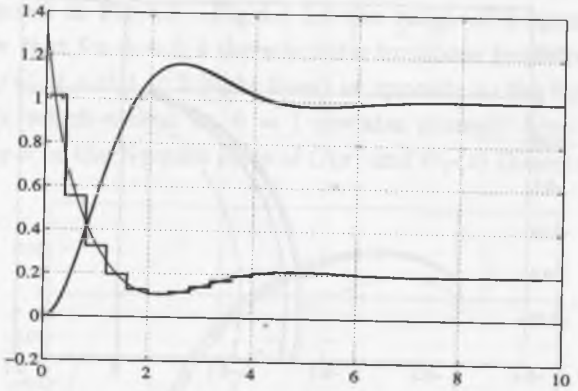

Fig. 4.6. Approximate model matching: $K = 5, a = 0.3, b = 0, H_3(z)$

4.11 Conclusion

The approximate pulse transfer functions derived here exhibit several important features. Because of their structure, they appear to be useful for identification of sampled-data systems and to deliver estimates of both discrete-time and continuous-time parameters. They also offer advantages in the theory of model matching and robust control. The accuracy of our approximations have been shown to be superior to those based on the δ-operator presented in (Goodwin et al., 1986).

5. Sampling Systems with Feedthrough

Discrete-time models of sampled-data control systems are addressed when both a continuous-time plant and a discrete-time controller have a feedthrough. [1] It is pointed out that in this case discrete-time models which can be found in most references should not be used in the closed-loop context. A new state-space model appropriate for the closed-loop modeling, and formulae for calculating the related discrete-time pulse transfer functions are derived. Intersample phenomena are studied and the feasibility of that model to describe systems with parasiting dynamics is emphasized. Examples from the literature illustrate the relevance of the issue.

5.1 Introduction

A model of a classical sampled-data control system as presented in Fig. 5.1 is considered . It consists of a single-input single-output linear continuous-time plant, a zero-order hold, two switches and a discrete-time control algorithm. A sampling instant at which the reading of the output is performed is denoted t_i^s, and t_i^m is a modulation instant at which the control signal changes its value.

The normal situation, depicted in Fig. 5.1, is when sampling takes place prior to modulation $t_i^s < t_i^m$.

Synchronous sampling is considered here, where $t_i^m \to t_i^s \to t_i = ih$. This means that the processing time τ (necessary for A/D conversion, computation of control and D/A conversion), which can be modeled by a delayed action of the second switch, is assumed to be negligible compared with the sampling period h.

The operation pattern of the digital closed-loop sampled-data control is then as follows. The value of the discrete-time control is calculated based on the present sample of the output and possibly some previous values of discrete-time signals. The hold device converts the discrete-time control signal into a discontinuous analog one, driving the plant between the sampling instants. The output of the controlled plant, whether continuous or not, is then sampled not earlier than at the next sampling instant.

[1] The chapter is based on (Błachuta, 1997e; Błachuta, 1997g) and (Błachuta, 1999a)

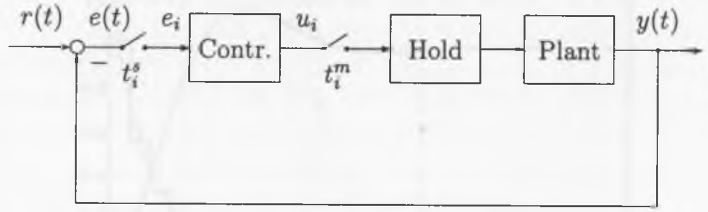

Fig. 5.1. Block diagram of a sampled-data control system, $t_i^s = ih$, $t_i^m = ih + \tau$

Practical instrumentation of this scheme involving elements of data conversion hardware, and microprocessor implementation of digital control strategies can be found in many text-books, e.g. (Forsythe & Goodall, 1991; Houpis & Lamont, 1985; Jacquot, 1994; Williamson, 1991).

In (Kučera, 1991; Kwakernaak & Sivan, 1972) models of sampled systems are introduced which, in contrast to the situation depicted in Fig. 5.1, base on the reversed order of events where updating of control precedes output reading. This is in discrepancy with practical solutions and can lead to serious problems discussed further.

A general case when the model of a system to be controlled is allowed to have a direct coupling between input and output is addressed. Such system can be seen as a model in which some parasitic dynamics have been neglected for simplification. If there is a zero-order hold and a feedthrough is present then the controlled variable is discontinuous at the sampling instants. This can be referred to as a simplified mathematical model of a physical system whose continuous-time output changes rapidly in response to a jump in control.

A survey of both classical and modern literature shows that this particular but important point is usually missing or solved incorrectly.

In almost all references that admit discontinuous output, the actual samples are assigned to the right side limits of the output signal at sampling instants. This was recognized as a convention in (Ackermann, 1985) and leads to open loop models called here $\mathcal{D}^+$.

In (Ackermann, 1985; Dahleh & Diaz-Bobillo, 1995; Franklin et al., 1990; Jacquot, 1994; Saberi, Sannuti & Chen, 1995) the authors make an assumption that there is no direct coupling between input and output of the plant. The results are therefore correct but the problem considered here is not solved. In other references (Åström et al., 1984; Åström & Wittenmark, 1997; Houpis & Lamont, 1985; Isermann, 1989; Jury, 1958; Kwakernaak & Sivan, 1972; Ogata, 1987; Santina, Stubberud & Hostetter, 1994) a direct transmission

in the continuous-time plant is admitted and their results will become incorrect when the feedback loop is closed.

The aim of the chapter is: (a) to show that the closed loop models $\mathcal{M}^+$ constructed formally from a discrete-time controller with a feedthrough and $\mathcal{D}^+$ are not able to describe any sampled-data feedback system, (b) to advocate discrete-time models $\mathcal{D}^-$ and $\mathcal{M}^-$ related to the left side sampling, claimed to be feasible for the closed-loop modeling of sampled-data control systems with feedthrough, and (c) to compare the properties of $\mathcal{D}^-$ and $\mathcal{D}^+$.

It should be noted that in (Williamson, 1991) skewed sampling models can be found which simplify to $\mathcal{D}^-$. A model having the same structure as $\mathcal{D}^-$ also emerged in (Esfandiari & Khalil, 1989) in the context of the robust stability of singularly perturbed systems.

The chapter is organized as follows. The main result is presented in section 5.2. In section 5.3 the discrete-time model derived in section 5.2 is shown to be a limit of a commonly used discrete-time model of a continuous-time system without any feedthrough whose parameters change so as to approach a system with a feedthrough. Then the input-output models are derived in section 5.4 with emphasis put on modified pulse transfer functions and their use for the closed-loop modeling. The approach in which a system with the hold element absorbed in the continuous-time part is fed by Dirac impulses is presented in section 5.5, where also some pioneering results in the area (Kuzin, 1962; Tsypkin, 1958; Węgrzyn, 1960; Węgrzyn, 1963, 1970, 1980) and recent paper (Gessing, 1996) are briefly surveyed. Conclusions are drawn in section 5.6.

5.2 The Proposed Model

5.2.1 Preliminaries

The plant is described by the following set of state-space equations:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \tag{5.1}$$

$$y(t) = \boldsymbol{c}'\boldsymbol{x}(t) + du(t) \tag{5.2}$$

or by the transfer function $G(s)$:

$$G(s) = \boldsymbol{c}'(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d = G_0(s) + G(\infty). \tag{5.3}$$

Here $\boldsymbol{A}$ is an $n \times n$ matrix, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors and d is a scalar. $G(s)$ is a rational function in the variable s and, assuming the system (5.1)-(5.2) to be both controllable and observable, the degrees of the numerator and the denominator polynomials are m and n, respectively, with $m \leq n$.

The discrete-time control algorithm considered is of the form:

$$x^c_{i+1} = F^c x^c_i + g^c(r_i - y_i) \tag{5.4}$$
$$u_i = c^{c\prime} x^c_i + d^c(r_i - y_i), \tag{5.5}$$

where i denotes the i-th sampling instant, $i = 0, 1, 2\ldots$; the controller state x^c_i is a discrete-time state vector of dimension n^c, r_i is a reference, y_i is a sample of the output at $t_i = ih$, and matrix F^c and vectors g^c, c^c are of appropriate dimensions. The feedthrough term d^c is assumed to be nonzero. Equations (5.4)-(5.5) cover all classical digital controllers including the proportional one.

5.2.2 Sampling and causality

Since the output $u(t)$ from the zero-order hold is discontinuous at t_i, the output $y(t)$ is also discontinuous at t_i if $d \neq 0$. Both the left-side and the right-side limits: $y(t_i^-) = \lim_{\epsilon\to 0} y(ih - \epsilon)$ and $y(t_i^+) = \lim_{\epsilon\to 0} y(ih + \epsilon)$, $\epsilon > 0$ are well defined and, due to (5.2) and continuity of the state vector $x(t)$, there is:

$$y(t_i^-) = c' x(t_i) + du(t_i^-) \tag{5.6}$$
$$y(t_i^+) = c' x(t_i) + du(t_i^+), \tag{5.7}$$

where (5.6) expresses the response to the control signal u_{i-1} and (5.7) to u_i.

For a normal chronology sampler, equation (5.6) is the sampling equation with $y(t_i^-)$ being a cause and $y(t_i^+)$ an effect of the control u_i calculated from (5.5) based on $y_i = y(t_i^-)$.

In reversed chronology, (Kučera, 1991; Kwakernaak & Sivan, 1972), equation (5.7) is referred to as a sampling equation with u_i being the cause and $y_i = y(t_i^+)$ an effect.

Using (5.7) together with (5.4)-(5.5) requires that an algebraic loop is solved. This is not possible in real time. Obviously, at least a one-step delay in the controller is then required for causality, which implies that equation (5.7) can be tentatively used as a possible sampling equation only if there is no feedthrough in the controller, i.e. when $d^c = 0$ in (5.5). This is not the case in (Houpis & Lamont, 1985; Kwakernaak & Sivan, 1972), where noncausal configurations of the closed-loop system can be found.

5.2.3 The classical model $\mathcal{D}^+$

In the literature (Ackermann, 1985; Åström & Wittenmark, 1997; Franklin et al., 1990; Houpis & Lamont, 1985; Isermann, 1989; Ogata, 1987; Phillips & Nagle, 1990; Santina et al., 1994; Williamson, 1991), the following set of equations:

$$x_{i+1} = F x_i + g u_i \tag{5.8}$$
$$y_i = c' x_i + d u_i, \tag{5.9}$$

where

$$F = e^{Ah},\ g = \int_0^h e^{Av} b dv \tag{5.10}$$

based on (5.7), is usually referred to as a discrete-time state-space model, say $\mathcal{D}^+$, of (5.1)-(5.2). The associated formula

$$H^+(z) = (1 - z^{-1})\mathcal{Z}\{\frac{G(s)}{s}\} \tag{5.11}$$

is commonly used (Ackermann, 1985; Åström et al., 1984; Åström & Wittenmark, 1997; Forsythe & Goodall, 1991; Franklin et al., 1990; Isermann, 1989; Jacquot, 1994; Jury, 1958; Kuo, 1970; Ogata, 1987; Phillips & Nagle, 1990; Santina et al., 1994; Williamson, 1991) for the calculation of the pulse transfer function $H(z)$. Here $\mathcal{Z}\{G(s)/s\}$ denotes the $\mathcal{Z}$ transform of the sequence $g_1(ih), i = 0, 1, 2\ldots$, where $g_1(t) = \mathcal{L}^{-1}[G(s)/s]$ is the plant step response.

The models $\mathcal{D}^+$ and $H^+(z)$ are also present in the popular CACSD packages (CC, MATLAB, MATRIX$_X$), where they appear as a result of the 'convert', 'c2d' or 'c2dm' commands when an option of the step invariant transform is chosen.

5.2.4 The new model $\mathcal{D}^-$

It is important to notice that because of violating the causality $\mathcal{D}^+$ must not be used to model the sampled-data system of Fig. 5.1. Moreover, it will be shown that due to its high sensitivity to any parasiting dynamics, the normal chronology of sampling and the resulting model $\mathcal{D}^-$ are superior to those with a reversed chronology even if a strictly proper controller is applied.

Based on the sampling equation (5.6), the discrete-time model, say $\mathcal{D}^-$, of the system (1)-(2) with a zero-order hold has the form:

$$x_{i+1} = F x_i + g u_i \tag{5.12}$$
$$y_i = c' x_i + d u_{i-1}. \tag{5.13}$$

The transfer function $H^-(z)$ that results from (5.12)-(5.13) is

$$H^-(z) = (1 - z^{-1})(\mathcal{Z}\{\frac{G(s)}{s}\} - G(\infty)). \tag{5.14}$$

Formula (5.14) can also be written in an equivalent form in which $G(\infty)$ is replaced by the initial value, $g_1(0^+)$, of the plant step response $g_1(t)$. The passage from (5.12)-(5.13) to (5.14) is shown in section IV A.

The difference between the closed loop models $\mathcal{M}^-$ and $\mathcal{M}^+$ that base on $\mathcal{D}^-$ and $\mathcal{D}^+$ respectively is shown in the following example.

Example 5.2.1. The step response of two models $\mathcal{M}^-$ and $\mathcal{M}^+$ of a sampled-data control system with a proportional digital controller with $k = 2$, $h = 0.5$ that base on

$$G(s) = \frac{1}{s+1} - 0.2 \tag{5.15}$$

$$H^-(z) = \frac{1-d}{z-d} - \frac{0.2}{z} \tag{5.16}$$

$$H^+(z) = \frac{1-d}{z-d} - 0.2 \tag{5.17}$$

is depicted in Fig. 5.2 and compared with the actual solution. It is clear that the output from $\mathcal{M}^+$ differs greatly from the actual output.

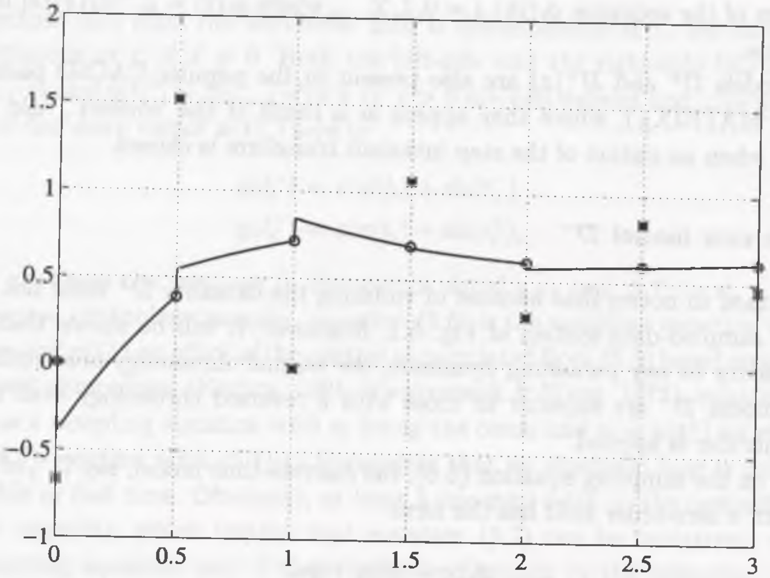

Fig. 5.2. $\mathcal{M}^-$ (circles), $\mathcal{M}^+$ (asterisks), and analog simulation (line)

5.2.5 Remarks on discrete systems

Equations of type (5.8)-(5.9) are often met in the framework of a purely discrete argument which does not necessarily represent time in the strict physical sense, so that the causality is not violated. Closed-loop discrete-time systems are mathematically well defined if the well-posedness conditions (Dahleh & Diaz-Bobillo, 1995; Saberi et al., 1995) are fulfilled. This is, however, a different issue and the requirement that the control action does

not influence the measurement from which it was calculated is inherent to any realistic sampled-data control system (Ackermann, 1985; Kučera, 1991).

One example of a purely discrete-time system is a discrete-time approximation (Kowalczuk, 1993) of a continuous-time system performed for the sake of digital computations and/or simulations. Then equations of the form (5.8)-(5.9) together with (5.4)-(5.5) imply that an algebraic loop is to be solved iteratively for u_i at each i, where i denotes a step of a recursive procedure rather than a sampling instant.

Algebraic loops can easily be eliminated by a simple transformation of the equations involved. Then in the particular case of the so called step invariant transform (Williamson, 1991), from (5.8)-(5.9) the following discrete-time equation:

$$\boldsymbol{x}_{i+1} = [\boldsymbol{F} - \frac{\boldsymbol{g}\boldsymbol{c}'}{1+d}]\boldsymbol{x}_i + \frac{1}{1+d}\boldsymbol{g}r_i. \tag{5.18}$$

results for a continuous-time closed-loop system consisting of a plant (5.1)-(5.2) and a unity feedback. The well-posedness condition in this case is $d \neq -1$. Although claimed in (Houpis & Lamont, 1985) to be a model of a sampled-data system, $\mathcal{M}^+$ in (5.18) obviously does not describe any sampled system, and for a sampled-data system with a unity feedback a correct model $\mathcal{M}^-$ is:

$$\boldsymbol{x}^*_{i+1} = [\boldsymbol{F}^* - \boldsymbol{g}^*\boldsymbol{c}^{*\prime}]\boldsymbol{x}^*_i + \boldsymbol{g}^* r_i \tag{5.19}$$

with $y_i = \boldsymbol{c}^{*\prime}\boldsymbol{x}^*_i$, $\boldsymbol{x}^*_i = [\boldsymbol{x}'_i, u_{i-1}]'$, $\boldsymbol{F}^* = \mathrm{diag}\{\boldsymbol{F}, 0\}$, $\boldsymbol{g}^* = [\boldsymbol{g}', 1]'$, $\boldsymbol{c}^* = [\boldsymbol{c}', d]'$. Note that the system in (5.19) is always well posed.

It is interesting to note that algebraic loop solving is performed in the simulation package SIMULINK even for systems with a zero-order hold. This makes simulation of sampled data systems with feedthrough incorrect.

5.3 State-Space Models and Intersample Phenomena

In this section, it will be shown that for $\mathcal{D}^-$ sampling commutes with limit operations that convert a regular system into the one with a feedthrough. This property is essential for modeling and discretizing systems with parasitic dynamics using simplified models with a feedthrough.

5.3.1 Systems with negligible dynamics

Let the system of equations of type (5.1)-(5.2) with $d = 0$ be transformed to the following Jordan form:

$$\dot{x}_1(t) = J_1 x_1(t) + b_1 u(t) \tag{5.20}$$
$$\dot{x}_2(t) = J_2 x_2(t) + b_2 u(t) \tag{5.21}$$
$$y(t) = c_1' x_1(t) + c_2' x_2(t), \tag{5.22}$$

where $\dim x_1 = n_1$, $\dim x_2 = n_2$; J_1 and J_2 are quasi-diagonal Jordan matrices with the eigenvalues arranged with increasing moduli and $n_1 + n_2 = n$. Let the second subsystem be fast compared with the first one, and the eigenvalues $\lambda_{2,i}$ obey $\mathrm{Re}\,\lambda_{2,i} < 0$ for $i = 1, 2, \ldots n_2$. Define $\lambda = \min_i |\mathrm{Re}\lambda_{2,i}|$, $i = 1, 2, \ldots n_2$. Then for λ large enough, when neglecting the transients of $x_2(t)$, the second subsystem can be considered to be algebraic and the following approximation of (5.20)-(5.22) holds:

$$\dot{x}_1(t) = J_1 x_1(t) + b_1 u(t) \tag{5.23}$$
$$y(t) \approx c_1' x_1(t) + d_2 u(t) \tag{5.24}$$

with $d_2 = -c_2' J_2^{-1} b_2$. It has been shown in (Esfandiari & Khalil, 1989) that the following system:

$$x_{1,i+1} = F_1 x_{1,i} + g_1 u_i \tag{5.25}$$
$$y_i \approx c_1' x_{1,i} + d_2 u_{i-1} \tag{5.26}$$

with

$$F_1 = e^{J_1 h}, \; g_1 = \int_0^h e^{J_1 v} b_1 dv \tag{5.27}$$

approximates a discrete-time version of the model in (5.20)-(5.22). After dropping the subscript 1, the above system is equivalent to the model $\mathcal{D}^-$ of (5.12)-(5.13). Model (5.25)-(5.26) was identified in (Esfandiari & Khalil, 1989) as the one which assures robust stability of a closed-loop system with negligible unmodeled dynamics. This result has a very simple and natural input-output interpretation presented in section 5.4.2.

The effect of a proper choice between both transfer functions H^+ and H^- on a simple closed loop system with negligible dynamics is studied in the following example.

Example 5.3.1. Consider a linear system with the transfer functions

$$G_a(s) = \frac{a}{s+a}, \; H_a(z) = \frac{1-d}{z-d}, \; d = e^{-ah}. \tag{5.28}$$

For a control system comprising a proportional digital controller with gain k, a zero-order hold and a plant as in (5.28), the characteristic polynomial $p(z)$ and the range of stabilizing k are:

$$p(z) = z - d + k(1-d), \; -1 < k < \frac{1+d}{1-d}. \tag{5.29}$$

Any value $0 \leq d < 1$ can be inserted into the relations in (5.29). For $d = 0$ one gets $H(z) = z^{-1}$, and:

$$p^-(z) = z + k, \; -1 < k < 1. \tag{5.30}$$

It is clear that for a large enough the expressions in (5.30) can be regarded as good approximations of that in (5.29). When using $H(z) = 1$, which results from formula (5.11), one gets

$$p^+(z) = 1 + k. \tag{5.31}$$

Now, from (5.31) one could infer that there are no transients in the control process and that tracking could be performed arbitrary exactly by choosing k large enough. This contradicts (5.30) from which it follows that for $k \geq 1$ the system becomes unstable. This supports the claim that $H^-(z)$ should be used for the closed loop modeling instead of $H^+(z)$.

Observe that letting $a \to \infty$ in (5.28) results in

$$G_a(s) \to G_\infty(s) = 1$$

and

$$H_a(z) \to H_\infty(z) = z^{-1}.$$

It is claimed in (Åström & Wittenmark, 1997) that $H_\infty(z)$ is not the pulse transfer function of $G_\infty(s)$, which according to this reference should be $H_\infty(z) = 1$. This led Åström & Wittenmark (1997) to the conclusion that sampling and limit operations do not commute for (5.28). This conclusion is not correct. Indeed, when using $H^-(z)$ sampling commutes with limit operation.

5.3.2 Model with negligible delay

The influence of the processing time on the loop behavior can be modeled by a continuous time-delay τ in the control variable:

$$\dot{x}(t) = Ax(t) + bu(t-\tau) \tag{5.32}$$
$$y(t) = c'x(t) + du(t-\tau). \tag{5.33}$$

Assume that the time delay τ is less than the sampling period h. Then (Åström & Wittenmark, 1997) the following set of equations:

$$x_{i+1} = Fx_i + g_0 u_i + g_1 u_{i-1} \tag{5.34}$$
$$y_i = c'x_i + du_{i-1} \tag{5.35}$$

is a discrete-time model of (5.32)-(5.33), where

$$F = e^{Ah}, \; g_0 = \int_0^{h-\tau} e^{Av} b dv, \; g_1 = e^{A(h-\tau)} \int_0^{\tau} e^{Av} b dv. \tag{5.36}$$

From equations (5.36) and (5.10) we have $g_0 \to g$ and $g_1 \to 0$ as $\tau \to 0$. Hence equations (5.34)-(5.35) take in limit the form of (5.12)-(5.13) defining $\mathcal{D}^-$

5.3.3 Values in between sampling

Intersample values, i.e. values for $t = ih + \sigma h,\ 0 \leq \sigma \leq 1$, of the output signal can be obtained by the following modification (Åström & Wittenmark, 1997) of the output equation:

$$y(ih + \sigma h) = \boldsymbol{c}^*(\sigma)'\boldsymbol{x}_i + d^*(\sigma)u_i, \tag{5.37}$$

where

$$\boldsymbol{c}^*(\sigma)' = \boldsymbol{c}' e^{\boldsymbol{A}\sigma h} \tag{5.38}$$

$$d^*(\sigma) = d + \boldsymbol{c}' \int_0^{\sigma h} e^{\boldsymbol{A}v} \boldsymbol{b} dv. \tag{5.39}$$

When substituting $\sigma \to 0$ into (5.37)-(5.39) model $\mathcal{D}^+$ is obtained. However, when there are any unmodeled dynamics in the system then the output of the original system is continuous, $y(ih^+) = y(ih^-)$, which means that for small σ equation (5.37) may model the output incorrectly. For $\sigma = 1$ one gets:

$$y_{i+1} = \boldsymbol{c}'\boldsymbol{F}\boldsymbol{x}_i + (d + \boldsymbol{c}'\boldsymbol{g})u_i = \boldsymbol{c}'\boldsymbol{x}_{i+1} + du_i, \tag{5.40}$$

which is consistent with (Williamson, 1991) and again leads to $\mathcal{D}^-$.

5.4 Input-Output Models

5.4.1 Transfer functions

Observe that from (5.12)-(5.13) the transfer function $H(z)$ can be expressed as follows

$$H(z) = \boldsymbol{c}'(z\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{g} + dz^{-1} = H_0(z) + G(\infty)z^{-1}. \tag{5.41}$$

Denote $\boldsymbol{x}_1(t)$ the step response of the system (1) with the zero initial condition and $g_1^0(t) = \boldsymbol{c}'\boldsymbol{x}_1(t) = \mathcal{L}^{-1}[G_0(s)/s]$. Then

$$g_1^0(ih) = \boldsymbol{c}' \int_0^{ih} e^{\boldsymbol{A}v} \boldsymbol{b} dv = \boldsymbol{c}' \sum_{j=0}^{i-1} \boldsymbol{F}^j \boldsymbol{g} = \boldsymbol{c}'(\boldsymbol{I} - \boldsymbol{F}^i)(\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{g}. \tag{5.42}$$

Now, after performing some calculations, the following formula results:

$$H_0(z) = (1 - z^{-1})\mathcal{Z}_h\{\mathcal{L}^{-1}[\frac{G_0(s)}{s}]\}. \tag{5.43}$$

On the other hand

$$(1 - z^{-1})\mathcal{Z}_h\{\mathcal{L}^{-1}[\frac{G(s)}{s}]\} = H_0(z) + G(\infty). \tag{5.44}$$

Finally, comparing (5.44) with (5.41) yields (5.14).

5.4.2 Approximations and unmodeled dynamics

Let the transfer functions: $G(s)$ of (5.20)-(5.21) and $H(z)$ of (5.12)-(5.13) be given in the expanded form, i.e.:

$$G(s) = \sum_{i=1}^{n} r_i \frac{\lambda_i}{s + \lambda_i} \tag{5.45}$$

$$H(z) = \sum_{i=1}^{n} r_i \frac{1 - d_i}{z - d_i},\ d_i = e^{-\lambda_i h}, \tag{5.46}$$

where, according to equations (5.20)-(5.22), $r_i\lambda_i = b_i c_i$ and for the simplicity of the notation it is assumed that $\lambda_i \neq \lambda_j, i \neq j,\ i, j = 1, 2 \ldots n$. Suppose that $d_i \approx 0$ for $i = n_1 + 1, \ldots n$. Then $H(z)$ can be approximated as follows:

$$H(z) \approx \bar{H}(z) = \sum_{i=1}^{n_1} r_i \frac{1 - d_i}{z - d_i} + dz^{-1},\ d = \sum_{i=n_1^n+1} r_i \tag{5.47}$$

and $\bar{H}(z)$ is the transfer function of the model in (5.25)-(5.26).

5.4.3 Links with modified pulse transfer functions

Taking (5.8) and (5.37) into account, a modified discrete-time transfer function $H_\sigma(z, \sigma)$ which relates intersample values of the output with the discrete-time input can be defined as

$$H_\sigma(z, \sigma) = \frac{Y(z, \sigma)}{U(z)} = (1 - z^{-1})\mathcal{Z}\{g_1(ih + \sigma h)\}$$

$$= \boldsymbol{c}^*(\sigma)'(z\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{g} + d^*(\sigma), \tag{5.48}$$

where

$$Y(z, \sigma) = \mathcal{Z}\{y(ih + \sigma h)\},\ g_1(t) = \mathcal{L}^{-1}[\frac{G(s)}{s}]. \tag{5.49}$$

The modified pulse transfer function defined in (5.48) can be found e.g. in (Ackermann, 1985; Kuzin, 1962; Tsypkin, 1958). Another (Åström & Wittenmark, 1997; Houpis & Lamont, 1985; Jury, 1958; Phillips & Nagle, 1990; Williamson, 1991) definition of the modified discrete-time transfer function $H_\mu(z, \mu)$ is based on a delayed output $y(ih - \tau)$, $\tau = h - \mu h,\ 0 \leq \mu \leq 1$:

$$H_\mu(z, \mu) = \frac{Y(z, \mu)}{U(z)} = (1 - z^{-1})\mathcal{Z}\{g(ih - h + \mu h)\} = z^{-1}H_\sigma(z, \mu). \tag{5.50}$$

From (5.48) and (5.50) it is easily seen that the following links exist between the transfer functions:

$$H^+(z) = H_\sigma(z,0) = zH_\mu(z,0) \tag{5.51}$$

$$H^-(z) = z^{-1}H_\sigma(z,1) = H_\mu(z,1). \tag{5.52}$$

It is interesting to note that although Jury (1958) had (5.52) at his disposal he chose $H^+(z)$ given in (5.51) rather than $H^-(z)$ as the pulse transfer function of a system with feedthrough.

5.4.4 Closed-loop models

Denote $1 + K^-(z)$ the return difference with $K^-(z) = D(z)H^-(z)$, where $D(z)$ is the controller transfer function. One is now able to write the following standard closed-loop transfer function $T^-(z)$:

$$T^-(z) = \frac{K^-(z)}{1+K^-(z)} = \frac{Y(z,0^-)}{R(z)} \tag{5.53}$$

where $Y(z,0^-) = \mathcal{Z}\{y(ih^-)\}$. One may however, wish to have a transfer function $T_\sigma(z,\sigma)$ relating $Y(z,\sigma) = \mathcal{Z}\{y(ih+\sigma h)\}$ with $R(z)$. It can be expressed as follows:

$$T_\sigma(z,\sigma) = \frac{D(z)H_\sigma(z,\sigma)}{1+K^-(z)} = \frac{Y(z,\sigma)}{R(z)}. \tag{5.54}$$

From (5.51) and (5.54) it follows that in the closed loop context $H^+(z)$ can only be used as a part of the following causal formula:

$$T^+(z) = \frac{D(z)H^+(z)}{1+K^-(z)} = \frac{Y(z,0^+)}{R(z)} \tag{5.55}$$

Unfortunately, due to high sensitivity of $H^+(z)$ to the unmodeled dynamics, (5.55) models the system behavior in a rather unreliable way. The sensitivity of $\mathcal{D}^+$ to the unmodeled dynamics is illustrated in Example 5.4.1.

Example 5.4.1. Consider two closed-loop sampled-data control systems, $\mathcal{S}_1$ and $\mathcal{S}_2$, with plants having transfer functions $G_1(s)$ and $G_2(s)$:

$$G_1(s) = \frac{1}{s+1} - \frac{0.2}{0.01s+1} \tag{5.56}$$

$$G_2(s) = \frac{1}{s+1} - \frac{0.2}{0.2s+1}. \tag{5.57}$$

Each of them is fitted with a proportional controller with the gain $k = 3$. The responses of both systems to the unit step-wise set point change when the sampling period h equals to 1 are presented in Fig. 5.3. Both open-loop systems can be approximated by the same model:

$$\bar{G}(s) = \frac{1}{s+1} - 0.2. \tag{5.58}$$

Although the behaviors of systems $\mathcal{S}_1$ and $\mathcal{S}_2$ at sampling instants are almost the same, the intersample behaviors of both systems are different, particularly shortly after control updating. Note that the the discrete-time values of the output calculated from $\bar{T}^-(z)$ approximate actual values very exactly whereas those based on $\bar{T}^+(z)$ are quite wrong. This supports model $\mathcal{D}^-$ and invalidates $\mathcal{D}^+$ also in this context.

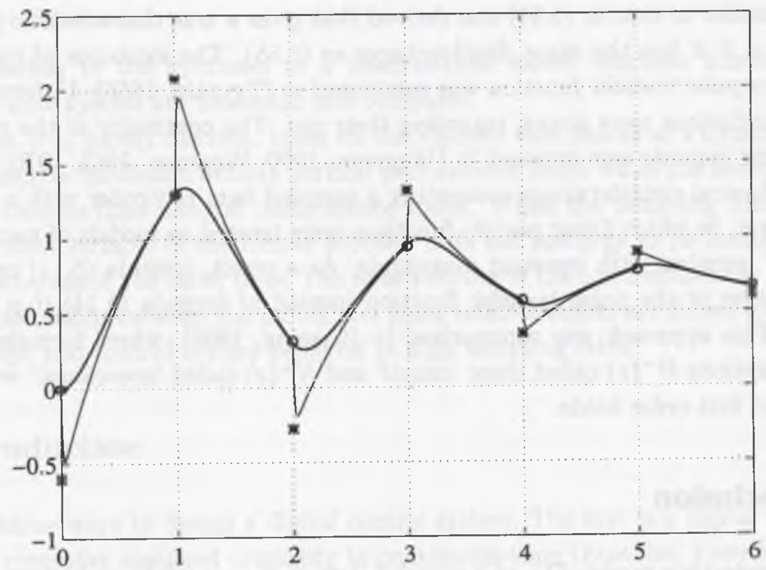

Fig. 5.3. $\mathcal{S}1$ (light line), $\mathcal{S}2$ (bold line), $\bar{T}^-$ (circles), and $\bar{T}^+(z)$ (asterisks)

Example 5.4.2. In (Isermann, 1989), an example is given of a differentiator:

$$G(s) = \frac{T_d s}{1+Ts}, \tag{5.59}$$

whose discrete-time transfer functions $H^+(z)$ and $H^-(z)$ are

$$H^+(z) = \frac{T_d}{T}\frac{z-1}{z-d}, \quad H^-(z) = \frac{T_d}{T}\frac{d(z-1)}{z(z-d)}, \; d = e^{-h/T}. \tag{5.60}$$

It is interesting to notice that unlike $H^+(z)$, $\lim_{h\to\infty} H^-(z) = 0$, which is consistent with a low-frequency approximation $G(s) \approx 0$, and $\lim_{h\to 0} H^-(z) = (T_d/T)z^{-1}$, which is consistent with a high-frequency approximation $G(s) \approx T_d/T$. Isermann (1989) defines only $H^+(z)$ as a pulse transfer function of (5.59).

5.5 Remarks on Other Approaches

In most references digital systems are modeled in such a way that inputs have the form of a train of Dirac impulses, $u(t) = \sum_i \delta(t-t_i)u_i$, which enter certain linear system representing a hold followed by the plant in series. The model of the hold can then be absorbed by the plant model in the transfer function form. Within this framework, the problem of closed-loop sampled-data systems with a discontinuous output was pioneered in (Kuzin, 1962; Tsypkin, 1958; Węgrzyn, 1960; Węgrzyn, 1963, 1970, 1980). In (Kuzin, 1962) an equation similar to that in (5.54) was derived that gives a true characteristic polynomial but for $\sigma = 0$ it has the same disadvantages as (5.55). The existence of two different types of the pulse transfer function was mentioned in (Tsypkin, 1958). Unfortunately, no binding conclusions were drawn regarding their use. The continuity of the response at the sampling instants was imposed in (Węgrzyn, 1960; Węgrzyn, 1963, 1970, 1980) as a result of physical considerations concerning a sampled data controller with a falling bar galvanometer, in which Dirac pseudo-functions were treated as models of narrow width-modulated impulses with constant magnitude. As a result, formula (5.11) can be used for calculation of the pulse transfer function instead of formula (5.14) if $g_1(0) = 0$ is assumed. This approach was summarized in (Gessing, 1996), where formulas for both transfer functions $H^-(z)$ called there 'causal' and $H^+(z)$ called 'non-causal' were derived for zero and first order holds.

5.6 Conclusion

A new discrete-time model, $\mathcal{D}^-$, of a sampled-data system consisting of a zero-order hold and a linear plant with a feedthrough, defined in (5.12)-(5.13), has been presented and compared with the classical model $\mathcal{D}^+$ given in (5.8)-(5.9).

It has been shown that because of violation of the closed-loop causality the classical model $\mathcal{D}^+$ related to the right-side limit of the output signal with the transfer function $H^+(z)$ is not feasible for feedback modeling if there is a feedthrough in both the plant and controller.

The new model, $\mathcal{D}^-$, related to the left-side limit of a discontinuous output signal has been shown to be appropriate for modeling of feedback systems. Its transfer function $H^-(z)$ appears to be vital for both the return difference and the characteristic polynomial of the closed-loop system.

$\mathcal{D}^-$, whose sensitivity to the unmodeled dynamics is small is also better suited for state estimation and observer-based controllers than $\mathcal{D}^+$, whose sensitivity is extremely high.

6. Hybrid LQR Design

Two approaches to the synthesis of a discrete-time model reference controller for a continuous-time system are presented and compared. [1]

The first one, purely discrete, bases on the discrete-time model of a dynamic system and on a discrete quadratic infinite horizon performance index while the second is based on the continuous-time integral performance index. When the sampling time tends to zero the control variable in the former problem does not converge to its continuous time prototype whereas in the latter does. The relative order of the continuous-time plant itself and the relationship between the model and plant relative orders are shown to be crucial for the design and control system behavior at high sampling rates.

6.1 Introduction

There are three ways to design a digital control system. The first is a digital redesign of an analog controller designed originally in continuous-time (Franklin, Powell & Emani-Naeini, 1986; Ieko, Ochi, Kanai, Hori & Okamoto, 1996; Kuo & Peterson, 1973; Shieh, Kasavaraju & Tsai, 1995; Yackell, Kuo & Singh, 1974). It is the simplest one but while giving satisfactory results at high sampling rates it fails when the sampling rate is low. The second, referred to as a purely discrete-time approach, requires the continuous-time plant to be discretized prior to defining the control task with regard to the system behavior at sampling instants. Although most frequently met in text-books, e.g. (Åström & Wittenmark, 1997; Franklin et al., 1990), it overlooks intersample behavior and might fail at high sampling rates. The third way, called hybrid approach, is to perform direct digital design taking the intersample behavior into account. It has been receiving increasing recognition for the last years and there are several investigations along this line. Various approaches to the optimal $\mathcal{H}_2$ problem can be found in (Bamieh & Pearson, 1992; Chen & Francis, 1991; Hagiwara & Araki, 1995; Hara, Fujioka & Kabamba, 1994; Kabamba & Hara, 1993; Khargonekar & Sivashankar, 1991). Other interesting hybrid models and approaches are presented in (Yamamoto, 1994) and (Lampe & Rosenwasser, 1993; Rosen-

[1] The chapter is based on (Błachuta, 1992; Błachuta, 1997*b*) and (Błachuta, 1997*c*)

wasser & Lampe, 1997), and intersample behavior in the ripple-free deadbeat control context is considered in (Sirisena, 1985; Urikura & Nagata, 1987). An overview of the issue can be found in (Chen & Francis, 1995; Hara, Yamamoto & Fujioka, 1996) and a related software package is presented in (Hara, Yamamoto & Fujioka, 1997).

Due to sampling and control signal modulation the performance of sampled-data control systems is usually poorer than that of continuous-time ones. Therefore it is reasonable to expect that increasing the sampling rate should result in the continuous-time performance recovery. It will be shown here that this is not the case if the controller is designed in pure discrete-time, when 'ringing' of the control signal is observed, which leads to an unacceptable intersample ripple of the output, (Baron, 1989; Chen & Francis, 1995; Hara et al., 1996). In contrast to this, a hybrid design method yielding controllers which converge to continuous-time controllers at high sampling rates and have good properties for a wide range of sampling periods is proposed. To this end, a simple LQR framework of (Dorato & Levis, 1971) based on a discretized continuous-time performance index (Błachuta, 1982) designed so as to approach the required output dynamics is employed.

A model reference control task will be defined in such manner that the output is required to fulfill a predefined differential or difference equation, or to be close to its solution while the overall closed-loop system is stable. Because of space limitations, only a deterministic regulator problem will be considered. It can also be seen as a solution to a problem with a step-wise changing reference or disturbance, and as a starting point to a more general tracking problem under stochastic disturbances.

The chapter is organized as follows. In the next section a hybrid control problem with a discrete-time controller and a continuous-time performance index will be stated. In section 6.3 it will be shown that a solution can be found to the continuous-time problem irrespectively of the system order and locations of the system zeros. Three different types of problems are then distinguished depending on the relationship between the system relative order and the order of the reference differential equation. In section 6.4 it is shown that a discrete-time performance index which can be considered as a discrete-time counterpart of the underlying continuous-time index produces a solution which does not converge to the continuous-time one, and the source of controller 'ringing' is revealed. In section 6.5 a solution to the hybrid problem is found and shown to converge to the continuous-time one when the sampling rate increases. The results are illustrated by an example in section 6.6 and concluding remarks are presented in section 6.7.

6.2 Problem statement

The system to be controlled is defined by the following set of state-space equations:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u, \ \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{6.1}$$

$$y = \boldsymbol{d}'\boldsymbol{x}, \tag{6.2}$$

where $\boldsymbol{x}$ is an $n \times 1$ state vector, u is a scalar control variable and y is a scalar output. Matrix $\boldsymbol{A}$ and vectors $\boldsymbol{b}$ and $\boldsymbol{d}$ are time invariant and have appropriate dimensions. Vector $\boldsymbol{x}_0$ is an arbitrary initial condition. The system in (6.1)-(6.2) is assumed to be both controllable and observable.

Markov parameters m_j of the continuous-time system, which are coefficients of the infinite expansion of the transfer function $G(s)$,

$$G(s) = \boldsymbol{d}'(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{B(s)}{A(s)} = \sum_{i=0}^{\infty} \frac{m_i}{s^i} \tag{6.3}$$

can also be expressed in the state-space terms as $m_j = \boldsymbol{d}'\boldsymbol{A}^{j-1}\boldsymbol{b}$, $j > 0$. Provided $b_0 \neq 0$, it is well known that $m_0 = 0, \ m_1 = 0, \ \ldots \ m_{k-1} = 0, \ m_k = b_0$, where $k = m - n$ is the relative order of the system.

Assume that a discrete-time sampled data control algorithm

$$u_i = -\boldsymbol{k}'\boldsymbol{x}_i, \ \boldsymbol{x}_i = \boldsymbol{x}(t_i) \tag{6.4}$$

with a zero order hold is to be applied, so that

$$u(t) = u_i, \ t \in (t_i, \ t_{i+1}], \ t_{i+1} - t_i = h, \ i = 0, 1 \ldots$$

The output $y(t)$ of the controlled system is required to be close to the solution $y_r(t)$ of a reference differential equation:

$$\sum_{i=0}^{r} c_i y_r^{(r-i)}(t) = 0, \ y_r^{(i)}(0) = y^{(i)}(0), i = 0 \ldots r-1, \tag{6.5}$$

where r is an integer and the closed-loop system is required to be stable. Denoting

$$C(s) = \sum_{i=0}^{r} c_i s^{r-i} = \prod_{i=1}^{r}(s - s_i), \tag{6.6}$$

then on assumption that $s_i \neq s_j$ for $i, j = 1, 2 \ldots r, \ i \neq j$, the solution of (6.5) is defined by

$$y_r(t) = \sum_{j=1}^{r} \psi_j e^{s_j t}, \tag{6.7}$$

where ψ_i depend on initial conditions.

Function $e_r(t) = C(p)y(t)$, where p is a differential operator can serve as a measure of discrepancy between $y(t)$ and $y_r(t)$. It is therefore reasonable to require that the following quadratic performance index is minimized:

$$I_c = \int_0^{\infty} e_r^2(t)dt. \tag{6.8}$$

6.3 Continuous-time control

The control problem (6.8) with (6.1)-(6.2) can be transformed to the classical continuous time LQ formulation (Kwakernaak & Sivan, 1972). To this end, $e_r(t)$ should be expressed as a function of the state variable $\boldsymbol{x}$. Depending on the relationship between r and the relative order $k = n - m$ three different problems can be distinguished:

(i) if $r < k$ then $e_r(t) = \boldsymbol{f}_r'\boldsymbol{x}(t)$

(ii) if $r = k$ then $e_r(t) = \boldsymbol{f}_r'\boldsymbol{x}(t) + b_0 u(t)$

(iii) if $r > k$ then $e_r(t) = \boldsymbol{f}_r'\boldsymbol{x}(t) + \sum_{j=0}^{r-k} \phi_{r-j} u^{(j)}(t)$,

where

$$\boldsymbol{f}_r = \sum_{j=0}^{r} c_{r-j}\boldsymbol{d}_j,\ \phi_i = \sum_{j=0}^{i} m_j c_{i-j}, \tag{6.9}$$

$$\boldsymbol{d}_j = \boldsymbol{A}'\boldsymbol{d}_{j-1},\ \boldsymbol{d}_0 = \boldsymbol{d},\ j = 1, 2 \ldots \tag{6.10}$$

The character of the solution strongly depends on the relationship between the order r of the reference differential equation and the relative order k of the system. For $r = k$, a linear state-space feedback is a solution to the problem while for $r > k$ a dynamic controller, which can be viewed as a state-feedback from an augmented state, is needed. When $r < k$ then a highly impractical singular control algorithm is obtained, with the control signal consisting of $\delta(t) \ldots \delta^{(q-1)}(t)$-type impulses followed by a state feedback (Błachuta, 1982; Clements & Anderson, 1978; Sirisena, 1968), where $q = k - r$ is called the *order of singularity*. It is worth noting that a performance index with $r = k - 1$, i.e. $q = 1$, similar to that in (6.8), whose integrand is augmented with a weighted square of the control signal term is considered in (Hashimoto, Yoneya & Togari, 1989) and (Yoneya, Hashimoto & Togari, 1992) as a tool for the continuous-time model reference system design. A disadvantage of that method is that to arrive at a desired dynamics the control weighting is to be set small which results in an impulsive behavior of the control signal for small t unless the initial condition is close to the singular hyperplane, (O'Malley & Jameson, 1975; O'Malley & Jameson, 1977).

6.3.1 Regular problem

Assume $r = k$. Performance index (6.8) can now be expressed in the terms of the state vector x and the control variable u as follows:

$$I_c = \int_0^\infty (\boldsymbol{x}'\boldsymbol{Q}_c\boldsymbol{x} + 2\boldsymbol{x}'\boldsymbol{h}_c u + \lambda_c u^2)dt. \tag{6.11}$$

Matrix $\boldsymbol{Q}_c$, vector $\boldsymbol{h}_c$ and scalar λ_c are determined by the formulae:

$$\boldsymbol{Q}_c = \boldsymbol{f}_r\boldsymbol{f}_r',\ \boldsymbol{h}_c = \boldsymbol{f}_r b_0,\ \lambda_c = b_0^2. \tag{6.12}$$

A stable solution can be obtained applying the Kalman LQ regulator theory (Kwakernaak & Sivan, 1972). The optimal control, u^o, is then of the form:

$$u^o(t) = -\boldsymbol{k}_c'\boldsymbol{x}(t),\ \boldsymbol{k}_c = \frac{1}{\lambda_c}(\boldsymbol{h}_c + \boldsymbol{P}\boldsymbol{b}) \tag{6.13}$$

with $\boldsymbol{P}$ being a stabilizing solution of the following algebraic matrix Riccati equation:

$$\boldsymbol{A}_*'\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_* - \boldsymbol{P}\boldsymbol{b}r_c^{-1}\boldsymbol{b}'\boldsymbol{P} = 0, \tag{6.14}$$

where

$$\boldsymbol{A}_* = \boldsymbol{A} - \lambda_c^{-1}\boldsymbol{b}\boldsymbol{h}_c'. \tag{6.15}$$

A trivial solution $\boldsymbol{P} = 0$ is stabilizing if and only if the system to be controlled is minimum phase. This results from the closed loop state equation matrix (equal to $\boldsymbol{A}_*$), whose eigenvalues are the roots of the equation $C(s)B(s) = 0$. Otherwise, a stabilizing solution can be constructed from the eigenvectors and left-hand half plane eigenvalues of the Hamiltonian matrix $\boldsymbol{H}$ (Kwakernaak & Sivan, 1972):

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A}_* & -\lambda_c^{-1}\boldsymbol{b}\boldsymbol{b}' \\ 0 & -\boldsymbol{A}_*' \end{bmatrix}. \tag{6.16}$$

From (6.16), the eigenvalues of $\boldsymbol{H}$ are related to the roots of $C(s)$ and $B(s)$ in a very simple way:

$$\begin{aligned} \det(s\boldsymbol{I} - \boldsymbol{H}) &= \det(s\boldsymbol{I} - \boldsymbol{A}_*)\det(s\boldsymbol{I} + \boldsymbol{A}_*') \\ &= C(s)B(s)C(-s)B(-s). \end{aligned} \tag{6.17}$$

Denote $B(s) = B^-(s)B^+(s)$ where all roots of $B^-(s)$ lie in the left-hand half plane and all roots of $B^+(s)$ lie in the right-hand half plane. Then the optimal stabilizing solution of the Riccati equation is constructed from all roots of $C(s), B^-(s)$ and $B^+(-s)$. The characteristic polynomial of the closed loop system:

$$\dot{\boldsymbol{x}} = (\boldsymbol{A}_* - \frac{1}{\lambda_c}\boldsymbol{b}\boldsymbol{b}'\boldsymbol{P})\boldsymbol{x} \tag{6.18}$$

takes the form:

$$\det(s\boldsymbol{I} - \boldsymbol{A}_* - \frac{1}{\lambda_c}\boldsymbol{b}\boldsymbol{b}'\boldsymbol{P}) = C(s)B^-(s)B^+(-s). \tag{6.19}$$

6.3.2 The shape of control

For the system in (6.1)-(6.2) is deterministic and we study the free response, which only depends on the initial conditions, then the closed-loop solution and the open-loop solution are equivalent. The shape of an optimal control variable $u^0(t)$ for a minimum-phase system can now be found using the expression

$$y(t) = \sum_{i=1}^{n} \rho_i e^{\sigma_i t} + \int_0^t g(t-\tau)u^0(\tau)d\tau, \tag{6.20}$$

where σ_i, $i = 1, 2, \ldots n$ are the roots of the denominator $A(s)$ of the transfer function $G(s)$, ρ_i, $i = 1, 2, \ldots n$, are constants depending on initial conditions and

$$g(t) = \sum_{i=1}^{n} \kappa_i e^{\sigma_i t} \tag{6.21}$$

is a weighting function, with $\kappa_i \neq 0$, $i = 1, 2 \ldots n$. Let the optimal control be of the form

$$u^0(t) = \sum_{j=1}^{n} \vartheta_j e^{\lambda_j t} \tag{6.22}$$

and $y(t) = y_r(t)$. Parameters ϑ_j, λ_j for $j = 1, 2 \ldots n$ and c_i for $i = 1, 2, \ldots n$ are to be determined. Upon calculating the Laplace transformation of both sides of (6.20) and taking (6.7) into account we get

$$\sum_{i=1}^{r} \frac{\psi_i}{s - s_j} = \sum_{j=1}^{n} \frac{\rho_i}{s - \sigma_j} + G(s)u(s), \tag{6.23}$$

which can be rewritten as

$$\Psi(s)C^{-1}(s) = [R(s) + B(s)u(s)]A^{-1}(s), \tag{6.24}$$

and finally

$$u(s) = \frac{A(s)\Psi(s) - R(s)C(s)}{B(s)C(s)}. \tag{6.25}$$

To arrive at (6.5), r parameters ψ_i of the polynomial $\Psi(s)$ should be set so that the order of the numerator of the expression in (6.25) is reduced from its generic value $n + r - 1$ to $n - 1$. The order of the denominator equals to $m + r$. Then for $r < k$ (i.e. $q > 0$) we have:

$$u(s) = \sum_{i=0}^{q-1} \omega_i s^i + \sum_{i=1}^{m+r} \frac{\vartheta_i}{s - \lambda_i} \tag{6.26}$$

The above equation shows the appearance of δ-like impulses in the singular control. For $r = k$ ($q = 0$) one gets a regular solution:

$$u(s) = \sum_{i=1}^{n} \vartheta_i (s - \lambda_i)^{-1}, \tag{6.27}$$

where $\lambda_j = s_j$, $j = 1, 2 \ldots r$, $\lambda_i = \tau_i$, $i = 1, 2 \ldots m$ and τ_i are roots of the polynomial $B(s)$.

Further simplifications of the control signal occur when $A(s)$ and $C(s)$ contain common roots.

6.4 Purely discrete-time approach

The discrete-time model of the plant (6.1)-(6.2) is as follows:

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{g}u_i,\ \boldsymbol{x}_0 = \boldsymbol{x}(0) \tag{6.28}$$

$$y_i = \boldsymbol{d}'\boldsymbol{x}_i, \tag{6.29}$$

where

$$\boldsymbol{F} = e^{\boldsymbol{A}h},\ \boldsymbol{g} = \int_0^h e^{\boldsymbol{A}\tau}\boldsymbol{b}d\tau. \tag{6.30}$$

The relative order of the discrete-time system (6.28)-(6.29) with the transfer function

$$H(z) = \boldsymbol{d}'(z\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{g} = \frac{b(z)}{a(z)} \tag{6.31}$$

is independent of the relative order k of the continuous-time counterpart and generically equals to 1.

Denote $z_j = e^{s_j h}$, $j = 1, 2 \ldots r$. Then the sampled version of (6.7) is

$$y_r(i) = \sum_{j=1}^{r} \psi_j (z_j)^i. \tag{6.32}$$

As a result, the sampled reference output fulfills the following difference equation

$$c(z)y_r(i) = 0, \tag{6.33}$$

where

$$c(z) = \sum_{i=0}^{r} \gamma_i z^{r-i} = \prod_{i=1}^{r}(z - z_i). \tag{6.34}$$

It is therefore reasonable to define the performance index for the purely discrete-time problem in the form of:

$$I_d = \sum_{i=0}^{\infty} e_r^2(i),\ e_r(i) = c(z)y_i. \tag{6.35}$$

When expressing $e_r(i)$ in the state-space terms, three different problems can be distinguished:

(i) if $r=0$ then $e_0(i)=\boldsymbol{\theta}_0'\boldsymbol{x}_i$

(ii) if $r=1$ then $e_1(i)=\boldsymbol{\theta}_1'\boldsymbol{x}_i+\beta_0 u_i$

(iii) if $r>1$ then $e_r(i)=\boldsymbol{\theta}_r'\boldsymbol{x}_i+\sum_{j=0}^{r-1}\varphi_{r-j}u_{i+j}$,

where

$$\boldsymbol{\theta}_r=\sum_{j=0}^{r}\gamma_{r-j}\boldsymbol{\delta}_j,\ \varphi_i=\sum_{j=0}^{i}\mu_j\gamma_{i-j} \tag{6.36}$$

$$\boldsymbol{\delta}_j=\boldsymbol{F}'\boldsymbol{\delta}_{j-1},\ \boldsymbol{\delta}_0=\boldsymbol{d},\ j=1,2\ldots \tag{6.37}$$

It is important to notice that the above classification does not depend on the relative order k of the continuous-time plant. While the problems in (i) and (ii) lead to well defined discrete LQ singular and nonsingular control problems, the problem in (iii) has no causal solution because current values of $\boldsymbol{x}_i$ depend on future values of control variable u_i in that problem.

6.4.1 Zero order problem

For $r=0$ we have:

$$I_d=\sum_{i=0}^{\infty}\boldsymbol{x}_i'\boldsymbol{Q}_d\boldsymbol{x}_i,\ \boldsymbol{Q}_d=\boldsymbol{d}\boldsymbol{d}'. \tag{6.38}$$

The optimal control u_i^o is given by $u_i^o=-\boldsymbol{k}_d'\boldsymbol{x}_i$ where $\boldsymbol{k}_d=\boldsymbol{F}'\boldsymbol{P}\boldsymbol{g}/(\boldsymbol{g}'\boldsymbol{P}\boldsymbol{g})$ and $\boldsymbol{P}$ is a solution of the discrete Riccati equation

$$\boldsymbol{P}=\boldsymbol{F}'(\boldsymbol{P}-\frac{\boldsymbol{P}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}}{\boldsymbol{g}'\boldsymbol{P}\boldsymbol{g}})\boldsymbol{F}+\boldsymbol{Q}_d. \tag{6.39}$$

A trivial solution which belongs to the set of positive symmetric solutions of (6.39) is $\boldsymbol{P}=\boldsymbol{Q}_d=\boldsymbol{d}\boldsymbol{d}'$ for which $\boldsymbol{k}_d=\beta_0^{-1}\boldsymbol{F}'\boldsymbol{d}$ and the closed loop system matrix takes the form

$$\boldsymbol{F}_*=\boldsymbol{F}-\boldsymbol{g}\boldsymbol{k}_d' \tag{6.40}$$

with the characteristic polynomial $p(z)$:

$$p(z)=\det(z\boldsymbol{I}-\boldsymbol{F}_*)=\beta_0^{-1}zb(z). \tag{6.41}$$

From (6.39) it is seen that $\boldsymbol{P}=\boldsymbol{d}\boldsymbol{d}'$ is a stabilizing solution if and only if the roots of $b(z)$ are inside the unit circle. Otherwise, when $b(z)=b^-(z)b^+(z)$ and $b(z)^+$ has all roots outside the unit circle, a stabilizing solution to (6.46) is to be found such that

$$p(z)=\beta_0^{-1}zb^-(z)b^+(z^{-1})z^{m^+}, \tag{6.42}$$

where $m^+=\deg b^+(z)$.

6.4.2 First order problem

For $r=1$ there is:

$$I_d=\sum_{i=0}^{\infty}(\boldsymbol{x}_i'\boldsymbol{Q}_d\boldsymbol{x}_i+2\boldsymbol{h}_d\boldsymbol{x}_iu_i+\lambda_d u_i^2) \tag{6.43}$$

with

$$\boldsymbol{Q}_d=\boldsymbol{\theta}_1\boldsymbol{\theta}_1',\ \boldsymbol{h}_d=\beta_0\boldsymbol{\theta}_1,\ \lambda_d=\beta_0^2. \tag{6.44}$$

The optimal control is $u_i^o=-\boldsymbol{k}_d'\boldsymbol{x}_i$, where

$$\boldsymbol{k}_d=\frac{1}{\lambda_d}\boldsymbol{h}_d+\frac{\boldsymbol{F}_*'\boldsymbol{P}}{\lambda_d+\boldsymbol{g}'\boldsymbol{P}\boldsymbol{g}}\boldsymbol{g} \tag{6.45}$$

and $\boldsymbol{P}$ is a solution of the discrete Riccati equation

$$\boldsymbol{P}=\boldsymbol{F}_*'(\boldsymbol{P}-\frac{\boldsymbol{P}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}}{\lambda_d+\boldsymbol{g}'\boldsymbol{P}\boldsymbol{g}})\boldsymbol{F}_* \tag{6.46}$$

with

$$\boldsymbol{F}_*=\boldsymbol{F}-\lambda_d^{-1}\boldsymbol{g}\boldsymbol{h}_d',\ \boldsymbol{Q}_{d*}=\boldsymbol{Q}_d-\lambda_d^{-1}\boldsymbol{h}_d\boldsymbol{h}_d'=0. \tag{6.47}$$

The solution to this problem depends on eigenvalues of the matrix $\boldsymbol{F}_*$. For $\boldsymbol{P}=\boldsymbol{0}$, we have

$$p(z)=\det(z\boldsymbol{I}-\boldsymbol{F}_*)=\beta_0^{-1}(\gamma_1 z+1)b(z). \tag{6.48}$$

Thus for a stable polynomial $b(z)$, $\boldsymbol{P}=0$ is a stabilizing solution of the Riccati equation (6.46), and then $u_i=-\lambda_d^{-1}\boldsymbol{h}_d'\boldsymbol{x}$ is a stable optimal controller. Otherwise, a stabilizing solution to (6.46) is to be found such that

$$p(z)=\beta_0^{-1}(\gamma_1 z+1)b^-(z)b^+(z^{-1})z^{m^+}. \tag{6.49}$$

6.4.3 Higher order problems

To avoid non-causality an additional delay must be introduced into the control path so that v_i is a new control variable and $u_i=v_{i-k}$. One is then able to write:

$$e_r(i)=\boldsymbol{\theta}_r'\boldsymbol{x}_i+\sum_{j=0}^{r-1}\varphi_{r-j}v_{i+j-k}. \tag{6.50}$$

If $k=r-1$ then $e_r(i)$ depends on current and previous values $v_i\ldots v_{i-r+1}$ of the new control signal only.

6.4.4 High sampling rates phenomena

According to (Åström et al., 1984) a peculiarity of $H(z)$ is that m *intrinsic zeros* of $b(z)$ tend to 1, and $k-1$ *discretization zeros* tend to the zeros of the so called reciprocal or Euler polynomial $\mathcal{E}_k(z)$ as the sampling period $h \to 0$. It has been shown in Chapter 2 that the zeros of $\mathcal{E}_k(z)$, ζ_i and ζ_i^{-1}, are negative real so that:

$$\mathcal{E}_k(z)=\begin{cases} \prod_{i=1}^{\frac{k-1}{2}} (z-\zeta_i)(z-\zeta_i^{-1}), & k\text{-odd} \\ (z+1)\prod_{i=1}^{\frac{k}{2}-1} (z-\zeta_i)(z-\zeta_i^{-1}), & k\text{-even.} \end{cases} \tag{6.51}$$

As a result, $\mathcal{E}_k(z)$ have zeros on or outside the unit circle for $k \geq 2$. Due to (Åström et al., 1984; Hagiwara et al., 1993; Błachuta, 1997*f*), from (6.51) it results that for $k > 2$ the polynomial $b(z)$ becomes unstable if $h \leq h_{crit}$. Unstable zeros are replaced by their reciprocals in the characteristic polynomial, (6.42) and (6.49), so that the resulting closed loop system remains stable but as they tend to be negative real highly undesirable 'ringing' of the control signal appears. Moreover, since

$$\beta_0 = b_0 h^k/k! + o(h^k), \tag{6.52}$$

according to Theorem 2.3.1 the magnitudes of the control signal become large for small values of h. This shows that the relative order of the system is responsible for the above phenomena as supposed in (Hara et al., 1996). A remedy against unstable pole-zero cancellation, which could also be used in our problem, has been proposed in (Goodwin et al., 1986; Tesfaye & Tomizuka, 1995). It consists in replacing the original pulse transfer function by its approximation in the Euler operator domain, where the discretization zeros, which lie far from the origin, are ommited. Unfortunately, as shown in Chapter 4 this approximation is quite poor unless h is very small.

6.5 Hybrid design of a discrete-time controller

6.5.1 Problem solution

Let us assume that the control task is the same as in section 6.3, i.e. it is defined by the integral

$$I = \int_0^\infty (\boldsymbol{x}'\boldsymbol{Q}_c\boldsymbol{x} + 2\boldsymbol{x}'\boldsymbol{h}_c u + \lambda_c u^2)dt, \tag{6.53}$$

in which matrix $\boldsymbol{Q}_c$, vector $\boldsymbol{h}_c$ and scalar λ_c are determined by the formulae:

$$\boldsymbol{Q}_c = \boldsymbol{f}_r\boldsymbol{f}_r', \ \boldsymbol{h}_c = \boldsymbol{f}_r m_r, \ \lambda_c = m_r^2 \tag{6.54}$$

but the controller is discrete-time as defined in section 6.2. Now, taking into account that the intersample values of the state vector $\boldsymbol{x}(t)$ are given by the formula

$$\boldsymbol{x}(t) = \boldsymbol{F}(\tau)\boldsymbol{x}_i + \boldsymbol{g}(\tau)u_i, \ \text{for}\ t = ih+\tau,\ \tau \in (0,h] \tag{6.55}$$

with

$$\boldsymbol{F}(\tau) = e^{\boldsymbol{A}\tau},\ \boldsymbol{g}(\tau) = \int_0^\tau e^{\boldsymbol{A}v}\boldsymbol{b}dv, \tag{6.56}$$

we are able to replace the integral performance index of (6.53) by a discrete summation index (Dorato & Levis, 1971; Kwakernaak & Sivan, 1972; Kuo & Peterson, 1973):

$$J = h\sum_{i=1}^{\infty}(\boldsymbol{x}_i'\boldsymbol{Q}\boldsymbol{x}_i + 2\boldsymbol{h}'\boldsymbol{x}_i u_i + \lambda u_i^2) \tag{6.57}$$

with

$$\boldsymbol{Q} = \frac{1}{h}\int_0^h e^{\boldsymbol{A}'\tau}\boldsymbol{Q}_c e^{\boldsymbol{A}\tau}d\tau \tag{6.58}$$

$$\boldsymbol{h} = \boldsymbol{h}_c + \frac{1}{h}\int_0^h e^{\boldsymbol{A}'\tau}\boldsymbol{Q}_c\boldsymbol{g}(\tau)d\tau \tag{6.59}$$

$$\lambda = \lambda_c + \frac{1}{h}\int_0^h [\boldsymbol{g}'(\tau)\boldsymbol{Q}_c\boldsymbol{g}(\tau) + 2\boldsymbol{h}_c'\boldsymbol{g}(\tau)]d\tau. \tag{6.60}$$

The integrals in (6.58)-(6.60) can easily be calculated using algorithms from (van Loan, 1977). It is worth noting that vector $\boldsymbol{h}$ and scalar λ are nonzero even if both $\boldsymbol{h}_c$ and λ_c are zero, thus producing a nonsingular discrete-time problem even if the continuous-time counterpart is singular. The optimal control u_i^o is given by the formula:

$$u_i^o = -\boldsymbol{k}'\boldsymbol{x}_i, \tag{6.61}$$

where

$$\boldsymbol{k} = \frac{1}{\lambda}\boldsymbol{h} + \frac{\boldsymbol{F}_*'\boldsymbol{S}}{\lambda + \boldsymbol{g}'\boldsymbol{S}\boldsymbol{g}}\boldsymbol{g} \tag{6.62}$$

and $\boldsymbol{S}$ is a solution of the discrete Riccati equation

$$\boldsymbol{S} = \boldsymbol{Q}_* + \boldsymbol{F}_*'(\boldsymbol{S} - \frac{\boldsymbol{S}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{S}}{\lambda + \boldsymbol{g}'\boldsymbol{S}\boldsymbol{g}})\boldsymbol{F}_*, \tag{6.63}$$

where

$$\boldsymbol{F}_* = \boldsymbol{F} - \lambda^{-1}\boldsymbol{g}\boldsymbol{h}',\ \boldsymbol{Q}_* = \boldsymbol{Q} - \lambda^{-1}\boldsymbol{h}\boldsymbol{h}'. \tag{6.64}$$

The solution to this problem depends not only on the eigenvalues of the matrix $\boldsymbol{F}_*$ but also on the remaining parameters λ and $\boldsymbol{Q}_*$.

6.5.2 Limiting behavior of the solution

Let us now study the limiting behavior of the optimally controlled system when $h \to 0$. From (6.58) it follows:

$$\boldsymbol{Q} = \boldsymbol{Q}_c + (\boldsymbol{A}'\boldsymbol{Q}_c + \boldsymbol{Q}_c\boldsymbol{A})\frac{h}{2} + O(h^2). \tag{6.65}$$

When calculating approximations to $\boldsymbol{h}$ and λ we have different results depending on $q = k - r$.

For $q = 0$ we get

$$\boldsymbol{h} = \boldsymbol{h}_c + \boldsymbol{Q}_c\boldsymbol{b}\frac{h}{2} + O(h^2) \tag{6.66}$$

$$\lambda = \lambda_c + \boldsymbol{h}_c'\boldsymbol{b}h + \frac{1}{3}\boldsymbol{b}'\boldsymbol{Q}_c\boldsymbol{b}h^2 + O(h^3), \tag{6.67}$$

and finally

$$\boldsymbol{Q}_* = (\boldsymbol{A}_*'\boldsymbol{Q}_c + \boldsymbol{Q}_c\boldsymbol{A}_*)\frac{h}{2} + O(h^2), \tag{6.68}$$

where

$$\boldsymbol{A}_* = \boldsymbol{A} - \lambda^{-1}\boldsymbol{b}\boldsymbol{h}_c'. \tag{6.69}$$

Using

$$\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{A}h + O(h^2),\ \boldsymbol{g} = \boldsymbol{b}h + O(h^2) \tag{6.70}$$

the Riccati equation can be transformed as follows:

$$(\boldsymbol{F}_*')^{-1}\boldsymbol{S}(\boldsymbol{F}_*)^{-1} = (\boldsymbol{F}_*')^{-1}\boldsymbol{Q}_*(\boldsymbol{F}_*)^{-1} + \boldsymbol{S} - \frac{\boldsymbol{S}\boldsymbol{b}\boldsymbol{b}'\boldsymbol{S}h^2}{\lambda + \boldsymbol{b}'\boldsymbol{S}\boldsymbol{b}h^2} + O(h^4). \tag{6.71}$$

Taking $\boldsymbol{P} = \boldsymbol{S}h$ and $(\boldsymbol{F}_*)^{-1} = \boldsymbol{I} - \boldsymbol{A}_*h + O(h^2)$ equation (6.71) becomes:

$$\boldsymbol{A}_*'\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_* - \boldsymbol{P}\boldsymbol{b}\lambda_c^{-1}\boldsymbol{b}'\boldsymbol{P} = 0 \tag{6.72}$$

when $h \to 0$, and the optimal gain will be

$$\boldsymbol{k} = \frac{1}{\lambda_c}(\boldsymbol{P}\boldsymbol{b} + \boldsymbol{h}_c) = \boldsymbol{k}_c. \tag{6.73}$$

Investigation of the limiting behavior of solutions to problems with $q > 0$ remains a still open question. Based on numerical examples (Baron, 1989) and a continuity argument, it is conjectured that the discrete-time solution tends to produce a singular control with δ-like impulses at the beginning followed by a quite regular control afterwards.

6.6 Example

A continuous-time plant with the transfer function $K(s)$:

$$K(s) = \frac{s + 0.06}{(s + 0.4)(s^2 - 0.35s + 0.15)} \tag{6.74}$$

is considered for $h = 1.0$. The results obtained for $k = 0$ and

$$I_1 = \int_0^\infty (1 - y(t))^2 dt, \tag{6.75}$$

with a new design method and those for

$$I_2 = \sum_0^\infty (1 - y(i))^2 \tag{6.76}$$

produced by the purely discrete-time method are displayed in Fig.1. In the latter case it is clearly seen that although the results at the sampling instants are excellent, the intersample behavior is unacceptable, which is due to control signal 'ringing'. The results obtained by our method are in a great contrast with those of the digital one. Fig.2 displays the results of the hybrid method when the reference model is of the first order (i.e by one smaller than the actual relative order $k = 2$) with a zero at $s = -0.4$. It is important to note that in the hybrid framework control signals which are quite impulsive at the beginning become very soon 'smooth' for problems with $q > 0$.

6.7 Conclusion

It has been shown that the purely discrete-time approach to the control systems design suffers from severe disadvantages when the sampling rate becomes high. They demonstrate as 'ringing' and high magnitudes of the control signal. These phenomena are caused by the properties of the sampling zeros of pulse transfer functions at high sampling rates.

The proposed method of a hybrid discrete-time controller design does not exhibit these disadvantages. Provided that the order of the desired output model equals to the relative order of the continuous-time system, the control signal tends to a smooth continuous-time function when the sampling rate increases.

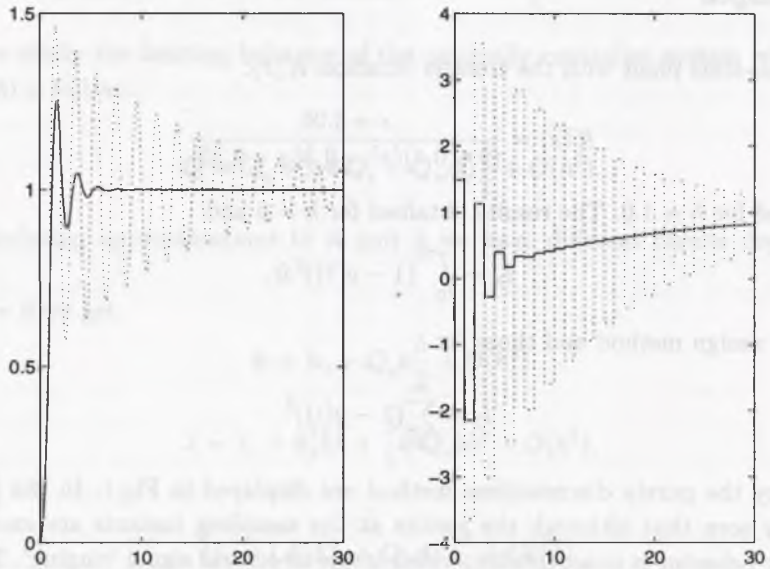

Fig. 6.1. Output and control signals for hybrid (solid) and discrete (dotted) designs, $q = 2$

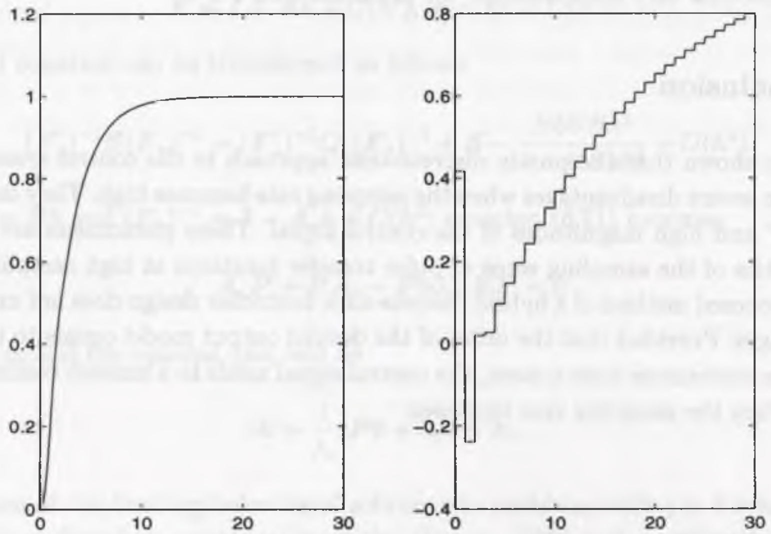

Fig. 6.2. Output and control signals for hybrid design, $q = 1$

Part II

Stochastic Systems

7. Modeling Sampled Stochastic Processes

A class of second-order continuous-time stochastic processes, which can be thought as models of disturbances, is characterized and the issue of their sampling is discussed. [1] As a result of sampling, discrete second-order random processes described by linear time-invariant state-space models are obtained. Equivalent representations with the number of noise inputs reduced to one are presented. In contrast to the innovations approach these representations have time-invariant parameters. The relationship with ARMA models is discussed and the Representations Theorem is generalized to a class of nonstationary processes. Finally, the identification issue of continuous-time processes is discussed.

7.1 Continuous-Time Stochastic Processes

7.1.1 Process models

A wide class of stochastic processes can be described by the following system of equations:

$$d\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t)dt + \boldsymbol{c}d\xi(t),\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{7.1}$$

$$z(t) = \boldsymbol{d}'\boldsymbol{x}(t). \tag{7.2}$$

Here $z(t)$ is a scalar process, $\boldsymbol{x}(t)$ is an n-dimensional state vector, $\boldsymbol{A}$ is a matrix with constant entries, $\boldsymbol{c}$ and $\boldsymbol{d}$ are vectors, and $\xi(t)$ is a standard Wiener process (Gikhman & Skorokhod, 1969; Gikhman & Skorokhod, 1972) with Gaussian increments

$$\mathrm{E}\,[\xi(t)] = 0,\ \mathrm{E}\,[\xi^2(t)] = t^2. \tag{7.3}$$

The symbol d stands for Ito differential. The initial condition $\boldsymbol{x}_0$ is a normally distributed random vector, $\boldsymbol{x}_0 \sim \mathcal{N}(\boldsymbol{m}_0, \boldsymbol{Q}_0)$, i.e. $\boldsymbol{m}_0 = \mathrm{E}\,(\boldsymbol{x}_0)$ and $\boldsymbol{Q}_0 = \mathrm{E}\,[(\boldsymbol{x}_0 - \boldsymbol{m}_0)(\boldsymbol{x}_0 - \boldsymbol{m}_0)']$.

It will be assumed that the system (7.1)-(7.2) is both controllable and observable, i.e.

$$\mathrm{rank}\,[\boldsymbol{c}, \boldsymbol{A}\boldsymbol{c},\ \ldots \boldsymbol{A}^{n-1}\boldsymbol{c}] = n, \tag{7.4}$$

[1] The chapter is based on (Błachuta, 1994; Błachuta, 1996*d*; Błachuta, 1996*e*; Błachuta & Polański, 1986*b*; Błachuta & Polański, 1987) and (Błachuta & Polański, 1990)

$$\operatorname{rank}[d, A'd, \ldots A'^{n-1}d] = n. \tag{7.5}$$

Equation (7.1), which is a stochastic differential equation, can be understood as a symbolical notation of the following integral equation:

$$x(t) = x_0 + \int_0^t Ax(s)ds + \int_0^t cd\xi(s), \tag{7.6}$$

where the second integral in formula (7.6) is the Ito stochastic integral (Gikhman & Skorokhod, 1969; Gikhman & Skorokhod, 1972). The solution of equation (7.6) has the following form:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}cd\xi(s). \tag{7.7}$$

In the technical literature the system (7.1)-(7.2) is sometimes presented in a less strict form as a system of ordinary differential equations driven by a continuous-time stationary white noise $\dot{\xi}(t)$:

$$\frac{d}{dt}x(t) = Ax(t) + c\dot{\xi}(t) \tag{7.8}$$

$$z(t) = d'x(t), \tag{7.9}$$

where

$$\mathrm{E}[\dot{\xi}(t)\dot{\xi}(\tau)] = \delta(t-\tau). \tag{7.10}$$

7.1.2 Characteristics of stochastic processes

The expected value $m(t)$ and covariance $Q_t(\tau)$ are expressed by the formulae:

$$m(t) = \mathrm{E}[x(t)] = e^{At}m_0 \tag{7.11}$$

$$Q_t(\tau) = \mathrm{E}[x(t) - m(t)][(x(t+\tau) - m(t+\tau)]' = Q(t)e^{A'\tau}, \tag{7.12}$$

where

$$Q(t) = \mathrm{E}[x(t) - m(t)][(x(t) - m(t)]' = e^{At}Q_0e^{A't} + \int_0^t e^{A(t-s)}cc'e^{A'(t-s)}ds, \tag{7.13}$$

fulfills a differential matrix Lyapunov equation:

$$\dot{Q}(t) = AQ(t) + Q(t)A' + cc' \tag{7.14}$$

with the initial value Q_0.

According to (7.11)-(7.12) the expected value $\mu(t)$ and autocorrelation function $\rho_t(\tau)$ of the process $z(t)$ are defined by the following relationships:

$$\mu(t) = d'm(t) = d'e^{At}m_0 \tag{7.15}$$

$$\rho_t(\tau) = d'Q_t(\tau)d = d'Q(t)e^{A'\tau}d. \tag{7.16}$$

7.1.3 Stationary processes

Since the process $z(t)$ defined in equations (7.1)-(7.2) is completely characterized by two first moments, the necessary and sufficient condition of stationarity is that the expected value $\mu(t)$ and correlation function $\rho_t(\tau)$ do not depend on current time t.

It is well known that process $z(t)$ defined in (7.1)-(7.2) with (7.4)-(7.5) is stationary if and only if:

- matrix A is stable, i.e. for $i = 1, 2 \ldots n$ there is $\operatorname{Re}\lambda_i(A) < 0$,
- expected value of the initial condition $m_0 = 0$,
- covariance matrix Q_0 is a solution of the following algebraic Lyapunov equation:

$$AQ + QA' = -cc'. \tag{7.17}$$

Under the above conditions the expected value of the process equals to zero and the correlation function is:

$$\rho(\tau) = d'Q_0e^{A'\tau}d. \tag{7.18}$$

As a result, the correlation function $\rho(\tau)$ can be expressed in the following form:

$$\rho(\tau) = \sum_{i=1}^{N} a_i p_i(\tau)e^{\lambda_i \tau}, \tag{7.19}$$

where $p_i(\tau)$ are finite degree polynomials and the real parts of λ_i are strictly negative.

Since the spectral density $\Sigma(\omega)$ and the autocorrelation function are related by:

$$\Sigma(\omega) = \int_{-\infty}^{+\infty} \rho(t)e^{-j\omega t}dt, \tag{7.20}$$

then, based on (7.17), (7.18) and (7.20),

$$\Sigma(\omega) = d'(sI - A)^{-1}cc'(-sI - A')^{-1}d|_{s=j\omega}. \tag{7.21}$$

$\Sigma(\omega)$ is a real rational function and can be expressed in the form of:

$$\Sigma(\omega) = \frac{c(s)c(-s)}{a(s)a(-s)}|_{s=j\omega}, \tag{7.22}$$

where:

$$a(s) = \det(sI - A) \tag{7.23}$$

$$c(s) = d'[\operatorname{adj}(sI - A)]c \tag{7.24}$$

and

$$a(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n \tag{7.25}$$

$$c(s) = \qquad \gamma_1 s^{n-1} + \ldots + \gamma_n. \tag{7.26}$$

Due to (7.4)-(7.5) polynomials $a(s)$ and $c(s)$ are relatively prime.

From (7.22) it is seen that for a given spectral density function $\Sigma(\omega)$ of the process $z(t)$ and the fixed polynomial $a(s)$, there exist polynomials $c(s)$, and thus vectors $\boldsymbol{c}$, for which the system in (7.1)-(7.2) is a model of the process $z(t)$. Among them such vector $\boldsymbol{c}$ can be found that all roots of the polynomial $c(s)$ lie in the left half-plane. Such representation is called invertible. Given a spectral density function $\Sigma(\omega)$, polynomials $a(s)$ and $c(s)$ can be found from equations (7.1)-(7.2), (7.27) and (7.22) by a spectral factorization procedure (Åström, 1970). Given polynomials $a(s)$ and $c(s)$, the state space representation (7.1)-(7.2) can be easily constructed by using canonical forms. The observer canonical form (Söderström, 1991) serves as an example, where

$$\boldsymbol{A} = \begin{bmatrix} -\alpha_1 & 1 & . & . & 0 \\ -\alpha_2 & 0 & 1 & . & 0 \\ \ldots & . & . & . & . \\ -\alpha_{n-1} & 0 & . & . & 1 \\ -\alpha_n & 0 & . & . & 0 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ . \\ \vdots \\ \gamma_n \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} 1 \\ 0 \\ . \\ \vdots \\ 0 \end{bmatrix}. \tag{7.27}$$

7.2 Sampling stochastic processes

A process $z(t)$ is usually observed by a sensor, which introduces its own errors. Sensors can be discrete- or continuous-time. In this section we are interested in getting a discrete-time model of a sensed and sampled stochastic process. To avoid the loss of observability a non-pathological sampling period is assumed.

7.2.1 Continuous-time sensor output

In (Kwakernaak & Sivan, 1972) one can find a model:

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{c}\dot{\xi}(t) \tag{7.28}$$

$$y(t) = \boldsymbol{d}'\boldsymbol{x}(t) + \dot{\zeta}(t) \tag{7.29}$$

based on model (7.9)-(7.10) with the output corrupted by a continuous-time white noise $\dot{\zeta}(t)$ with

$$\mathrm{E}\left[\dot{\xi}(t)\dot{\xi}(\tau)\right] = \delta(t-\tau), \ \mathrm{E}\left[\dot{\zeta}(t)\dot{\zeta}(\tau)\right] = \nu^2\delta(t-\tau). \tag{7.30}$$

Although the above model can be successfully used for filtration, prediction and smoothing, it is criticized since it is not mathematically strict, and it is not clear how to sample it.

To avoid mathematical incorrectness, in literature, e.g. (Kučera, 1972; Feuer & Goodwin, 1996), the following is proposed as the measurement equation:

$$dy(t) = z(t)dt + d\zeta(t) = \boldsymbol{d}'\boldsymbol{x}(t)dt + d\zeta(t). \tag{7.31}$$

This can also be written in the version of an integral equation :

$$y(t) = y(0) + \int_0^t \boldsymbol{d}'\boldsymbol{x}(s)ds + \int_0^t d\zeta(s), \tag{7.32}$$

where $\zeta(t)$ is a Wiener process. Equations (7.1) and (7.31) are sometimes written in the following less strict form

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{c}\dot{\xi}(t) \tag{7.33}$$

$$\frac{d}{dt}y(t) = \boldsymbol{d}'\boldsymbol{x}(t) + \dot{\zeta}(t). \tag{7.34}$$

7.2.2 Sampling: continuous-time noise model

In contemporary technical applications, the continuous-time signal being a realization of a continuous-time stochastic process is usually sampled, i.e. it is measured at discrete equi-distant time instants $t_i = hi$, where h is the sampling period and i is an integer, and then further processed digitally. Since, as follows from (7.30), the output of the model in (7.9)-(7.10) has infinite variance, sampling makes no sense for this model.

This problem is overcome when using the output defined in (7.31). Indeed, sampling equation (7.32) leads to:

$$y(t_i + h) = y(t_i) + \int_{t_i}^{t_i+h} \boldsymbol{d}'\boldsymbol{x}(s)ds + \int_{t_i}^{t_i+h} d\zeta(s). \tag{7.35}$$

According to Feuer & Goodwin (1996), this can be interpreted so that before being sampled the output signal is passed through an anti-aliasing filter with the transfer function

$$F(s) = \frac{1 - e^{hs}}{s} \tag{7.36}$$

Denote $y_i = y(t_i)$ and $\boldsymbol{x}_i = \boldsymbol{x}(t_i)$. For the model in (7.1) and (7.31) or, equivalently, (7.33)-(7.34), with the anti-aliasing filter (7.36) one gets:

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{w}_i \tag{7.37}$$

$$y_{i+1} = \boldsymbol{f}'\boldsymbol{x}_i + y_i + r_i \tag{7.38}$$

with

$$F = e^{Ah}, \; f' = d' \int_0^h e^{As} ds, \tag{7.39}$$

where w_i and r_i are zero mean Gaussian variables with

$$\mathrm{E}\begin{bmatrix} w_i w_j' & w_i r_j \\ r_i w_j' & r_i r_j \end{bmatrix} = \begin{bmatrix} W & \gamma \\ \gamma' & \rho^2 \end{bmatrix} \delta_{ij} \tag{7.40}$$

$$\begin{bmatrix} W & \gamma \\ \gamma' & \rho^2 \end{bmatrix} = \int_0^h e^{\bar{A}s} \begin{bmatrix} cc' & 0 \\ 0 & \nu^2 \end{bmatrix} e^{\bar{A}'s} ds, \; \bar{A} = \begin{bmatrix} A & 0 \\ d' & 0 \end{bmatrix}. \tag{7.41}$$

7.2.3 Sampling: discrete-time noise model

Although being mathematically well justified, the above model of section 7.2.2 has little technical meaning. As a result, the filter of (7.36) is not used in practical solutions.

Therefore, a very simple measurement model will be used in this work where it is assumed that samples of the process $\{z(t); t \geq 0\}$ of (7.2) are corrupted by a discrete-time white noise which models the measurement error.

In this case the measurement equation takes the form:

$$y_i = d' x_i + r_i, \tag{7.42}$$

where r_i is a discrete-time Gaussian white noise.

Process $\{y_i, i = 1, 2 \ldots\}$ is discrete-time and can be described by the following discrete-time system of stochastic equations (Kučera, 1972):

$$x_{i+1} = F x_i + w_i \tag{7.43}$$

$$y_i = d' x_i + r_i, \tag{7.44}$$

where the initial state $x_0 \in \mathbb{R}^n$ is a random vector with zero mean and covariance matrix

$$\mathrm{E}[x_0 x_0'] = Q_0. \tag{7.45}$$

The inputs w_i, and r_i are independent zero-mean white noises of appropriate dimensions with

$$\mathrm{E}[w_i, w_j'] = W \delta_{ij}, \; \mathrm{E}[r_i, r_j] = \rho^2 \delta_{ij}, \; \mathrm{E}[w_i, r_j] = 0 \tag{7.46}$$

for all $i, j \geq 0$, δ_{ij} denotes the Kronecker delta, and w_i is a vector-valued white Gaussian noise with covariance matrix W:

$$W = \int_0^h e^{As} cc' e^{A's} ds. \tag{7.47}$$

The random vectors x_0 and $[w_i', r_i]$ are uncorrelated for all $t \geq 0$.

7.2.4 Numerical issues

According to Chen & Francis (1995) the integral in (7.47) can be computed as

$$W = X_{22}' X_{12}, \tag{7.48}$$

where X_{12} and X_{22} result from the matrix exponential

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \exp\left\{ h \begin{bmatrix} -A & cc' \\ 0 & A' \end{bmatrix} \right\}. \tag{7.49}$$

Another way is as follows. Let Q denotes the covariance matrix of a stationary discrete-time process x_i fulfilling a discrete-time algebraic Lyapunov equation:

$$Q = FQF' + W. \tag{7.50}$$

Since vectors $x(t_i)$ and x_i are the same, then their covariance matrices are equal. From this property a method for computing the integral in (7.39) results, where first the Lyapunov equation (7.17) is solved for Q, and then (7.50) is employed to give:

$$W = Q - FQF'. \tag{7.51}$$

7.2.5 Nonpathological sampling

Although necessary for the pair (F, d) to be observable, the observability of (A, d) is not sufficient. In other words, the observability of a continuous-time system does not guaranties that the sampled system is observable.

Definition 7.2.1 (Nonpathological sampling). *The values of h which fulfill*

$$h \neq q\frac{\pi}{\omega_i}, \; q = 1, 2, \ldots, \tag{7.52}$$

where ω_i is the imaginary part of the i-th eigenvalue of matrix A, $\lambda_i(A) = \sigma_i + j\omega_i$ are called non-pathological.

To avoid possible loss of observability caused by sampling it is assumed that the sampling period h is nonpathological.

7.3 Kalman Filter and Innovations Representation

7.3.1 Kalman Filter

Denote $Y_i = \{y_0, y_1, \ldots, y_i\}$ a set of observations, $x_{i|i-1} = \mathrm{E}[x_i|Y_{i-1}]$ and $x_{i|i} = \mathrm{E}[x_i|Y_i]$ the predicted and filtered values of the state variable expressed as the expected values conditioned on available observations, and $\boldsymbol{\Sigma}_{i|i-1} = \mathrm{E}\{[\boldsymbol{x}_i - \boldsymbol{x}_{i|i-1}][\boldsymbol{x}_i - \boldsymbol{x}_{i|i-1}]'|Y_{i-1}\}$, $\boldsymbol{\Sigma}_{i|i} = \mathrm{E}\{[\boldsymbol{x}_i - \boldsymbol{x}_{i|i}][\boldsymbol{x}_i - \boldsymbol{x}_{i|i}]'|Y_i\}$ the related error covariances. The above values are produced by the Kalman filter which, for the model in (7.43)-(7.44), has the following form:

$$\hat{\boldsymbol{x}}_{i|i} = \hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{k}_i^f(y_i - \boldsymbol{d}'\hat{\boldsymbol{x}}_{i|i-1}) \tag{7.53}$$

$$\hat{\boldsymbol{x}}_{i+1|i} = \boldsymbol{F}\hat{\boldsymbol{x}}_{i|i}, \tag{7.54}$$

where

$$\boldsymbol{k}_i^f = \frac{\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}}{\rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}} \tag{7.55}$$

$$\boldsymbol{\Sigma}_{i|i} = \boldsymbol{\Sigma}_{i|i-1} + \frac{\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}\boldsymbol{d}'\boldsymbol{\Sigma}_{i|i-1}}{\rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}} \tag{7.56}$$

$$\boldsymbol{\Sigma}_{i+1|i} = \boldsymbol{F}\boldsymbol{\Sigma}_{i|i}\boldsymbol{F}' + \boldsymbol{W}. \tag{7.57}$$

Equations (7.54)-(7.57) play an important role in prediction of stochastic processes and are used for control and identification.

7.3.2 Innovations representations

Kalman filter can serve to design the so called innovations representation of a stochastic process. Denote $\bar{\boldsymbol{x}}_i = \hat{\boldsymbol{x}}_{i|i-1}$, $e_i = y_i - \boldsymbol{d}'\bar{\boldsymbol{x}}_i$, $\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}_{i|i-1}$. Then the innovations representation resulting from equations (7.54)-(7.57) has the form:

$$\bar{\boldsymbol{x}}_{i+1} = \boldsymbol{F}\bar{\boldsymbol{x}}_i + \boldsymbol{k}_i e_i, \ \bar{\boldsymbol{x}}_0 = 0 \tag{7.58}$$

$$y_i = \boldsymbol{d}'\bar{\boldsymbol{x}}_i + e_i, \tag{7.59}$$

where

$$\boldsymbol{k}_i = \frac{\boldsymbol{F}\boldsymbol{\Sigma}_i\boldsymbol{d}}{\rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}_i\boldsymbol{d}} \tag{7.60}$$

$$\boldsymbol{\Sigma}_{i+1} = \boldsymbol{W} + \boldsymbol{F}\boldsymbol{\Sigma}_i\boldsymbol{F}' - \frac{\boldsymbol{F}\boldsymbol{\Sigma}_i\boldsymbol{d}\boldsymbol{d}'\boldsymbol{\Sigma}_i\boldsymbol{F}'}{\rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}_i\boldsymbol{d}}, \ \boldsymbol{\Sigma}_0 = \boldsymbol{Q}_0. \tag{7.61}$$

Here e_i is a white noise with zero mean and time variable variance $\mathrm{E}[e_i^2] = \sigma_i^2$:

$$\sigma_i^2 = \rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}_i\boldsymbol{d}. \tag{7.62}$$

The advantage of the innovations representation of (7.58)-(7.59) as compared with the general representation of (7.43)-(7.44) is that it is driven by a scalar noise and the initial condition is deterministic. The disadvantage is that both the gain vector $\boldsymbol{k}_i$ and variance of e_i are time variable.

7.4 Simplified Representation

This section considers the problem of determining equivalent time invariant state-space representations with one noise input for the class of discrete random processes defined in (7.43)-(7.44) of section 7.2.3. For equivalence we require that models have the same output covariance properties.

The problem belongs to the field of stochastic realization theory (Kailath, 1968; Kailath & Frost, 1968) and (Akaike, 1975; Anderson & Moore, 1979; Åström, 1970; Badawi, Lindquist & Pavon, 1979; Doob, 1953; Lindquist & Picci, 1979; van der Shaft & Willems, 1984), and is of considerable importance in many areas including prediction, parameter estimation, and control. The proposed approach exploits the well-known properties of the matrix Riccati equations (Friedland, 1967; Willems, 1971; Kučera, 1973), previously used in stochastic realization theory in the context of continuous-time smoothing problems (Badawi et al., 1979).

7.4.1 Simplified model

Consider a time-invariant representation of a zero mean, second-order process $\{z_i; i \geq 0\}$

$$\boldsymbol{x}_{i+1}^* = \boldsymbol{F}\boldsymbol{x}_i^* + \boldsymbol{h}v_i \tag{7.63}$$

$$z_i = \boldsymbol{d}'\boldsymbol{x}_i^* + v_i, \tag{7.64}$$

where, at the initial instant, the state $\boldsymbol{x}_0^*$ is an n-dimensional, zero mean random vector with covariance matrix

$$\mathrm{E}[\boldsymbol{x}_0^*\boldsymbol{x}_0^{*\prime}] = \boldsymbol{Q}_0^*. \tag{7.65}$$

and the process v_i is a zero mean white noise with

$$\mathrm{E}[v_i v_j] = \sigma^2\delta_{ij} \tag{7.66}$$

for all $t, q \geq 0$. The random vector $\boldsymbol{x}_0^*$ and v_i are uncorrelated for all $i \geq 0$, and the matrix $\boldsymbol{F}$ and vector $\boldsymbol{d}$ are the same as in (7.43)-(7.44).

It is interesting to note that upon eliminating v_i from (7.63) one gets

$$\boldsymbol{x}_{i+1}^* = \boldsymbol{F}^*\boldsymbol{x}_i^* + \boldsymbol{h}z_i \tag{7.67}$$

$$z_i = \boldsymbol{d}'\boldsymbol{x}_i^* + v_i, \tag{7.68}$$

with

$$\boldsymbol{F}^* = \boldsymbol{F} - \boldsymbol{h}\boldsymbol{d}'. \tag{7.69}$$

From (7.68)-(7.69) it is seen that $\boldsymbol{x}_0^*$ is the only source of randomness in that model.

We shall show that under certain conditions, the representation in (7.63)-(7.64) is equivalent with that of (7.43)-(7.44).

7.4.2 Kalman Filter for Simplified Model

Kalman filter equations (Anderson & Moore, 1979) for system (7.63)-(7.64) or (7.67)-(7.68) have the form:

$$\hat{\boldsymbol{x}}^*_{i|i} = \hat{\boldsymbol{x}}^*_{i|i-1} + \boldsymbol{h}_i^f[y_i - \boldsymbol{d}'\hat{\boldsymbol{x}}^*_{i|i-1}] \tag{7.70}$$

$$\hat{\boldsymbol{x}}^*_{i+1|i} = \boldsymbol{F}^*\hat{\boldsymbol{x}}^*_{i|i} + \boldsymbol{h}y_i, \tag{7.71}$$

where

$$\boldsymbol{h}_i^f = \frac{\boldsymbol{S}_i\boldsymbol{d}}{\sigma^2 + \boldsymbol{d}'\boldsymbol{S}_i\boldsymbol{d}} \tag{7.72}$$

and matrix $\boldsymbol{S}_i$ satisfies

$$\boldsymbol{S}_{i+1} = \boldsymbol{F}^*\boldsymbol{S}_i\boldsymbol{F}^{*\prime} - \frac{\boldsymbol{F}^*\boldsymbol{S}_i\boldsymbol{d}\boldsymbol{d}'\boldsymbol{S}_i\boldsymbol{F}^{*\prime}}{\sigma^2 + \boldsymbol{d}'\boldsymbol{S}_i\boldsymbol{d}}, \quad \boldsymbol{S}_0 = \boldsymbol{Q}_0^*. \tag{7.73}$$

7.4.3 Innovations representation

The innovations representation for the model (7.63)-(7.64) reads

$$\hat{\boldsymbol{x}}^*_{i+1} = \boldsymbol{F}\hat{\boldsymbol{x}}^*_i + (\boldsymbol{h} + \boldsymbol{h}_i)\epsilon_i, \quad \hat{\boldsymbol{x}}^*_0 = 0 \tag{7.74}$$

$$z_i = \boldsymbol{d}'\hat{\boldsymbol{x}}^*_i + \epsilon_i, \tag{7.75}$$

where

$$\boldsymbol{h}_i = \frac{\boldsymbol{F}^*\boldsymbol{S}_i\boldsymbol{d}}{\sigma^2 + \boldsymbol{d}'\boldsymbol{S}_i\boldsymbol{d}}. \tag{7.76}$$

7.4.4 Equivalence criteria

Theorem 7.4.1. *For equivalence of the representations in (7.43)-(7.44) and (7.63)-(7.64), the following is sufficient and necessary:*

$$\boldsymbol{Q}_0^* = \boldsymbol{Q}_0 - \boldsymbol{\Sigma} \tag{7.77}$$

$$\boldsymbol{h} = \frac{\boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{d}}{\sigma^2} \tag{7.78}$$

$$\sigma^2 = \rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}\boldsymbol{d}, \tag{7.79}$$

where $\boldsymbol{\Sigma}$ is an arbitrary symmetric solution of the algebraic Riccati equation

$$\boldsymbol{\Sigma} = \boldsymbol{W} + \boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{F}' - \frac{\boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{d}\boldsymbol{d}'\boldsymbol{\Sigma}\boldsymbol{F}'}{\rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}\boldsymbol{d}}. \tag{7.80}$$

Proof. Upon subtracting (7.80) from (7.61) it is easily seen that

$$\boldsymbol{S}_{i+1} = \boldsymbol{F}\boldsymbol{S}_i\boldsymbol{F}' + \sigma^2\boldsymbol{h}\boldsymbol{h}' - \frac{(\boldsymbol{F}\boldsymbol{S}_i\boldsymbol{d} + \boldsymbol{h}\sigma^2)(\boldsymbol{F}\boldsymbol{S}_i\boldsymbol{d} + \boldsymbol{h}\sigma^2)'}{\sigma^2 + \boldsymbol{d}'\boldsymbol{S}_i\boldsymbol{d}}, \quad \boldsymbol{S}_0 = \boldsymbol{Q}_0^*. \tag{7.81}$$

which is equivalent with (7.73). Hence under conditions (7.77)-(7.79) there is

$$\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma} + \boldsymbol{S}_i. \tag{7.82}$$

Upon inserting (7.82) to (7.60) and taking (7.79) into account one gets:

$$\boldsymbol{k}_i = \frac{\boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{d} + \boldsymbol{F}\boldsymbol{S}_i\boldsymbol{d}}{\sigma^2 + \boldsymbol{d}'\boldsymbol{S}_i\boldsymbol{d}}. \tag{7.83}$$

Finally,

$$\boldsymbol{k}_i - \boldsymbol{h} = \frac{\boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{d} - \boldsymbol{h}\sigma^2 + \boldsymbol{F}^*\boldsymbol{S}_i\boldsymbol{d}}{\sigma^2 + \boldsymbol{d}'\boldsymbol{S}_i\boldsymbol{d}} \tag{7.84}$$

and (7.79), (7.76) lead to

$$\boldsymbol{k}_i = \boldsymbol{h} + \boldsymbol{h}_i, \ \mathrm{E}\,(e_i)^2 = \mathrm{E}\,(\epsilon_i)^2. \tag{7.85}$$

As a result the innovations representations (7.58)-(7.59) and (7.74)-(7.75) are equivalent. This proves sufficiency.

To prove necessity assume that the representations (7.43)-(7.44) and (7.63)-(7.64) are equivalent. Then the matrix $\boldsymbol{\Sigma} = \boldsymbol{Q}_0 - \boldsymbol{Q}_0^*$ fulfills equations (7.80), (7.78), and (7.79). This is clear by comparison of the covariance matrices

$$\mathrm{E}\,[y_iy_i] = \boldsymbol{d}'(\boldsymbol{F}^i\boldsymbol{Q}_0\boldsymbol{F}^{i\prime} + \sum_{k=0}^{i-1}\boldsymbol{F}^k\boldsymbol{W}\boldsymbol{F}^{k\prime})\boldsymbol{d} + \rho^2 \tag{7.86}$$

$$\mathrm{E}\,[y_iy_j] = \boldsymbol{d}'\boldsymbol{F}^{i-j}(\boldsymbol{F}^j\boldsymbol{Q}_0\boldsymbol{F}^{j\prime} + \sum_{k=0}^{j-1}\boldsymbol{F}^k\boldsymbol{W}\boldsymbol{F}^{k\prime})\boldsymbol{d} + \boldsymbol{d}'\boldsymbol{F}^{i-j-1}, i > j \tag{7.87}$$

$$\mathrm{E}\,[z_iz_i] = \boldsymbol{d}'(\boldsymbol{F}^i\boldsymbol{Q}_0^*\boldsymbol{F}^{i\prime} + \sigma^2\sum_{k=0}^{i-1}\boldsymbol{F}^k\boldsymbol{h}\boldsymbol{h}'\boldsymbol{F}^{k\prime})\boldsymbol{d} + \sigma^2 \tag{7.88}$$

$$\mathrm{E}\,[z_iz_j] = \boldsymbol{d}'\boldsymbol{F}^{i-j}(\boldsymbol{F}^j\boldsymbol{Q}_0^*\boldsymbol{F}^{j\prime} + \sigma^2\sum_{k=0}^{j-1}\boldsymbol{F}^k\boldsymbol{h}\boldsymbol{h}'\boldsymbol{F}^{k\prime})\boldsymbol{d} + \sigma^2\boldsymbol{d}'\boldsymbol{F}^{i-j-1}\boldsymbol{h}, i > j. \tag{7.89}$$

Comparison is performed in three steps.

(i) $i = 0, j = 0$,

$$\mathrm{E}\,[y_0y_0] - \mathrm{E}\,[z_0z_0] = \boldsymbol{d}'\boldsymbol{Q}_0\boldsymbol{d} + \rho^2 - \boldsymbol{d}'\boldsymbol{Q}_0^*\boldsymbol{d} - \sigma^2. \tag{7.90}$$

(ii) $i = 1, 2 \ldots, j = 0$,

$$\mathrm{E}\,[y_iy_0] - \mathrm{E}\,[z_iz_0] = \boldsymbol{d}'\boldsymbol{F}^i\boldsymbol{Q}_0\boldsymbol{d} - \boldsymbol{d}'\boldsymbol{F}^i\boldsymbol{Q}_0^*\boldsymbol{d} - \sigma^2\boldsymbol{d}'\boldsymbol{F}^{i-1}\boldsymbol{h}. \tag{7.91}$$

(iii) $i = 1, 2, \ldots, j = 1, 2 \ldots$

Denote $\delta Y_{ij} = \mathrm{E}[y_i y_j'] - \mathrm{E}[y_{i-1} y_{j-1}']$ and $\delta Z_{ij} = \mathrm{E}[z_i z_j'] - \mathrm{E}[z_{i-1} z_{j-1}']$. Then from (7.86)-(7.89) it results that

$$\delta Y_{ij} = \boldsymbol{d}' \boldsymbol{F}^{i-j} (\boldsymbol{F} \boldsymbol{Q}_0 \boldsymbol{F}' - \boldsymbol{Q}_0 + \boldsymbol{W})(\boldsymbol{F}^{j-1})' \boldsymbol{d},\ i \geq j \tag{7.92}$$

$$\delta Z_{ij} = \boldsymbol{d}' \boldsymbol{F}^{i-j} (\boldsymbol{F} \boldsymbol{Q}_0^* \boldsymbol{F}' - \boldsymbol{Q}_0^* + \sigma^2 \boldsymbol{h} \boldsymbol{h}')(\boldsymbol{F}^{j-1})' \boldsymbol{d},\ i \geq j. \tag{7.93}$$

From (7.90)-(7.93), taking the observability of $(\boldsymbol{F}, \boldsymbol{d})$ into account we get

$$\sigma^2 = \boldsymbol{d}'(\boldsymbol{Q}_0 - \boldsymbol{Q}_0^*)\boldsymbol{d} + \rho^2 \tag{7.94}$$

$$\boldsymbol{h} = \frac{\boldsymbol{F}(\boldsymbol{Q}_0 - \boldsymbol{Q}_0^*)}{\sigma^2} \boldsymbol{d} \tag{7.95}$$

$$\boldsymbol{F}(\boldsymbol{Q}_0 - \boldsymbol{Q}_0^*)\boldsymbol{F}' - (\boldsymbol{Q}_0 - \boldsymbol{Q}_0^*) + \boldsymbol{W} - \sigma^2 \boldsymbol{h}\boldsymbol{h}' = 0. \tag{7.96}$$

□

7.5 Relationships between Reduced Models

7.5.1 Positive definite solutions of Riccati equation

Among all symmetric solutions of Riccati equation (7.80), the positive definite one, $\boldsymbol{\Sigma}_+$, plays a fundamental role and can found by a couple of numerical algorithms. For example (Anderson & Moore, 1979), $\boldsymbol{\Sigma}_+$ can be found from the following matrix

$$\boldsymbol{\Phi} = \begin{bmatrix} (\boldsymbol{F}')^{-1} & (\boldsymbol{F}')^{-1} \boldsymbol{d}\boldsymbol{d}'/\rho^2 \\ \boldsymbol{W}(\boldsymbol{F}')^{-1} & \boldsymbol{F} + \boldsymbol{W}(\boldsymbol{F}')^{-1} \boldsymbol{d}\boldsymbol{d}'/\rho^2 \end{bmatrix} \tag{7.97}$$

using the factorization:

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda} & 0 \\ 0 & \boldsymbol{\Lambda}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{bmatrix}^{-1}, \tag{7.98}$$

where $\boldsymbol{\Lambda}$ is a Jordan matrix with $\lambda_j(\boldsymbol{\Lambda}) < 1, j = 1, 2 \ldots n$. Then

$$\boldsymbol{\Sigma}_+ = \boldsymbol{V}_{22} \boldsymbol{V}_{12}^{-1}. \tag{7.99}$$

7.5.2 Symmetrical solutions of Riccati equation

The relationship between an arbitrary symmetric solution $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_+$ is expressed in the following lemma.

Lemma 7.5.1. *Two symmetric solutions, an arbitrary $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_+ > 0$, of the Riccati equation (7.80) are related by*

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_+ + \boldsymbol{S}, \tag{7.100}$$

where $\boldsymbol{S}$ fulfills the following equation:

$$\boldsymbol{S} = \boldsymbol{F}_+^* \boldsymbol{S} \boldsymbol{F}_+^{*\prime} - \frac{\boldsymbol{F}_+^* \boldsymbol{S} \boldsymbol{d}\boldsymbol{d}' \boldsymbol{S} \boldsymbol{F}_+^{*\prime}}{\sigma_+^2 + \boldsymbol{d}' \boldsymbol{S} \boldsymbol{d}} \tag{7.101}$$

with

$$\boldsymbol{F}_+^* = \boldsymbol{F} - \boldsymbol{h}_+ \boldsymbol{d}' \tag{7.102}$$

$$\boldsymbol{h}_+ = \boldsymbol{F} \boldsymbol{\Sigma}_+ \boldsymbol{d} / \sigma_+^2 \tag{7.103}$$

$$\sigma_+^2 = \rho^2 + \boldsymbol{d}' \boldsymbol{\Sigma}_+ \boldsymbol{d}. \tag{7.104}$$

Proof. The proof is given in Appendix A.3.

Given $\boldsymbol{\Sigma}_+$, and $\boldsymbol{F}_+^*$ related with $\boldsymbol{\Sigma}_+$ by equation (7.102), the following lemma gives an analytic expression for matrix $\boldsymbol{S}$ in terms of eigenvalues of $\boldsymbol{F}_+^*$.

Lemma 7.5.2. *Assume that $\boldsymbol{T}$ is a transformation matrix such that*

$$\boldsymbol{F}_+^* = \boldsymbol{T}^{-1} \boldsymbol{\Lambda} \boldsymbol{T} \tag{7.105}$$

and

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & 0 \\ 0 & \boldsymbol{\Lambda}_2 \end{bmatrix}, \tag{7.106}$$

where $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are, respectively, $m \times m$ and $(n-m) \times (n-m)$ matrices in the Jordan form with an arbitrary $m \leq n$ and eigenvalues λ_j fulfilling the condition:

$$0 < |\lambda_j| < 1,\ j = 1, 2 \ldots n. \tag{7.107}$$

Then a matrix $\boldsymbol{S}$ of rank m, a solution of equation (7.101) related to this decomposition has the form

$$\boldsymbol{S} = \boldsymbol{T}^{-1} \boldsymbol{X} (\boldsymbol{T}^{-1})', \tag{7.108}$$

where

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.109}$$

Matrix X_1 is given by the formula

$$X_1 = -\sigma_+^2 [\sum_{j=0}^{\infty} (\Lambda_1')^j \delta_1 \delta_1' (\Lambda_1)^j]^{-1}, \tag{7.110}$$

where an m vector δ_1 results from the partition

$$(T^{-1})' d = \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}. \tag{7.111}$$

Proof. The proof is given in Appendix A.3.

Remark 7.5.1. $S = 0$ belongs to the solutions of equation (7.101).

7.5.3 Relationships between the representations

Theorem 7.5.1. *The relationship between an arbitrary representation (F^*, h, σ^2) and (F_+^*, h_+, σ_+^2) is expressed in terms of matrix S of equation (7.101) as follows*

$$\sigma^2 = \sigma_+^2 + d'Sd \tag{7.112}$$

$$h = h_+ + \frac{F_+^* Sd}{\sigma^2 + d'Sd} \tag{7.113}$$

$$F^* = F_+^* (I - \frac{Sdd'}{\sigma^2 + d'Sd}). \tag{7.114}$$

Proof. The proof is given in Appendix A.3.

Theorem 7.5.2. *Matrix F^* has m eigenvalues λ_j, $j = 1, 2 \ldots m$ equal to reciprocals of the eigenvalues of Λ_1 and $n - m$ eigenvalues equal to eigenvalues of Λ_2. Moreover*

$$\sigma^2 = (\det \Lambda_1)^2 \sigma_+^2. \tag{7.115}$$

Proof. The proof is given in Appendix A.3.

7.5.4 Invertibility

The simplified representation that are based on the positive definite Σ_+ can be written in an equivalent form

$$\bar{x}^*_{i+1} = F_+^* \bar{x}^*_i + h_+ z_i \tag{7.116}$$

$$\epsilon_i = z_i - d' \bar{x}^*_i \tag{7.117}$$

with

$$F_+^* = F - h_+ d'. \tag{7.118}$$

An important feature is that matrix F_+^* is stable:

$$|\lambda_j(F_+^*)| = |\lambda_j(\Lambda)| < 1. \tag{7.119}$$

Due to this property, given $Z_i = \{z_0, z_1, \ldots, z_i\}$ a sequence $\bar{\epsilon}_i = \{\bar{\epsilon}_0, \bar{\epsilon}_1, \ldots, \bar{\epsilon}_i\}$ is generated by the model in (7.116)-(7.117) such that $\bar{\epsilon}_i \to \epsilon_i$ as $i \to \infty$, where ϵ_i is an entry of $\epsilon_i = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_i\}$, a representation of the driving noise. This property is called invertibility.

7.5.5 Limiting innovations representation of the general representation

As time i tends to infinity then the solution Σ_i of the dynamic Riccati equation (7.61) converges to Σ_+:

$$\lim_{i \to \infty} \Sigma_i = \Sigma_+, \tag{7.120}$$

where Σ_+ is the positive definite symmetric solution of the algebraic Riccati equation (7.80), and the innovations representation in (7.58)-(7.59) takes the limiting form:

$$\bar{x}_{i+1} = F\bar{x}_i + k_+ \epsilon_i, \; \bar{x}_0 = 0 \tag{7.121}$$

$$y_i = d' \bar{x}_i + \epsilon_i \tag{7.122}$$

with

$$k_+ = \frac{F \Sigma_+ d}{\rho^2 + d' \Sigma_+ d} \tag{7.123}$$

and $\mathrm{E}(\epsilon_i^2) = \sigma_+^2 = \rho^2 + d'\Sigma_+ d$. A comparison between (7.121)-(7.122) and (7.63)-(7.64) with h calculated from (7.78) for $\Sigma = \Sigma_+$ shows that $k_+ = h_+$ and the parameters of the limiting innovations representation of the general model are the same as those of the invertible simplified model. The difference is that the simplified model in (7.63)-(7.64) is valid for all $i \geq 0$ while the model in (7.121)-(7.66) is only valid for $i \to \infty$.

7.5.6 Limiting innovations representation for the simplified model

From (7.82) and (7.120) it follows that the limiting solution S_i of the Riccati equation (7.73) fulfills:

$$\lim_{i \to \infty} S_i = \lim_{i \to \infty} \Sigma_i - \Sigma = \Sigma_+ - \Sigma. \tag{7.124}$$

As a result

$$\lim_{i \to \infty} (h + h_i) = h_+, \tag{7.125}$$

so that the limiting innovations representation becomes invertible. If h in (7.63) already equals to h_+ then $\lim_{i \to \infty} S_i = 0$ and $\lim_{i \to \infty} h_i = 0$.

7.5.7 Discussion of the results

Variances σ^2 are positive for all real symmetric solutions of (7.80). This is readily seen from equations (7.104) and (7.115). Moreover, Q_0^* fulfills the algebraic Lyapunov equation

$$Q_0^* = FQ_0^*F' + hh'\sigma^2 \tag{7.126}$$

for stationary processes and as a result $Q_0^* > 0$. For nonstationary processes the matrix Q_0^* is not guaranteed to be positive definite. Two questions arise.

(i) Does at least one simplified representation (7.63)-(7.64) having positive semidefinite matrix Q_0^* and equivalent with a full representation (7.43)-(7.44) exist?

(ii) Does it make sense to admit models with a negative semidefinite matrix Q_0^*?

As for the former, (i), the choice of the negative semidefinite solution Σ to the Riccati equation (7.80) assures positive semidefinitness of the matrix Q_0^*.

The assumption that the matrices Q_0^* are positive semidefinite is crucial in the situation when using the model (7.63)-(7.64) for the simulation of an output process. However, in the case of (ii), the models with not necessarily positive semidefinite matrices Q_0^* are also applicable. One example is the output process prediction, in which the physical interpretation for the state vector is not important.

7.6 Time Series Models

The ARMA model is commonly employed as a representation of a stationary time series. Similarly as in the continuous-time case the Representations Theorem (Åström, 1970) shows that there exists a class of covariance equivalent ARMA models having different MA parts. The relationship between reduced representations and ARMA models allows to characterize the set of all symmetric solutions to the Riccati equation (7.80) by roots of MA polynomial.

We will also show that the ARMA model with an appropriate initial condition can also describe a class of nonstationary processes. The Representation Theorem will be extended to this class of processes.

7.6.1 Relationship between simplified representations and ARMA models

Theorem 7.6.1. *The simplified representation*

$$x_{i+1}^* = Fx_i^* + hv_i \tag{7.127}$$
$$y_i = d'x_i^* + v_i \tag{7.128}$$

with $\mathrm{E}\{v_iv_j\} = \sigma^2\delta_{ij}$ *the initial condition* $x_0^* \sim \mathcal{N}(0, Q_0^*)$ *implies an ARMA model*

$$A(z)y_i = C(z)v_i \tag{7.129}$$

with the initial condition

$$\boldsymbol{y}_0 = \boldsymbol{\Omega} x_0^* + \boldsymbol{E}\boldsymbol{v}_0, \tag{7.130}$$

where the vectors $\boldsymbol{y}_0$ *and* $\boldsymbol{v}_0$ *contain first n values of output and input signals*

$$\boldsymbol{y}_0 = [y_0, y_1 \dots y_{n-1}]', \quad \boldsymbol{v}_0 = [v_0, v_1 \dots v_{n-1}]' \tag{7.131}$$

and the matrices $\boldsymbol{\Omega}$ *and* $\boldsymbol{E}$ *have the following forms*

$$\boldsymbol{\Omega} = \begin{bmatrix} d_0' \\ d_1' \\ d_2' \\ \vdots \\ d_{n-1}' \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 & . & 0 \\ e_1 & 1 & 0 & . & 0 \\ e_2 & e_1 & 1 & . & 0 \\ . & . & . & . & . \\ e_{n-1} & e_{n-2} & . & e_1 & 1 \end{bmatrix} \tag{7.132}$$

with

$$d_i = F'd_{i-1}, \quad d_0 = d, \quad e_i = d_{i-1}'h, \quad e_0 = 1. \tag{7.133}$$

Denoting

$$\boldsymbol{R} = \mathrm{E}\{\boldsymbol{y}_0\boldsymbol{y}_0'\} \tag{7.134}$$

there is

$$\boldsymbol{R} = \boldsymbol{\Omega} Q_0^* \boldsymbol{\Omega}' + \sigma^2 \boldsymbol{E}\boldsymbol{E}' \tag{7.135}$$

with

$$r_{ij} = d_i Q_0^* d_j' + \sigma^2\Big(\sum_{l=1}^{i} e_{l+i-j}e_l + e_{i-j}\Big), \; i \ge j \tag{7.136}$$

$$r_{ij} = d_j Q_0^* d_i' + \sigma^2\Big(\sum_{l=1}^{i} e_{l+j-i}e_l + e_{j-i}\Big), \; i \le j. \tag{7.137}$$

The polynomials $A(z)$ *and* $C(z)$ *are associated with matrices* F *and* F^* *by*

$$A(z) = z^n + a_1z^{n-1} + \ldots + a_n = \det(zI - F) \tag{7.138}$$
$$C(z) = z^n + c_1z^{n-1} + \ldots + c_n = \det(zI - F^*). \tag{7.139}$$

Proof. The proof follows upon straightforward calculations. □

Remark 7.6.1. Very often ARMA model is expressed in the backwards shift (lag) operator z^{-1}:

$$A(z^{-1})y_i = C(z^{-1})v_i, \tag{7.140}$$

and then

$$A(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_n z^{-n} = \det(\boldsymbol{I} - \boldsymbol{F} z^{-1}) \tag{7.141}$$

$$C(z^{-1}) = 1 + c_1 z^{-1} + \ldots + c_n z^{-n} = \det(\boldsymbol{I} - \boldsymbol{F}^* z^{-1}). \tag{7.142}$$

Remark 7.6.2. Given an ARMA model of equations (7.129) and (7.138)-(7.139) or (7.140)-(7.142) then the observer canonical state-space form can be constructed from the coefficients of the polynomials in (7.138)-(7.139) or (7.141)-(7.142) as follows:

$$\boldsymbol{F} = \begin{bmatrix} -a_1 & 1 & 0 & . & . & . & 0 \\ -a_2 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ -a_{n-1} & & & & & & 1 \\ -a_n & 0 & . & . & . & . & 0 \end{bmatrix}, \ \boldsymbol{h} = \begin{bmatrix} c_1 - a_1 \\ c_2 - a_2 \\ \vdots \\ c_n - a_n \end{bmatrix}, \ \boldsymbol{d} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{7.143}$$

7.6.2 Equivalent ARMA models

Assume that the model

$$A(z)y_i = C_+(z)v_{+i}, \ C_+(z) = \det(z\boldsymbol{I} - \boldsymbol{F}_+^*) \tag{7.144}$$

with a stable polynomial $C_+(z)$ and initial covariance matrix $\boldsymbol{R}_+$ driven by a white noise v_{+i} with the covariance σ_+^2 is given.

Theorem 7.6.2. *Denote* $\mathbb{M}$ *a class of ARMA models equivalent with (7.144). Then an ARMA model*

$$A(z)y_i = C(z)v_i \tag{7.145}$$

belongs to $\mathbb{M}$ *if and only if*

$$C(z) = \prod_{j=1}^{m}(z - \lambda_j^{-1}) \prod_{j=m+1}^{n}(z - \lambda_j) \tag{7.146}$$

$$\sigma^2 = \sigma_+^2 \prod_{j=1}^{m} \lambda_j^2 \tag{7.147}$$

$$\boldsymbol{R} = \boldsymbol{R}_+, \tag{7.148}$$

where $\lambda_j, j = 1, 2, \ldots n$ *are the roots of* $C_+(z)$ *and* $0 \le m \le n$.

Conditions (7.146)-(7.147) constitute the classical Representations Theorem (Åström, 1970), which is known to be valid in the case of stationary processes. Condition (7.148) extends the Representations Theorem to the case of arbitrary initial covariance matrix $\boldsymbol{R}$.

7.7 Remarks on Continuous-Time Process Identification

In point estimation theory, a function $\hat{\boldsymbol{\theta}}(\boldsymbol{y})$ of a random variable $\boldsymbol{y}$, whose distribution depends on an unknown parameter $\boldsymbol{\theta}$, is an unbiased estimator for $\boldsymbol{\theta}$ if its expected value satisfies

$$\mathrm{E}_\theta\{\hat{\boldsymbol{\theta}}(\boldsymbol{y})\} = \boldsymbol{\theta}, \tag{7.149}$$

where E_θ denotes expectation over the parametrized density function $p(\cdot;\boldsymbol{\theta})$ for the data. A natural measure of performance for a parameter estimator is the covariance of the estimation error, which for any unbiased estimator fulfills the following *Cramér-Rao Inequality*:

$$\mathrm{E}_\theta\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})'(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\} \ge \left\{ \mathrm{E}_\theta \left[\left(\frac{\partial \log p(\cdot;\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)' \left(\frac{\partial \log p(\cdot;\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right) \right] \right\}^{-1}. \tag{7.150}$$

A small covariance of the error is a desired property of the unbiased estimator, and it is achieved by the so called *Maximum Likelihood* estimators. Therefore ML estimators will be sought for our problem in the following.

7.7.1 Continuous-time model identification based on ARMA identification

Assuming that $\deg c(s) = m \le n - 1$ vector $\boldsymbol{\theta}$,

$$\boldsymbol{\theta} = [\alpha_1, \alpha_2 \ldots \alpha_n, \gamma_{n-m}, \ldots \gamma_n, \rho^2]' \tag{7.151}$$

of the parameters to be estimated contains $n + m + 1$ entries consisting of the coefficients of the polynomials $a(s)$, $c(s)$ and variance ρ^2 while vector $\boldsymbol{\theta}^*$,

$$\boldsymbol{\theta}^* = [a_1, a_2 \ldots a_n, c_1, c_2 \ldots c_n, \sigma^2]', \tag{7.152}$$

contains $2n + 1$ entries consisting of the coefficients of the polynomials $A(z)$, $C(z)$ and variance σ^2 of the discrete-time model (7.127)-(7.128). These two vectors are related by a nonlinear vector function $\mathcal{F}(\boldsymbol{\theta})$. Calculation of $\mathcal{F}(\boldsymbol{\theta})$ consists of:

- construction of the canonical form (7.27) for the continuous-time system (7.1)-(7.2),
- computation of $\boldsymbol{F}$ given $\boldsymbol{A}$ based on (7.39),
- computation of $\boldsymbol{W}$ using (7.48)-(7.49) or (7.51),
- determination of the simplified representation (7.127)-(7.128) based on (7.77)-(7.79),
- transformation of (7.127)-(7.128) to the canonical form (7.143).

Since methods for identification and parameter estimation of ARMA models are very well developed e.g. (Gardner, Harvey & Phillips, 1980; Hannan, 1988; Hannan & Kavalieris, 1983; Hannan & Rissanen, 1982)[2] the following idea emerged (Söderström, 1984; Söderström, 1991):

- estimate the vector θ^* of discrete-time parameters of an ARMA model,
- restore the vector θ of the parameters of the continuous-time model.

7.7.1.1 ARMA model identification. In order to make the Kalman filter equations for the model (7.127)-(7.128) independent of the variance σ^2 denote $\operatorname{cov}(x^*) = Q^*\sigma^2$, where Q^* is a solution of the algebraic Lyapunov equation:

$$Q^* = FQ^*F' + hh'. \tag{7.153}$$

Then the one step predictor is determined by

$$\hat{x}^*_{i+1|i} = F\hat{x}^*_{i|i-1} + (h + k_i)\epsilon_i, \quad \hat{x}^*_{0|-1} = 0 \tag{7.154}$$

$$k_i = \frac{F^* S^*_{i|i-1} d}{1 + d' S^*_{i|i-1} d} \tag{7.155}$$

$$S^*_{i+1|i} = F^*\left(S^*_{i|i-1} - \frac{S^*_{i|i-1} dd' S^*_{i|i-1}}{1 + d' S^*_{i|i-1} d}\right)F^{*\prime}, \quad S^*_{0|-1} = Q^*, \tag{7.156}$$

where $S^*_{i|i-1} = \operatorname{cov}(\tilde{x}^*_i)/\sigma^2$ and $\tilde{x}^*_i = x^*_i - \hat{x}^*_{i|i-1}$ and the variance σ_i^2 of the one step output prediction $\epsilon_i = y_i - d'\hat{x}^*_{i|i-1}$ is

$$\sigma_i^2 = \sigma^2\phi_i, \quad \phi_i = 1 + d' S^*_{i|i-1} d. \tag{7.157}$$

Denote $y = \{y_i, i = 1 \ldots N\}$ a sample of the proces. Since $\{\epsilon_i, i = 1, 2, \ldots N\}$ is a series of independent normally distributed random numbers with zero mean and variance σ_i, the likelihood function has the following form:

$$L^*(y, \theta^*) = (2\pi)^{-\frac{N}{2}}\left(\prod_{i=1}^{N}\sigma_i^2\right)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\frac{\epsilon_i^2}{\sigma_i^2}\right\}. \tag{7.158}$$

In practical calculations instead of maximization of (7.158) a function $l^*(y, \theta^*) = -\ln L^*(y, \theta^*)$ is minimized. Upon neglecting a constant the function to be minimized is expressed as follows:

$$l^*(y, \theta^*) = \frac{1}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + \frac{1}{2}\sum_{i=1}^{N}\frac{\epsilon_i^2}{\sigma_i^2}. \tag{7.159}$$

[2] An extensive survey can be found in Blachuta (1996e)

Taking (7.157) into account this gives:

$$l^*(y, \theta^*) = \frac{N}{2}\ln\sigma^2 + \frac{1}{2}\sum_{i=1}^{N}\ln\phi_i + \frac{1}{2\sigma^2}\sum_{i=1}^{N}\epsilon_i^2/\phi_i. \tag{7.160}$$

Calculating $(\partial/\partial\sigma^2)l^*(y, \theta^*)$ and equating the result to zero yields:

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}\epsilon_i^2/\phi_i. \tag{7.161}$$

Minimization of the function in (7.160) with respect to the remaining variables is equivalent to the minimization of the function:

$$l_0^*(y, \theta^*) = \frac{1}{N}\left(\prod_{i=1}^{N}\phi_i\right)^{\frac{1}{N}}\sum_{i=1}^{N}\epsilon_i^2/\phi_i, \tag{7.162}$$

where ϕ_i are determined by (7.157).

The function $l_0^*(y, \theta^*)$ can be minimized using the gradient and hessian calculated analytically or numerically. The estimated model need not necessarily be invertible, but in the case of invertible models $\phi_i \to 1$, and as a result starting from certain value of i for which $\phi_i < 1 + \epsilon$, where $\epsilon > 0$ is small enough, the innovations can be calculated directly from the polynomial ARMA model with $\phi_i = 1$ in (7.162). Switching to fast recursions greatly accelerates computations.

A disadvantage of that approach when used to sampled continuous-time systems is that due to the over-parametrization of θ^* the covariance of the estimator errors $\operatorname{cov}(\hat{\theta}^* - \theta^*)$ can be quite large.

7.7.1.2 Restoring continuous-time parameters. Restoring continuous-time parameters means calculating $\theta = \mathcal{F}^{-1}(\theta^*)$, or its approximation if for given θ^* the reciprocal function does not exist.

The part of the reciprocal procedure which consist in calculation of

$$A = h^{-1}\ln F \tag{7.163}$$

is simple and relies on the relationship between the zeros π_i of the polynomial $a(s)$ and the zeros z_i of the polynomial $A(z)$:

$$\pi_i = h^{-1}\ln z_i, \quad (i = 1, 2 \ldots n). \tag{7.164}$$

If z_i is real and positive then there is no problem. If it is negative real then the problem has no solution. If z_i is complex the solution is not unique and then a solution with the minimum imaginary part can be proposed. A survey of numerical algorithms to compute (7.163) can be found in (Sinha & Rao, 1991).

More severe problems are faced when estimating the coefficients of the polynomial $c(s)$ and the value of $\hat{\rho}^2$. The simplest way (Söderström, 1984; Söderström, 1991) consists in finding $\boldsymbol{\theta}$ from:

$$\hat{\boldsymbol{\theta}} = \arg\min\{[\hat{\boldsymbol{\theta}}^* - \mathcal{F}(\boldsymbol{\theta})]'\boldsymbol{W}[\hat{\boldsymbol{\theta}}^* - \mathcal{F}(\boldsymbol{\theta})]\} \tag{7.165}$$

for certain $\boldsymbol{W} > 0$. A survey of algorithms to calculate $\boldsymbol{\theta} = \mathcal{F}^{-1}(\boldsymbol{\theta}^*)$, valid for the case $\rho^2 = 0$, is given in Söderström (1984) and Söderström (1991). Unfortunately, when

$$||\hat{\boldsymbol{\theta}}^* - \mathcal{F}(\hat{\boldsymbol{\theta}})||_W \neq 0, \tag{7.166}$$

then no continuous-time counterpart exists. Since (7.165) is fulfilled with probability 1, the algorithm in (7.165) is then the only choice. Moreover, due to bad quality of $\hat{\boldsymbol{\theta}}^*$ estimator, $\operatorname{cov}(\hat{\boldsymbol{\theta}}^* - \boldsymbol{\theta}^*)$ is greater than that predicted by the Cramér-Rao bound in (7.150). This can be avoided by a direct continuous-time estimation.

7.7.2 Direct continuous-time model identification

Innovations process is described by the following set of recursive equations:

$$e_i = y_i - \boldsymbol{d}'\boldsymbol{x}_{i|i-1} \tag{7.167}$$

$$\hat{\boldsymbol{x}}_{i+1|i} = \boldsymbol{F}\hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{k}_i e_i,\ \hat{\boldsymbol{x}}_{0|-1} = 0 \tag{7.168}$$

$$\boldsymbol{h}_i = \frac{\boldsymbol{F}\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}}{\sigma_i^2} \tag{7.169}$$

$$\boldsymbol{\Sigma}_{i+1|i} = \boldsymbol{W} + \boldsymbol{F}(\boldsymbol{\Sigma}_{i|i-1} - \frac{\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}\boldsymbol{d}'\boldsymbol{\Sigma}_{i|i-1}}{\sigma_i^2})\boldsymbol{F}',\ \boldsymbol{\Sigma}_{0|-1} = \boldsymbol{Q} \tag{7.170}$$

$$\sigma_i^2 = \rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}. \tag{7.171}$$

Denote $\boldsymbol{y} = \{y_i, i = 1\ldots N\}$ a realization of the process (7.43)-(7.44). Since $\{e_i, i = 1, 2, \ldots N\}$ is a series of independent normally distributed random numbers with zero mean and variance σ_i^2, the likelihood function has the following form:

$$L(\boldsymbol{y}, \boldsymbol{\theta}) = (2\pi)^{-\frac{N}{2}}(\prod_{i=1}^{N}\sigma_i^2)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\frac{e_i^2}{\sigma_i^2}\right\} \tag{7.172}$$

Simple computation of the variance ρ^2 similarly as in (7.161) is not possible, and ρ^2 should be treated the same way as the remaining parameters when minimizing the function $l(\boldsymbol{y}, \boldsymbol{\theta}) = -\ln L(\boldsymbol{y}, \boldsymbol{\theta})$. An idea that takes advantage of well developed algorithms of ARMA identification and makes an effective alternative to direct numerical gradient computation of $L(\boldsymbol{y}, \theta)$ is as follows. Observe that

$$\boldsymbol{\theta} = \arg\min L^*[\boldsymbol{y}, \mathcal{F}(\boldsymbol{\theta})]. \tag{7.173}$$

Then the gradient of the function $L^*[\boldsymbol{y}, \mathcal{F}(\boldsymbol{\theta})]$ can be calculated as

$$\nabla_{\boldsymbol{\theta}} L^*[\boldsymbol{y}, \mathcal{F}(\boldsymbol{\theta})] = \frac{\partial \mathcal{F}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\nabla_{\boldsymbol{\theta}^*} L^*[\boldsymbol{y}, \boldsymbol{\theta}^*]. \tag{7.174}$$

The second factor can be found analytically using algorithms from (Kohn & Ansley, 1982; Burshtein, 1993). No analytic formulae are yet available to calculate the first factor but it can be computed numerically. In the case of a large dimension of $\boldsymbol{y}$ the above procedure is quite effective. However, it can face severe problems illustrated in an example.

Example 7.7.1. Consider a simple scalar system:

$$x_{i+1} = x_i, \quad x_0 = 0 \tag{7.175}$$

$$y_i = x_i + r_i \tag{7.176}$$

$$Er_i^2 = \rho^2 = \lambda. \tag{7.177}$$

The function $f(\lambda) = l(\boldsymbol{y}, \lambda)$ to be minimized is:

$$f(\lambda) = \frac{1}{2}N\ln\lambda + \frac{1}{2\lambda}\sum_{i=1}^{N}y_i^2 = \frac{N}{2}[\ln\lambda + \frac{r}{\lambda}], \tag{7.178}$$

where

$$r = \frac{1}{N}\sum_{i=1}^{N}y_i^2. \tag{7.179}$$

There is

$$\frac{\partial f(\lambda)}{\partial\lambda} = \frac{N}{2\lambda^2}[\lambda - r] \tag{7.180}$$

$$\frac{\partial^2 f(\lambda)}{\partial\lambda^2} = \frac{N}{2\lambda^3}[2r - \lambda]. \tag{7.181}$$

From equation (7.179) it is obvious that $\hat{\lambda} = \hat{\rho}^2 = r$. However, when applying Newton-Raphson iterations

$$\lambda_{k+1} = \lambda_k - [\frac{\partial^2 f(\lambda)}{\partial\lambda^2}]^{-1}\frac{\partial f(\lambda)}{\partial\lambda}|_{\lambda=\lambda_k}, \tag{7.182}$$

one gets:

$$\lambda_{k+1} = \lambda_k\frac{3r - 2\lambda_k}{2r - \lambda_k}. \tag{7.183}$$

Equation (7.183) has an equilibrium at $\lambda_k = r$. Introduce a new variable $\gamma_k = \lambda_k - r$ so that the equilibrium moves to $\gamma_k = 0$ and equation (7.183) becomes:

$$\gamma_{k+1} = \frac{2\gamma_k^2}{\gamma_k - r}. \tag{7.184}$$

From (7.184), a sufficient convergence condition $|\gamma_{k+1}| < |\gamma_k|$ is equivalent to:

$$\left|\frac{2\gamma_k}{r-\gamma_k}\right| < 1 \tag{7.185}$$

or

$$0 < \lambda_k < \frac{4}{3}r. \tag{7.186}$$

From (7.183) it results that for small values of λ_k convergence is very slow unless $\lambda_k \approx r$ when equation (7.184) becomes $\gamma_{k+1} \approx 0$, and that for λ_k only slightly excceeding r the iterations become divergent.

To summarize, the above method can efficiently be applied in three situations:

- if the measurements are accurate, $r_i = 0$ for all i,
- if the variance ρ^2 of the measurement noise is known,
- if there are good preliminary estimates of $\hat{\rho}^2$.

Assuming that sampling is fast and large sample is available problems with estimation of ρ^2 can be circumvented by applying an approximate procedure proposed by Maine & Iliff (1981). The procedure is based on an asymptotic Kalman predictor

$$\hat{x}_{i+1|i} = F\hat{x}_{i|i-1} + he_i, \ \hat{x}_{0|-1} = 0 \tag{7.187}$$

$$y_i = d'\hat{x}_{i|i-1} + e_i, \tag{7.188}$$

where

$$h = \frac{FSd}{\sigma^2}. \tag{7.189}$$

and S is a symmetric positive definite solution of

$$S = W + F(S - \frac{Sdd'S}{\sigma^2})F'. \tag{7.190}$$

Treating e_i as innovations with a time invariant variance σ^2, which is asymptotically justified for the invertible system (7.187)-(7.188), the likelihood function reduces to

$$L(y, \theta) = (\frac{2\sigma^2}{\pi})^{-\frac{N}{2}} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} e_i^2\right\}, \tag{7.191}$$

and the negative logarithm reads

$$l(y, \theta) = \frac{1}{2\sigma^2}\sum_{i=1}^{N} e_i^2 + \frac{N}{2}\ln\sigma^2. \tag{7.192}$$

Since h depends on σ^2, $e_i = e_i(\sigma^2)$ and calculating the gradient with respect to σ^2 yields:

$$\frac{2\sigma^4}{N}\frac{\partial}{\partial\sigma^2}l(y, \theta) = \sigma^2 - \frac{1}{N}\sum_{i=1}^{N} e_i^2 + \frac{2\sigma^2}{N}\left\{\sum_{i=1}^{N} e_i(\frac{\partial}{\partial\sigma^2}d'\hat{x}_{i|i-1})\right\}. \tag{7.193}$$

The last term is a zero mean random variable whose variance tends to zero as $N \to \infty$, so that for N large one can use the asymptotic result:

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N} e_i^2 \tag{7.194}$$

and then

$$\hat{\rho}^2 = \hat{\sigma}^2 - d'Sd. \tag{7.195}$$

To retain the parameters of A and c, S is now found from the continuous-time Riccati equation:

$$AS + SA' - \frac{Sdd'S}{\sigma^2} + cc' = 0. \tag{7.196}$$

Equation (7.196) is arrived at upon inserting the approximations:

$$F^{-1} \cong I - Ah, \ W \cong hcc' \tag{7.197}$$

into

$$F^{-1}SF'^{-1} = F^{-1}WF'^{-1} + S - \frac{Sdd'S}{\sigma^2}, \tag{7.198}$$

which is equivalent to (7.190). A great advantage of this formulation is the simplicity of (7.194)-(7.196).

For the algorithm bases on approximations, the results can be used as initial estimates for the exact ML algorithm based on (7.172) and/or (7.174).

7.8 Conclusions

A class of second-order continuous-time stochastic processes was investigated and the issue of their sampling was discussed. As a result of sampling discrete second-order random processes, described by linear time-invariant state-space models with a vector input were obtained. Furthermore, a set of simple representations (7.63)-(7.64) covariance equivalent with the model (7.43)-(7.44) was proposed. They rely on two sources of randomness. The first is a scalar noise v_i, and the second is the n-dimensional initial random vector x_0^*. These representations are distinct from the innovations representation of (7.43)-(7.44). However, they are time invariant, which is an advantage when using them in simulation, prediction, and parameter estimation.

As far as simulation is concerned, it surprisingly appears that the representation based on the negative semidefinite solution, Σ_-, of the algebraic Riccati equation (7.80) is the most important one. On the other hand, the representation which is related to the positive semidefinite solution, Σ_+, is most convenient for the output process prediction. In this case, as time i approaches infinity, the sequence S_i in (7.73) converges to the zero matrix. This leads to the simple structure of the formulas, where a separation into asymptotic and transient terms is attained.

Relationships between the simplified representations and ARMA models were discussed and the Representations Theorem was extended to a class of nonstationary stochastic processes.

The identification issue for continuous-time stochastic processes based on sampled data was discussed, and several methods, with the stress put on Maximum Likelihood estimators, were outlined.

The simplified representation was found useful also for parameter estimation of continuous-time processes.

8. State-Space Approach to Predictive Control

A clear and unified approach to the MV, LQG and GPC control problems based on the input-output and state-space representations of Box-Jenkins models will be presented. [1] Its two main advantages are: an integral action of the controller attained with a realistic stationary model of the disturbance, and a reduction of the computational complexity. Moreover, it will be shown that employing Chandrasekhar equations can improve the computational efficiency for receding-horizon control problems as compared to the use of Riccati equations. The above savings are particularly important for systems with a large value of the delay/sampling period ratio, and for high-order step-response models. The approach has also been shown to be an efficient design method for the optimal infinite horizon control systems.

8.1 Introduction

A quarter of century has passed since Åström (1970) started a direction of control theory based on the input-output description of discrete-time systems working under stochastic disturbances; the aim being design of controllers which optimize a receding horizon quadratic performance index. One of the best known and widely used algorithms of this class is the so called Generalized Predictive Control (GPC) (Clarke, Kanjilal & Mohtadi, 1985; Clarke, Mohtadi & Tuffs, 1987; Clarke & Mohtadi, 1989). In spite of its limitations (Grimble, 1992),this approach attracted an immense attention of control practitioners just from the beginning, which resulted in a large number of papers presenting further theoretical development and applications.

The LQG state-space theory was being developed independently, e.g. (Kwakernaak & Sivan, 1972), until Caines (1972) published a relevant paper investigating relationships between the Åström and the Kalman controllers for the ARMAX system. Then the state-space approach to this class of systems was practically abandoned until Lam (1982) published a paper dealing extensively with state-space design methods. This direction

[1] The chapter is based on (Błachuta, 1996*f*; Błachuta, 1996*h*; Błachuta, 1996*i*; Błachuta, 1998*b*) and (Błachuta, 1999*b*).

was then continued by Clarke et al. (1985). Both Lam and Clarke et al. claimed that the optimal filter is not the one which is used in the optimal control algorithm. Warwick (1987) resolved the dilemma by defining a state-space model of the process different than the commonly used 'innovations representation'. This point of view was then reinforced in (Warwick, 1990; Warwick, 1992) and (Warwick & Peterka, 1991). Błachuta (1987) identified the source of problems in Lam's papers and showed that what Lam proposed was in fact a non optimal filter followed by a non optimal feedback producing the asymptotically optimal controller by cancellation of two successive errors. The point was that a control law in the form of a linear feedback from the state estimate,

$$u_i = -k'\hat{x}_{i|i}, \tag{8.1}$$

was assumed by Lam, Clarke et al. and Warwick to be optimal under a zero set-point, which unfortunately is not the case when the system and measurement noises are correlated, (Uchida & Shimemura, 1976). State-space solutions to the control problem can be obtained by using either the one shot OLF (Open Loop Feedback) or Riccati equation based CL (Closed Loop) approach. An excellent book of Bitmead, Gevers & Wertz (1990) and a paper of Kwon & Byun (1989) treated the GPC and LQG problems in the state-space framework using the CL approach. Unfortunately, although attempt was made to force a coordinate-free theory, the system model used in (Bitmead et al., 1990) and (Kwon & Byun, 1989) for observer and controller synthesis is different than any state-space representation of the ARMAX model and the links between state-space and input-output approaches remain not always clear. A state space approach to GPC control of a state-space ARMA model similar to that used in this chapter was independently presented in (Matko, 1990) and (Kwon, Lee & Noh, 1992*b*) assuming a time-invariant filter. The above links are explained in more detail in the paper of Krauss, Daß & Bünte (1994) with extensions to multivariable processes in (Krauss, 1996; Krauss & Rake, 1994; Krauss & Rake, 1995), where relationships between the state-space model, the Kalman filter gain, the optimal predictor polynomial and the input-output model have been found after solving the control problem, and in the paper of Gambier & Unbehauen (1993), which unfortunately bases on a control algorithm of the form (8.1).

A common feature of (Clarke et al., 1985; Clarke et al., 1987; Clarke & Mohtadi, 1989) and (Bitmead et al., 1990) is a generically one-step delay model of the system to be controlled in which a delay $k > 1$ is accounted for by equating to zero several coefficients of certain polynomial. A peculiarity of this model is that to arrive at a controller which minimizes a single-stage cost, k iterations of the Riccati equation are necessary. There is, however, another phenomenon (noticed e.g. in (Clarke et al., 1985)) which demonstrates as a singularity of the associated Riccati equation if the control costing is zero and $k > 1$. It has been shown by Błachuta (1987) that the above is caused by assuming an implicit delay model and can be avoided by a simple reformulation of the system model and

performance index. A series of papers following and extending the lines of Caines published by Błachuta & Ordys (1984), Błachuta & Ordys (1987), Błachuta & Ordys (1988) and summarized in (Ordys, 1989), gave deeper insight into the relationships between optimal and asymptotically optimal regulators for one-stage performance indices. This chapter generalizes that approach to the LQG and GPC control problems. An extension of the earlier works of Błachuta and Ordys to a state-space description of the GPC can also be found in (Ordys & Clarke, 1993). Unfortunately, the above paper suffers from the implicit delay singularities, which preclude the possibility of studying some of those aspects of LQG and GPC that are discussed here. The chapter is organized as follows. First, a brief summary of the GPC in the input-output framework is given. Then the optimal filtration and prediction problems are discussed in connection with a state-space representation of an ARMAX model and their links with input-output predictors are presented. Next two types of solutions, the OLF solution and the CL one, to the control problem are presented with the stress being put on the solution that bases on Riccati equation. Finally, links between the state-space and polynomial approaches are highlighted and a comparison of advantages and disadvantages of various solutions is given.

8.2 Summary of the Polynomial Approach to GPC

For the sake of brevity here only a regulator problem with a zero set point is addressed. The problem considered is a slightly modified version of that of (Clarke et al., 1987), but extensions to the full problem with a nonzero reference signal are then easy to obtain.

8.2.1 Classical problem statement

The problem is as follows. For each time instant i find a sequence of control increments $\Delta u_i \ldots \Delta u_{i+N_u}$, with $\Delta u_j = 0$ for $j > N_u$, minimizing the following receding-horizon performance index:

$$I_i = E\{J_i\} = E\{\sum_{j=N_1}^{i+N_2-1} y_{i+j}^2 + \lambda \sum_{j=1}^{i+N_u-1} (\Delta u_{i+j-1})^2\} \tag{8.2}$$

for the ARIMAX plant:

$$\Delta\bar{A}(z^{-1})y_i = \bar{B}(z^{-1})\Delta u_{i-1} + \bar{C}(z^{-1})v_i, \tag{8.3}$$

where N_1, N_2 and N_u are certain integers; z^{-1} is a one step delay operator, $\Delta = 1 - z^{-1}$, and $\bar{A}(z^{-1})$, $\bar{B}(z^{-1})$ and $\bar{C}(z^{-1})$ are polynomials. Only Δu_i is applied at instant i and the whole procedure is then repeated as i increases. In this way, an integral action in the control loop is guaranteed but unfortunately, the implicit disturbance model becomes nonstationary:

$$\tilde{A}(z^{-1})y_i = \tilde{B}(z^{-1})u_{i-k} + \frac{\tilde{C}(z^{-1})}{\Delta}v_i. \tag{8.4}$$

Note that realistic disturbances are usually modelled by stationary stochastic processes characterized by their spectral density or correlation function. Therefore another methods of offset rejection are considered further.

8.2.2 Basic problem

The system model is assumed to fulfil the following ARMAX equation:

$$\tilde{A}(z^{-1})y_i = \tilde{B}(z^{-1})u_{i-k} + \tilde{C}(z^{-1})v_i, \tag{8.5}$$

where $A(z) = z^n\tilde{A}(z^{-1})$, $C(z) = z^n\tilde{C}(z^{-1})$, and $B(z) = z^m\tilde{B}(z^{-1})$ are monic polynomials; $k = n - m > 0$, and v_i is a normally distributed independent stochastic variable with $\mathrm{E}\{v_i\} = 0$, $\mathrm{E}\{v_iv_j\} = \delta_{ij}\sigma^2$. The value of the discrete-time delay k belongs to the model specification and, whether correct or not, is known at the design stage. The control objective is to minimize a moving-horizon performance index

$$I_i = \mathrm{E}\{J_i\}, \quad J_i = \sum_{j=i}^{i+N-1} y_{j+k}^2 + \lambda \sum_{j=i}^{i+N_u-1} u_j^2, \tag{8.6}$$

where $N_u \le N$ and additionally it is assumed that

$$u_j = 0 \text{ for } j > N_u. \tag{8.7}$$

Here N is called a cost horizon, N_u a control horizon and the index k represents a delay in the control path. The optimal control problem consists in finding an N_u vector $\boldsymbol{u}_i$ of current and future controls $\boldsymbol{u}_i = [u_i, u_{i+1}, \dots, u_{i+N_u-1}]'$ which, given information contained in the vector $\vec{\boldsymbol{y}}_i = [y_0, \dots, y_i, u_0, \dots, u_{i-1}]$, minimizes I_i, i.e.:

$$\boldsymbol{u}_i = \varphi_i(\vec{\boldsymbol{y}}_i) = \arg\min I_i|\vec{\boldsymbol{y}}_i. \tag{8.8}$$

The above problem statement is flexible enough to cover both LQG, with finite receding, one step (GMV) or infinite horizon, MPC problem (e.g. (Lee, Morari & Garcia, 1994)) as well as GPC control problems, with the distinguishing attribute of GPC being $N_u < N$ in contrast to LQG where $N_u = N$.

Predictions $\hat{y}_{i+j|i}$ of the output signal based on the information vector $\vec{\boldsymbol{y}}_i$ play the crucial role in the solution to the GPC problem. They are calculated from the ARMAX model as follows:

$$\hat{y}_{i+j|i} = \tilde{G}_j(z^{-1})u_{i+j-k} + \frac{\tilde{\Gamma}_j(z^{-1})}{\tilde{C}(z^{-1})}u_{i-k} + \frac{\tilde{F}_j(z^{-1})}{\tilde{C}(z^{-1})}y_i. \tag{8.9}$$

For the predictor (8.9) to be stable the ARMAX model in (8.5) must be invertible. The polynomials $\tilde{E}_j(z^{-1})$, $\tilde{F}_j(z^{-1})$, $\tilde{G}_j(z^{-1})$ and $\tilde{\Gamma}_j(z^{-1})$ fulfil the following Diophantine equations:

$$\tilde{C} = \tilde{E}_j\tilde{A} + z^{-j}\tilde{F}_j \tag{8.10}$$
$$\tilde{E}_j\tilde{B} = \tilde{G}_j\tilde{C} + z^{-j}\tilde{\Gamma}_j. \tag{8.11}$$

It is also well known that the coefficients e_i and g_i of the polynomials $\tilde{E}_j(z^{-1})$ and $\tilde{G}_j(z^{-1})$,

$$\tilde{E}_j(z^{-1}) = \sum_{l=0}^{j-1} e_l z^{-l}, \; \tilde{G}_j(z^{-1}) = \sum_{l=0}^{j-k} g_{k+l}z^{-l} \tag{8.12}$$

are the Markov parameters of system (8.5). Denoting $\hat{\boldsymbol{y}}_i = \left[\hat{y}_{i+1|i}, \hat{y}_{i+2|i}, \dots, \hat{y}_{i+N|i}\right]'$ then from (8.9) we have

$$\hat{\boldsymbol{y}}_i = \boldsymbol{G}\boldsymbol{u}_{i-k+1} + \boldsymbol{f}_i, \tag{8.13}$$

where the $N \times N_u$ matrix $\boldsymbol{G}$ and the N vector $\boldsymbol{f}_i$ of the free response predictions result from equation (8.9). The solution to the problem requires that the following system of linear algebraic equations is solved:

$$(\boldsymbol{G}'\boldsymbol{G} + \lambda\boldsymbol{I})\boldsymbol{u}_i = -\boldsymbol{G}'\boldsymbol{f}_{i+k-1}. \tag{8.14}$$

For numerical reasons, the value of the control horizon N_u should not be too large within this approach. A recursive solution of a system of equations similar to (8.14) can be found in (Kwon, Choi, Byun & Noh, 1992*a*).

8.3 State-Space Models

8.3.1 ARMAX and ARIMAX models in state space

It is assumed that the ARMAX system (8.5) is described in state space by a model which is an extension of the simplified representation (7.127)-(7.128) to contain the control input:

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{g}u_i + \boldsymbol{h}v_i \tag{8.15}$$
$$y_i = \boldsymbol{d}'\boldsymbol{x}_i + v_i \tag{8.16}$$

in which $\boldsymbol{F}$ is an $n \times n$ matrix; $\boldsymbol{g}$, $\boldsymbol{h}$ and $\boldsymbol{d}$ are n-vectors, and the initial condition, $\boldsymbol{x}_0$, is a normal random vector with $\mathrm{E}\{\boldsymbol{x}_0\} = \boldsymbol{m}_0$, $\mathrm{cov}\{(\boldsymbol{x}_0 - \boldsymbol{m}_0)\} = \sigma^2\boldsymbol{Q}_0$, being independent from the disturbances, i.e. $\mathrm{E}(\boldsymbol{x}_0v_i) = 0, i = 0, 1, \dots$

Introduce vectors

$$d_0 = d,\ d_j = F'd_{j-1},\ j = 1, 2, \ldots \tag{8.17}$$

Then the Markov parameters g_j, e_j for $j = 0, 1, \ldots$ can be expressed as:

$$g_0 = 0,\ g_j = d'_{j-1}g = d'F^{j-1}g,\ j > 0 \tag{8.18}$$

$$e_0 = 1,\ e_j = d'_{j-1}h = d'F^{j-1}h,\ j > 0 \tag{8.19}$$

and for k-step time delay in the control channel we have

$$g_0 = 0,\ g_1 = 0, \ldots, g_{k-1} = 0,\ g_k = b_0 \neq 0. \tag{8.20}$$

For the stochastic process to be stationary the subsystem controllable from v_i must be stable and $Q_0 \geq 0$ should fulfil the following discrete algebraic Lyapunov equation:

$$Q_0 = FQ_0F' + hh'. \tag{8.21}$$

Usually a different problem statement can be found in the literature, e.g. (Clarke et al., 1985; Clarke et al., 1987; Clarke & Mohtadi, 1989; Ordys, 1993), where a problem is defined using increments of the control signal $\Delta u_i = u_i - u_{i-1}$ rather than the actual control input u_i:

$$x^{\Delta}_{i+1} = F^{\Delta}x^{\Delta}_i + g^{\Delta}\Delta u_i + h^{\Delta}v_i \tag{8.22}$$

$$y_i = d^{*\prime}x^{\Delta}_i + v_i \tag{8.23}$$

and the matrix F^{Δ} necessarily has at least one eigenvalue $\lambda(F^{\Delta}) = 1$. The above model is a state-space model of the ARIMAX model in (8.4). It is not any more stationary, and equation (8.21) does not make sense for this model.

8.3.2 Output prediction

The optimal j-step ahead predictor $\hat{y}_{i+j|i}$, which is based on information contained in $\vec{y}_i$, is defined as a conditional mean $\hat{y}_{i+j|i} = \mathrm{E}\,(y_{i+j}|\vec{y}_i)$, for $j = 1, 2, \ldots$ and can be expressed in terms of $\hat{x}_{i|i} = E(x_i|\vec{y}_i)$, supplied by a Kalman filter, as follows:

$$\hat{y}_{i+j|i} = \hat{y}^u_{i+j|i} + \hat{y}^f_{i+j|i} = \sum_{l=0}^{j-k} g_{j-l}u_{i+l} + d'_{j-1}(F^*\hat{x}_{i|i} + hy_i). \tag{8.24}$$

A variant of (8.24) for $j = k$ can be found in (Błachuta & Ordys, 1984; Błachuta & Ordys, 1987). From (8.24), the vector of predictions $\hat{\boldsymbol{y}}_i = \left[\hat{y}_{i+1|i}, \hat{y}_{i+2|i}, \ldots, \hat{y}_{i+N|i}\right]'$ can be collected as:

$$\hat{\boldsymbol{y}}_i = Gu_{i-k+1} + f_i, \tag{8.25}$$

where the $N \times Nu$ matrix G is given as $G(i,j) = g_{i-j+k}$.

From (8.24) it follows that the free predictions, $\hat{y}^f_{i+j|i}$, fulfil the difference equation

$$\hat{y}^f_{i+j|i} + a_1\hat{y}^f_{i+j-1|i} + \ldots + a_n\hat{y}^f_{i+j-n|i} = 0, \tag{8.26}$$

for $j > n$, which after initialization by the first n free predictions from (8.24) can be used e.g. for fast calculation of the entries of the vector f_i in (8.14).

8.3.3 State filtration and prediction

On substituting v_i from equation (8.16) to equation (8.15) one gets

$$x_{i+1} = F^*x_i + gu_i + hy_i \tag{8.27}$$

$$y_i = d'x_i + v_i \tag{8.28}$$

with $F^* = F - hd'$. It is readily seen that given measurements $\vec{y}_i$, an unknown initial vector x_0 with the mean m_0 and the covariance matrix $\sigma^2 Q_0$ is the only source of uncertainty in equation (8.27). According to Anderson & Moore (1979) Kalman filter equations for system (8.27)-(8.28) can be written in the form:

(i) (measurement update)

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + k^f_i[y_i - d'\hat{x}_{i|i-1}],\ \hat{x}_{0|-1} = m_0 \tag{8.29}$$

(ii) (prediction)

$$\hat{x}_{i+1|i} = F^*\hat{x}_{i|i} + gu_i + hy_i \tag{8.30}$$

with the Kalman filter gain, k^f_i, given in the formula

$$k^f_i = \frac{\Sigma_i d}{1 + d'\Sigma_i d}, \tag{8.31}$$

where

$$\Sigma_i = \mathrm{cov}\,(\tilde{x}_{i|i-1})/\sigma^2 \tag{8.32}$$

is a scaled covariance matrix of the one-step predictor error $\tilde{x}_{i|i-1} = x_i - \hat{x}_{i|i-1}$. Σ_i is determined by a recursive Riccati equation

$$\Sigma_{i+1} = F^*(\Sigma_i - \frac{\Sigma_i dd'\Sigma_i}{1 + d'\Sigma_i d})F^{*\prime},\ \Sigma_0 = Q_0. \tag{8.33}$$

Inserting $\hat{x}_{i|i}$ calculated from (8.29) to (8.30) yields the following equations of a one-step state predictor:

$$\hat{x}_{i+1|i} = F\hat{x}_{i|i-1} + gu_i + h_i\epsilon_i,\ \hat{x}_{0|-1} = m_0, \tag{8.34}$$

where

$$h_i = h + k_i^p \tag{8.35}$$

and the innovations $\epsilon_i = y_i - d'\hat{x}_{i|i-1}$ constitute a zero mean white noise process with

$$\operatorname{var}(\epsilon_i^2) = \sigma_i^2 = (1 + d'\Sigma_i d)\sigma^2. \tag{8.36}$$

Vector k_i^p results from the formula

$$k_i^p = Fk_i^f = \frac{F\Sigma_i d}{1 + d'\Sigma_i d}. \tag{8.37}$$

Certain authors, e.g. Meditch, (1969), prefer writing equations (8.29)-(8.30) in the reversed order, and/or the Riccati equation for $S_i = \operatorname{cov}(\tilde{x}_{i|i})$, where $\tilde{x}_{i|i} = x_i - \hat{x}_{i|i}$, instead of Σ_i (Ordys, 1993). The main differences are complicated forms of expressions for initial values $\hat{x}_{0|0}$ and S_0 as opposed to initial values $\hat{x}_{0|-1}$ and Σ_0. Thus, there is no point in using equations of Meditch and equations (8.29)-(8.33) should be preferred.

8.3.4 Time-invariant filter and state predictor

Provided that the ARMAX model is invertible, i.e. $C(z) = \det(zI - F^*)$ is a stable polynomial, we have $\lim_{i\to\infty} \Sigma_i = 0$ and $\lim_{i\to\infty} k_i^f = 0$. This means that asymptotically, as time i tends to infinity, both the predicted, $\hat{x}_{i|i-1}$, and the filtered, $\hat{x}_{i|i}$, values of the state vector become equal. The same is true if only $C_2(z)$ is stable in the B-J model. Denoting $\hat{x}_{i|i} = \hat{x}_{i|i-1} = \hat{x}_i$, from (8.29)-(8.30) the equation of the asymptotically optimal state filter becomes:

$$\hat{x}_{i+1} = F^*\hat{x}_i + gu_i + hy_i. \tag{8.38}$$

Although this result is present in (Caines, 1972), some authors Lam (1982), Clarke et al. (1985) do not accept it, trying to derive another filter, in the form of equations (8.29)-(8.30) but with k_i^f replaced by k_f, where k_f is an arbitrary vector that fulfills the equation

$$h = Fk_f. \tag{8.39}$$

In view of uniqueness of the Kalman filter, the filter with $k_f \neq 0$ obviously does not constitute the optimal filter, even asymptotically.

8.3.5 Time-invariant output predictor

Definition 8.3.1. *The time invariant predictor is defined as*

$$\hat{y}^a_{i+j|i} = \sum_{l=0}^{j-k} g_{j-l}u_{i+l} + d'_{j-1}(F^*\hat{x}_i + hy_i), \tag{8.40}$$

where $\hat{x}_i$ is the asymptotically optimal state estimate calculated from (8.38).

Theorem 8.3.1. *The time invariant predictor given in eq. (8.40) is input-output equivalent with the predictor of (8.9)*

Proof. It is clear that

$$\sum_{l=0}^{j-k} g_{j-l}u_{i+l} = \bar{G}_j(z^{-1})u_{i+j-k} \tag{8.41}$$

with $\bar{G}_j(z^{-1})$ determined in (8.12). This proves the first part of formula (8.40). To prove the second part insert

$$\hat{x}_i = (zI - F^*)^{-1}(gu_i + hy_i) \tag{8.42}$$

to (8.40). Observe that

$$F^*\hat{x}_i + hy_i = z(zI - F^*)^{-1} - gy_i. \tag{8.43}$$

The rest of the proof bases on the identities

$$d_j'(zI - F^*)^{-1}h = z^j - \frac{A(z)E_{j+1}(z)}{C(z)} \tag{8.44}$$

$$d_j'(zI - F^*)^{-1}g = \frac{B(z)E_{j+1}(z)}{C(z)} - G_j(z), \tag{8.45}$$

where $E_j(z) = z^{j-1}\bar{E}_j(z^{-1})$ and $G_j(z) = z^{j-1}\bar{G}_j(z^{-1})$. To check the identities in (8.44) and (8.45) observe that:

$$(zI - F^*)^{-1} = (zI - F)^{-1}[I + hd'(zI - F)^{-1}]^{-1} \tag{8.46}$$

$$\left[I + hd'(zI - F)^{-1}\right]^{-1} h = \frac{A(z)}{C(z)}h \tag{8.47}$$

$$\left[I + hd'(zI - F)^{-1}\right]^{-1} g = g - \frac{B(z)}{C(z)}h. \tag{8.48}$$

Now, taking

$$F^{j-1}(zI - F)^{-1} = z^{j-1}(zI - F)^{-1} - \sum_{l=0}^{j-2} F^{j-l}z^l \tag{8.49}$$

into account yields (8.44)-(8.45). □

8.4 State-Space Solutions to the Control Problem

The original problem statement is that of the Open Loop Feedback i.e. it is assumed that the control law $u_i = f_i(\vec{y}_i)$, based on information contained in $\vec{y}_i$ should minimize the performance index (8.6) under assumption that no measurements will be available at the future time instants belonging to the control horizon. This procedure is being repeated at each instant i. A Closed Loop control algorithm is derived under assumption that new measurements will be available at any time instant belonging to the control horizon N. Although these two approaches use different algebra, it is well known that equivalence exists between OLF and CL algorithms for receding-horizon LQG problems such that both algorithms give the same control law at the current time instant i. This results directly from the Certainty Equivalence, which states that to find the optimal solution to an LQ stochastic control problem a deterministic control problem in which any stochastic variables are treated as they were exactly known is to be solved and then any uncertain variables should be replaced by their optimal estimates given $\bar{\boldsymbol{y}}_i$.

8.4.1 OLF solution

When using the OLF solution then from (8.25) it results that

$$u_i = -(\boldsymbol{G}'\boldsymbol{G} + \lambda \boldsymbol{I})^{-1}\boldsymbol{G}'\boldsymbol{D}(\boldsymbol{F}^*\hat{\boldsymbol{x}}_{i|i} + \boldsymbol{h}y_i), \tag{8.50}$$

where $\boldsymbol{D}$ is an $N \times n$ matrix, $\boldsymbol{D}' = [\boldsymbol{d}_{k-1} \dots \boldsymbol{d}_{k+N-1}]$, and the control algorithm is of the following form:

$$u_i = -\boldsymbol{k}_c'[\boldsymbol{F}^*\hat{\boldsymbol{x}}_{i|i} + \boldsymbol{h}y_i], \tag{8.51}$$

where the vector $\boldsymbol{k}_c'$ can be expressed in terms of system parameters as the first row of the matrix $(\boldsymbol{G}'\boldsymbol{G} + \lambda \boldsymbol{I})^{-1}\boldsymbol{G}'\boldsymbol{D}$. Apparently, the control law (8.51) is different than that of equation (8.1). For $k = 1$, a result similar to that of equation (8.51) can be found in (Ordys & Clarke, 1993) and (Krauss et al., 1994).

8.4.2 Solution by Riccati equation

Minimization of the performance index of equation (8.6) with respect to equation (8.7) is equivalent to minimization of

$$I_i = E \sum_{j=i}^{i+N-1} (y_{j+k}^2 + \lambda_j u_j^2) \tag{8.52}$$

with λ_j, see e.g. (Clarke et al., 1987), being defined as follows:

$$\text{for } j = i, \dots i + N_u - 1,\ \lambda_j = \lambda = \text{const} \tag{8.53}$$

$$\text{for } j = i + N_u, \dots i + N - 1,\ \lambda_j \to \infty. \tag{8.54}$$

At this point, the GPC control problem has been embedded into the framework of LQG problems and its solution is given by the following theorem:

Theorem 8.4.1. *The optimal control law has the form:*

$$u_i = -\boldsymbol{k}_c'[\boldsymbol{F}^*\hat{\boldsymbol{x}}_{i|i} + \boldsymbol{h}y_i] \tag{8.55}$$

$$\boldsymbol{k}_c = \frac{\boldsymbol{P}_0\boldsymbol{g}}{\lambda + \boldsymbol{g}'\boldsymbol{P}_0\boldsymbol{g}}, \tag{8.56}$$

where $\boldsymbol{P}_0$ is calculated from the following set of recursive equations:

(i) (Lyapunov)

$$\boldsymbol{P}_j = \boldsymbol{F}'\boldsymbol{P}_{j+1}\boldsymbol{F} + \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}',\ \boldsymbol{P}_N = \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}' \tag{8.57}$$

for $j = N-1, \dots, N_u$, and

(ii) (Riccati)

$$\boldsymbol{P}_j = \boldsymbol{F}'(\boldsymbol{P}_{j+1} - \frac{\boldsymbol{P}_{j+1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_{j+1}}{\lambda + \boldsymbol{g}'\boldsymbol{P}_{j+1}\boldsymbol{g}})\boldsymbol{F} + \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}' \tag{8.58}$$

for $j = N_u - 1, \dots 0$, where the vector $\boldsymbol{d}_{k-1}$ results from the recursion:

$$\boldsymbol{d}_0 = \boldsymbol{d},\ \boldsymbol{d}_j = \boldsymbol{F}'\boldsymbol{d}_{j-1},\ j = 1, 2, \dots k-1. \tag{8.59}$$

Remark 8.4.1. Notice that due to the special form of system equations (8.15)-(8.16), the control law in eq. (8.55), which is a function not only of the state estimate but also of a current reading y_i is somewhat different than the usual linear state feedback.

Proof. From (8.15)-(8.16) and (8.59) it follows that

$$y_{j+k} = \boldsymbol{d}_{k-1}'(\boldsymbol{F}\boldsymbol{x}_j + \boldsymbol{h}v_j) + g_k u_j + \sum_{l=0}^{k-1} e_l v_{j+k-l}, \tag{8.60}$$

where g_i and e_i are the corresponding Markov parameters:

$$g_0 = 0,\ g_j = \boldsymbol{d}_{j-1}'\boldsymbol{g} = \boldsymbol{d}'\boldsymbol{F}^{j-1}\boldsymbol{g},\ j > 0 \tag{8.61}$$

$$e_0 = 0,\ e_j = \boldsymbol{d}_{j-1}'\boldsymbol{g} = \boldsymbol{d}'\boldsymbol{F}^{j-1}\boldsymbol{g},\ j > 0 \tag{8.62}$$

and for a k-step time delay in the control channel there is:

$$g_0 = 0,\ g_1 = 0,\ \dots,\ g_{k-1} = 0,\ g_k = b_0 \neq 0. \tag{8.63}$$

The performance index J_i in (8.6) with (8.7) is equivalent to the following:

$$I_i = \mathrm{E}\{J_i\} = \mathrm{E}\{\sum_{j=i}^{i+N-1} y_{j+k}^2 + \lambda_j u_j^2\}, \tag{8.64}$$

where

$$\lambda_i = \lambda \text{ for } i = 0, 1, \ldots N_u - 1 \text{ and} \tag{8.65}$$

$$\lambda_i \to \infty \text{ for } i = N_u, \ldots N - 1. \tag{8.66}$$

On substituting (8.60) into the above performance index and averaging the terms containing noise which is in future with respect to any time instant j one gets

$$I_i = \mathrm{E}\{J_i'\} + N\sigma^2 \sum_{l=1}^{k-1} e_l^2 \tag{8.67}$$

$$J_i' = \sum_{j=i}^{i+N-1} \{[\boldsymbol{d}_{k-1}'(\boldsymbol{F}\boldsymbol{x}_j + \boldsymbol{h}v_j) + b_0 u_j]^2 + \lambda_j u_j^2\}. \tag{8.68}$$

A solution to the deterministic problem (8.15) with the performance index (8.68) can be found based on the Hamiltonian:

$$H_j = [\boldsymbol{d}_{k-1}'(\boldsymbol{F}\boldsymbol{x}_j + \boldsymbol{h}v_j) + b_0 u_j]^2 + \lambda_j u_j^2 + 2\boldsymbol{p}_{j+1}'(\boldsymbol{F}\boldsymbol{x}_j + \boldsymbol{g}u_j + \boldsymbol{h}v_j). \tag{8.69}$$

Assume that the adjoint variable $\boldsymbol{p}_j$ is of the form

$$\boldsymbol{p}_j = \partial H/\partial \boldsymbol{x}_j = (\boldsymbol{P}_j - \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}')\boldsymbol{x}_j + \boldsymbol{f}_j \tag{8.70}$$

with $\boldsymbol{p}_{i+N} = \boldsymbol{0}$. The optimal control minimizes the Hamiltonian, i.e. it can be calculated from

$$u_j = -\boldsymbol{k}_j^{c\prime}(\boldsymbol{F}\boldsymbol{x}_j + \boldsymbol{h}v_j) - \frac{\boldsymbol{g}'\boldsymbol{f}_{j+1}}{\lambda_j + \boldsymbol{g}'\boldsymbol{P}_{j+1}\boldsymbol{g}} \tag{8.71}$$

$$\boldsymbol{k}_j^c = \frac{\boldsymbol{P}_{j+1}\boldsymbol{g}}{\lambda_j + \boldsymbol{g}'\boldsymbol{P}_{j+1}\boldsymbol{g}} \tag{8.72}$$

$$\begin{aligned} \boldsymbol{f}_j &= \boldsymbol{F}'[\boldsymbol{I} - \frac{\boldsymbol{P}_{j+1}\boldsymbol{g}\boldsymbol{g}'}{\lambda_j + \boldsymbol{g}'\boldsymbol{P}_{j+1}\boldsymbol{g}}](\boldsymbol{f}_{j+1} + \boldsymbol{P}_{j+1}\boldsymbol{h}v_j) \end{aligned} \tag{8.73}$$

$$\boldsymbol{P}_j = \boldsymbol{F}'(\boldsymbol{P}_{j+1} - \frac{\boldsymbol{P}_{j+1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_{j+1}}{\lambda_j + \boldsymbol{g}'\boldsymbol{P}_{j+1}\boldsymbol{g}})\boldsymbol{F} + \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}' \tag{8.74}$$

with $\boldsymbol{f}_{i+N} = \boldsymbol{0}$ and $\boldsymbol{P}_{i+N} = \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}'$, for $j = i + N - 1, \ldots, i$. As a result, for current time i we have

$$u_i = -\boldsymbol{k}_i^c(\boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{h}v_i) + \sum_{k=1}^{N-1} \theta_{i,k} v_{i+k}, \tag{8.75}$$

where $\theta_{i,k}$ do not depend on state or noise. On applying the Certainty Equivalence Principle (Uchida & Shimemura, 1976), i.e. replacing stochastic variables by their estimates based on information available at time i, equations (8.55)-(8.58) are obtained. □

A solution to the problem with $k = 1$ can be found in (Ordys & Clarke, 1993). However, when used for problems with $k > 1$, it may lead to serious problems if $\lambda = 0$ (for details see (Błachuta, 1987)). Equivalence of the vectors $\boldsymbol{k}_c$ calculated using different approaches has been explicitly shown in (Kwon et al., 1992*a*).

8.4.3 Infinite horizon problems

Although GPC is stated as a finite receding horizon problem, its concerns are associated with the stability and minimum variance performance properties of the infinite horizon performance index. However, when presented as an LQG problem, it can also be stated as an infinite horizon problem. Assuming that the matrix $\boldsymbol{F}$ is stable we are able to solve a problem with a finite N_u and $N \to \infty$. Then instead of solving the Dynamic Lyapunov Equation (8.57) the following Algebraic Lyapunov equation is to be solved:

$$\boldsymbol{P} = \boldsymbol{F}'\boldsymbol{P}\boldsymbol{F} + \boldsymbol{d}\boldsymbol{d}' \tag{8.76}$$

and its solution is to be used for the initialization of the Riccati equation (8.58). If $N_u = N$ and $N \to \infty$ then to find the optimal controller the following Algebraic Riccati Equation is to be solved:

$$\boldsymbol{P} = \boldsymbol{F}'(\boldsymbol{P} - \frac{\boldsymbol{P}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}}{\lambda + \boldsymbol{g}'\boldsymbol{P}\boldsymbol{g}})\boldsymbol{F} + \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}'. \tag{8.77}$$

Remark 8.4.2. In (Grimble, 1990; Grimble, 1995; Ordys & Grimble, 1996) and (Taube & Lampe, 1992) an idea of a Dynamic Performance Predictive controller is presented which aims at combining the properties of GPC and LQG to retain the 'tuning knobs' of GPC and to maintain the stability of LQG. The performance index $I_{LQGPC} = \mathrm{E}\{J_{LQGPC}\}$ defining the so called LQGPC problem is believed to guarantee this task, where

$$J_{LQGPC} = \lim_{T\to\infty} \frac{1}{T}\sum_{i=0}^{T} J_i = \lim_{T\to\infty} \frac{1}{T}\sum_{i=0}^{T}(\sum_{j=i}^{i+N-1} y_{j+k}^2 + \lambda \sum_{j=i}^{i+N_u-1} u_j^2). \tag{8.78}$$

Upon rearranging the terms, (8.78) can, however, be written as follows:

$$J_{LQGPC} = N \lim_{T\to\infty} \frac{1}{T}\sum_{j=0}^{T}(y_{j+k}^2 + \lambda' u_j^2), \quad \lambda' = \frac{N_u}{N}\lambda \tag{8.79}$$

showing that the problem is equivalent to an ordinary LQG problem. Thus the GPC 'tuning knobs', N and N_u, lose their original meaning and particular approaches of (Grimble, 1995; Ordys & Grimble, 1996; Hangstrup, Ordys & Grimble, 1997) and (Taube & Lampe, 1992) are not necessary to solve the problem with performance index (8.78).

8.5 Relations Between State-Space and Polynomial Solutions

8.5.1 Polynomial input-output form of the control algorithm

Theorem 8.5.1. *Polynomials $D(z)$ and $F(z)$ of the input-output polynomial algorithm*

$$D(z)u_i = -F(z)y_i \tag{8.80}$$

are related to the state-space terms: $\boldsymbol{F}^*, \boldsymbol{g}, \boldsymbol{h}$ and $\boldsymbol{P}_0$ as follows

$$D(z) = \lambda C(z) + zg'\boldsymbol{P}_0 \operatorname{adj}(z\boldsymbol{I} - \boldsymbol{F}^*)\boldsymbol{g} \tag{8.81}$$
$$F(z) = zg'\boldsymbol{P}_0 \operatorname{adj}(z\boldsymbol{I} - \boldsymbol{F}^*)\boldsymbol{h}. \tag{8.82}$$

Proof. The proof follows by combining a control algorithm of the form (8.51) with the asymptotic filter (8.38). □

Remark 8.5.1. From equations (8.81)-(8.82), simple formulae can be obtained for a one-stage performance index:

$$\boldsymbol{P}_0 = \boldsymbol{d}_{k-1}\boldsymbol{d}'_{k-1}, \quad \boldsymbol{k}_c = \frac{b_0}{\lambda + b_0^2}\boldsymbol{d}_{k-1}. \tag{8.83}$$

Due to (Błachuta & Ordys, 1984), from (8.81)-(8.82), taking the identities (8.44)-(8.45) and (8.20) into account one is able to write:

$$D(z) = \frac{\lambda}{b_0}C(z) + zE_k(z)B(z) \tag{8.84}$$
$$F(z) = z[z^{k-1}C(z) + E_k(z)A(z)]. \tag{8.85}$$

The above formulae express the regulator of Clarke and Hastings-James (Clarke & Hastings-James, 1971). It is interesting to note that if $N_u = N$ and $\lambda = 0$ then $\boldsymbol{P}_0 = \boldsymbol{d}_{k-1}\boldsymbol{d}'_{k-1}$ of equation (8.83) is also the solution of the Riccati equation (8.58).

8.5.2 The characteristic polynomial

Theorem 8.5.2. *The characteristic polynomial of the closed-loop system composed of the plant (8.15), filter (8.38) and controller (8.51) has the form:*

$$p(z) = \frac{C(z)}{\lambda + g'\boldsymbol{P}_0 g}[\lambda A(z) + zg'\boldsymbol{P}_0 \operatorname{adj}(z\boldsymbol{I} - \boldsymbol{F})g]. \tag{8.86}$$

Proof. The state-space equation of a closed-loop system composed of the plant (8.15), filter (8.38) and controller (8.51) can be written in the form:

$$\boldsymbol{X}_{i+1} = \boldsymbol{\Phi}\boldsymbol{X}_i + \boldsymbol{g}v_i, \tag{8.87}$$

where

$$\boldsymbol{X}'_i = (\boldsymbol{x}'_i, \hat{\boldsymbol{x}}'_i) \quad , \quad \boldsymbol{g}' = (\boldsymbol{g}'_1, \boldsymbol{g}'_2), \tag{8.88}$$

and

$$\begin{array}{ll} \boldsymbol{\Phi}^{11} = (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}'_c)\boldsymbol{F} & , \quad \boldsymbol{\Phi}^{12} = (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}'_c)\boldsymbol{h}\boldsymbol{d}' \\ \boldsymbol{\Phi}^{21} = 0 & , \quad \boldsymbol{\Phi}^{22} = \boldsymbol{F}^* \\ \boldsymbol{g}^1 = (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}'_c)\boldsymbol{h} & , \quad \boldsymbol{g}^2 = 0 \end{array} \tag{8.89}$$

Hence the characteristic polynomial of the system is:

$$\begin{aligned} p(z) &= \det(z\boldsymbol{I} - \boldsymbol{\Phi}) = \det(z\boldsymbol{I} - \boldsymbol{F}^*)\det(z\boldsymbol{I} - \boldsymbol{F} + \boldsymbol{g}\boldsymbol{k}'_c\boldsymbol{F}) \\ &= \frac{C(z)}{\lambda + g'\boldsymbol{P}_0 g}[\lambda A(z) + zg'\boldsymbol{P}_0 \operatorname{adj}(z\boldsymbol{I} - \boldsymbol{F})g]. \end{aligned} \tag{8.90}$$

□

Remark 8.5.2. For one-stage performance indices, from (8.83), (8.44) and (8.90) it follows:

$$p(z) = \frac{1}{\lambda + b_0^2}C(z)[\lambda A(z) + b_0 z^k B(z)]. \tag{8.91}$$

If $\lambda = 0$ then the stability of the closed-loop system is determined by the stability of $B(z)$. Due to Remark 8.5.1, equation (8.91) is valid for any $N = N_u$ if $\lambda = 0$. This also clarifies the observed stability problems of GPC for small values of λ.

Remark 8.5.3. On calculating $p(z^{-1})p(z)$ with $p(z)$ of (8.90) and $\boldsymbol{P}_0$ of equation (8.77) one gets that the characteristic polynomial of the infinite-horizon LQG problem fulfills

$$p(z^{-1})p(z) = C(z^{-1})C(z)[\lambda A(z^{-1})A(z) + B(z^{-1})B(z)]. \tag{8.92}$$

Thus the stabilizing solution of the Riccati equation (8.77) produces a stable characteristic polynomial $p(z)$ in spite of the value of λ which could have been found by spectral factorization of (8.92). Another method to achieve stability with a finite horizon performance index is the so called GPC with terminal state constraint (Chisci, Lombardi, Mosca & Rossiter, 1996; de Nicolao, Magni & Scattolini, 1996; de Nicolao & Strada, 1997). Solutions to infinite cost horizon problems with control constraints can be found in (Scokaert, 1997) and (Scokaert & Rawlings, 1998)

8.6 Covariance Characteristics of the Control System

In this section formulae for variances of both the output and control variables will be derived in two cases. The first, simpler, one is when optimal Kalman predictor is applied. We have then the following theorem.

Theorem 8.6.1.

$$\sigma_{yi}^2 = \operatorname{var} y_i = \boldsymbol{d}'\boldsymbol{V}_i\boldsymbol{d} + \sigma_i^2 \tag{8.93}$$

$$\sigma_{ui}^2 = \operatorname{var} u_i = \boldsymbol{k}_c'[\boldsymbol{F}\boldsymbol{V}_i\boldsymbol{F}' + \boldsymbol{h}_i\boldsymbol{h}_i'\sigma_i^2]\boldsymbol{k}_c \tag{8.94}$$

$$\boldsymbol{V}_{i+1} = (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}_c')[\boldsymbol{F}\boldsymbol{V}_i\boldsymbol{F}' + \boldsymbol{h}_i\boldsymbol{h}_i'\sigma_i^2](\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}_c')' \tag{8.95}$$

with $\boldsymbol{V}_0 = \boldsymbol{0}$.

Proof. Observe that from (8.55) and (8.34)-(8.35), when expressing the control law in terms of the predicted state variable $\hat{\boldsymbol{x}}_{i|i-1}$ rather than in terms of the filtered state variable $\hat{\boldsymbol{x}}_{i|i}$, the following system of equations can be written:

$$y_i = \boldsymbol{d}'\hat{\boldsymbol{x}}_{i|i-1} + \epsilon_i \tag{8.96}$$

$$u_i = -\boldsymbol{k}_c[\boldsymbol{F}\hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{h}_i\epsilon_i] \tag{8.97}$$

$$\hat{\boldsymbol{x}}_{i+1|i} = (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}_c')[\boldsymbol{F}\hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{h}_i\epsilon_i]. \tag{8.98}$$

Now, taking into account that the state predictions and innovations are independent, one is able to write (8.93)-(8.95). □

Equations (8.93)-(8.95) are only valid when the optimal filter is applied. As a matter of fact, the dimension of the state of a closed loop system containing a plant, a filter and a controller is twice as great as the dimension of the state space of the plant alone. Hence, in the second case, we have the following theorem.

Theorem 8.6.2. *The variances* σ_{yi}^2 *and* σ_{ui}^2 *are expressed by:*

$$\sigma_{yi}^2 = \boldsymbol{d}'[\boldsymbol{W}_i^{11} + \boldsymbol{W}_i^{12} + \boldsymbol{W}_i^{21} + \boldsymbol{W}_i^{22}]\boldsymbol{d} + \sigma^2 \tag{8.99}$$

$$\sigma_{ui}^2 = \boldsymbol{k}_c'[\boldsymbol{F}\boldsymbol{W}_i^{11}\boldsymbol{F}' + \boldsymbol{F}\boldsymbol{W}_i^{12}\boldsymbol{d}\boldsymbol{c}_i' + \boldsymbol{h}_i\boldsymbol{d}'\boldsymbol{W}_i^{21}\boldsymbol{F}' + \boldsymbol{h}_i\boldsymbol{d}'\boldsymbol{W}_i^{22}\boldsymbol{d}\boldsymbol{c}_i' + \boldsymbol{h}_i\boldsymbol{h}_i'\sigma^2]\boldsymbol{k}_c, \tag{8.100}$$

where the covariance $\boldsymbol{W}_i = \operatorname{cov}(\hat{\boldsymbol{x}}_{i|i-1}', \tilde{\boldsymbol{x}}_{i|i-1})'$ *of the augmented state follows from a recursive equation*

$$\boldsymbol{W}_{i+1} = \boldsymbol{\Phi}_i\boldsymbol{W}_i\boldsymbol{\Phi}_i' + \boldsymbol{\gamma}_i\boldsymbol{\gamma}_i'\sigma^2 \tag{8.101}$$

with the initial condition $\boldsymbol{W}_0$ *being a block matrix with*

$$\boldsymbol{W}_0^{11} = 0,\ \boldsymbol{W}_0^{12} = 0,\ \boldsymbol{W}_0^{21} = 0,\ \boldsymbol{W}_0^{22} = \boldsymbol{Q}_0. \tag{8.102}$$

$\boldsymbol{\Phi}_i$ *and* $\boldsymbol{\gamma}_i$ *in (8.101) are block matrices with*

$$\begin{aligned} \boldsymbol{\Phi}_i^{11} &= (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}_c')\boldsymbol{F}, & \boldsymbol{\Phi}_i^{12} &= (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}_c')\boldsymbol{h}_i\boldsymbol{d}' \\ \boldsymbol{\Phi}_i^{21} &= 0, & \boldsymbol{\Phi}_i^{22} &= \boldsymbol{F} - \boldsymbol{h}_i\boldsymbol{d}' \\ \boldsymbol{\gamma}_i^1 &= (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}_c')\boldsymbol{h}_i, & \boldsymbol{\gamma}_i^2 &= \boldsymbol{h} - \boldsymbol{h}_i. \end{aligned} \tag{8.103}$$

Proof. The system of equations (8.96)-(8.98) is augmented by the prediction error $\tilde{\boldsymbol{x}}_{i|i-1}$. Then expressing ϵ_i as $\epsilon_i = \boldsymbol{d}'\tilde{\boldsymbol{x}}_{i|i-1} + v_i$ yields

$$\hat{\boldsymbol{x}}_{i+1|i} = (\boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}_c')(\boldsymbol{F}\hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{h}_i\boldsymbol{d}'\tilde{\boldsymbol{x}}_{i|i-1} + \boldsymbol{h}_i v_i) \tag{8.104}$$

$$\tilde{\boldsymbol{x}}_{i+1|i} = (\boldsymbol{F} - \boldsymbol{h}_i\boldsymbol{d}')\tilde{\boldsymbol{x}}_{i|i-1} + (\boldsymbol{h} - \boldsymbol{h}_i)v_i \tag{8.105}$$

$$y_i = \boldsymbol{d}'\hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{d}'\tilde{\boldsymbol{x}}_{i|i-1} + v_i \tag{8.106}$$

$$u_i = -\boldsymbol{k}_c(\boldsymbol{F}\hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{h}_i\boldsymbol{d}'\tilde{\boldsymbol{x}}_{i|i-1} + \boldsymbol{h}_i v_i). \tag{8.107}$$

□

Equations (8.101)-(8.100) are valid whether optimal or asymptotically optimal filter is applied. Derivation of equations that are valid when there is a mismatch between the plant and model structure and/or parameters is also possible in a similar way. Due to the use of predictor equations instead of filter equations, the formulae obtained are much simpler than those presented in Ordys (1993).

8.7 Box-Jenkins Model and Offset Rejection

In this section some structured models will be studied as a basis of LQG and GPC.

8.7.1 Input-output approach

Perhaps the most general and versatile model of a linear plant with a stochastic disturbance has the following form in the input-output framework:

$$\tilde{A}_0(z^{-1})y_i = \frac{\tilde{B}_1(z^{-1})}{\tilde{A}_1(z^{-1})}u_{i-k} + \frac{\tilde{C}_2(z^{-1})}{\tilde{A}_2(z^{-1})}v_i, \tag{8.108}$$

where $\deg \bar{A}_0 = n_0$, $\deg \bar{A}'_1 = n'_1$, $\deg \bar{B}'_1 = m'_1$ and $\deg \bar{A}'_2 = \deg \bar{C}'_2 = n'_2$. The two channels: the control channel and the disturbance are partly separated and the polynomial $\bar{A}_0$ represents this part of the dynamics which are shared by both of them. Since the model in (8.108) is presented in (Ljung and Söderström, 1983) it will be called the Ljung-Söderström model (L-S).

The Box-Jenkins (B-J) model is defined as follows:

$$y_i = \frac{\bar{B}_1(z^{-1})}{\bar{A}_1(z^{-1})}u_{i-k} + \frac{\bar{C}_2(z^{-1})}{\bar{A}_2(z^{-1})}v_i, \tag{8.109}$$

where $\deg \bar{A}_1 = n_1$, $\deg \bar{B}_1 = m_1$ and $\deg \bar{A}_2 = \deg \bar{C}_2 = n_2$. The B-J model consists of two parts, one being a model of the control channel while the second is a model of a stochastic disturbance. Assuming that the B-J model is obtained from (8.108), then: $\bar{A}_1 = \bar{A}_0\bar{A}'_1$, $\bar{A}_2 = \bar{A}_0\bar{A}'_2$. Finally, given an L-S model, an equivalent ARMAX model can be found in which

$$\bar{A} = \bar{A}_0\bar{A}'_1\bar{A}'_2,\ \bar{B} = \bar{B}_1\bar{A}'_2,\ \bar{C} = \bar{C}_2\bar{A}'_1. \tag{8.110}$$

The L-S model is defined by $n_0 + n'_1 + m_1 + 2n'_2 + 1$ coefficients of the polynomials in (8.109), the B-J model by $2n_0 + n'_1 + m_1 + 2n'_2 + 1$ coefficients while the ARMAX model (8.5) of order $n_0 + n'_1 + n'_2$ requires $n_0 + 2n'_1 + m_1 + 3n'_2 + 1$, i.e. $n'_1 + n'_2$ parameters more than the L-S model. The L-S model is thus preferred for its parsimony. The predictor equation for the L-S model is of the form

$$\hat{y}_{i+j|i} = G_j u_{i+j-k} + \frac{\Gamma_j}{\bar{A}'_1\bar{C}_2}u_{i-k} + \frac{F_j}{\bar{C}_2}y_i, \tag{8.111}$$

where the polynomials $G_j(z^{-1})$, $\Gamma_j(z^{-1})$ and $F_j(z^{-1})$ are the solutions of the following Diophantine equations:

$$\bar{C}_2 = E_j\bar{A}_0\bar{A}'_2 + z^{-j}F_j \tag{8.112}$$
$$\bar{A}'_2\bar{B}_1E_j = G_j\bar{A}'_1\bar{C}_2 + z^{-j}\Gamma_j. \tag{8.113}$$

A variation of the L-S model that leads to a controller with an integral action is as follows:

$$\bar{A}_0(z^{-1})y_i = \frac{\bar{B}_1(z^{-1})}{\Delta\bar{A}'_1(z^{-1})}\Delta u_{i-k} + \frac{\bar{C}_2(z^{-1})}{\bar{A}'_2(z^{-1})}v_i. \tag{8.114}$$

By setting certain polynomials equal to 1, equation (8.108) may represent any known input-output models, including an integrated step-response. As a result, solutions to various problems, e.g. different versions of MPC, are included in equations (8.111)-(8.113). It is to be emphasized that, unlike the ARMAX based standard approach, introduction of the Δ operator does not cause either nonstationarity or nonivertibility of the stochastic part of the model in (8.114).

8.7.2 State-space approach

For the system of equations in (8.15)-(8.16) to represent a B-J model it should be written in the following decomposed form:

$$\boldsymbol{x}^1_{i+1} = \boldsymbol{F}^1\boldsymbol{x}^1_i + \boldsymbol{g}^1u_i,\ \operatorname{cov}(\boldsymbol{x}^1_0 - \boldsymbol{m}^1_0) = 0 \tag{8.115}$$
$$\boldsymbol{x}^2_{i+1} = \boldsymbol{F}^2\boldsymbol{x}^2_i + \boldsymbol{h}^2v_i,\ \operatorname{cov}(\boldsymbol{x}^2_0 - \boldsymbol{m}^2_0) = \boldsymbol{Q}^2_0 \tag{8.116}$$
$$y_i = \boldsymbol{d}^{1'}\boldsymbol{x}^1_i + \boldsymbol{d}^{2'}\boldsymbol{x}^2_i + v_i. \tag{8.117}$$

Given a polynomial B-J model (8.109), then the vectors and matrices defining (8.115)-(8.117) can be constructed using e.g. canonical forms. Also the step-response model can easily be incorporated in (8.115), leading to a state-space solution similar to that of (Lee et al., 1994).

Assuming that the control channel is of type $l \geq 1$, B-J model allows for bias rejection while retaining the stationarity of the stochastic part. This is attained either when the plant itself is of type l, or by using the actuator of type l (e.g. electrical servomotor) or finally by using a discrete-time integrator with the transfer function $1/(1 - z^{-1})$ and treating it as a part of the plant model. Moreover, as shown in (Błachuta, 1996f), B-J model is more parsimonious than its ARMAX equivalent, and provides better computational efficiency, for details see (Błachuta, 1996f).

In the B-J model of (8.115)-(8.117) vector $\boldsymbol{x}^1_i$ is deterministic and only $\boldsymbol{x}^2_i$ requires Kalman filtering.

8.7.2.1 State filter for B-J-type model. Assuming that the model of a system is in the form of (8.115)-(8.117), then the filter equations for the control channel are trivial,

$$\hat{\boldsymbol{x}}^1_{i|i} = \hat{\boldsymbol{x}}^1_{i|i-1},\ \hat{\boldsymbol{x}}^1_{0|-1} = \boldsymbol{m}^1_0 \tag{8.118}$$
$$\hat{\boldsymbol{x}}^1_{i+1|i} = \boldsymbol{F}^1\hat{\boldsymbol{x}}^1_{i|i} + \boldsymbol{g}^1u_i \tag{8.119}$$

and a nontrivial filter is only necessary for the disturbance channel:

$$\hat{\boldsymbol{x}}^2_{i|i} = \hat{\boldsymbol{x}}^2_{i|i-1} + \boldsymbol{k}^{f2}_i[y_i - \boldsymbol{d}^{1'}\hat{\boldsymbol{x}}^1_{i|i-1} - \boldsymbol{d}^{2'}\hat{\boldsymbol{x}}^2_{i|i-1}] \tag{8.120}$$
$$\hat{\boldsymbol{x}}^2_{i+1|i} = \boldsymbol{F}^{*2}\hat{\boldsymbol{x}}^2_{i|i} + \boldsymbol{h}^2(y_i - \boldsymbol{d}^{1'}\hat{\boldsymbol{x}}^1_{i|i}) \tag{8.121}$$

with $\hat{\boldsymbol{x}}^2_{0|-1} = \boldsymbol{m}^2_0$ and the Kalman filter gain, $\boldsymbol{k}^{f2}_i$,

$$\boldsymbol{k}^{f2}_i = \frac{\boldsymbol{\Sigma}^2_i\boldsymbol{d}^2}{1 + \boldsymbol{d}^{2'}\boldsymbol{\Sigma}^2_i\boldsymbol{d}^2}, \tag{8.122}$$

where $\boldsymbol{\Sigma}^2_i$ is defined by a set of recursive equations

$$S_i^2 = \Sigma_i^2 - \frac{\Sigma_i^2 d^2 d^{2\prime} \Sigma_i^2}{1 + d^{2\prime}\Sigma_i^2 d^2}, \; \Sigma_{i+1}^2 = F^{*2} S_i^2 F^{*2\prime} \tag{8.123}$$

Here $F^{*2} = F^2 - h^2 d^{2\prime}$, $\Sigma_0^2 = Q_0^2$ and Q_0^2 is a solution of

$$Q_0^2 = F^2 Q_0^2 F^{2\prime} + h^2 h^{2\prime}. \tag{8.124}$$

8.7.2.2 Controller for B-J-type model. Assuming that the model of the system is in the form of (8.115)-(8.117), the controller equation is of the following form:

$$u_i = -k_c^{1\prime} F^1 \hat{x}_{i|i}^1 - k_c^{2\prime}[F^{*2}\hat{x}_{i|i}^2 + h^2 y_i] \tag{8.125}$$

with

$$k_c^1 = \frac{P_0^{11} g^1}{\lambda + g^{1\prime} P_0^{11} g^1}, \; k_c^2 = \frac{P_0^{21} g^1}{\lambda + g^{1\prime} P_0^{11} g^1}, \tag{8.126}$$

where P_0^{11} is a solution of the following recursive equations:

$$P_j^{11} = F^{1\prime} P_{j+1}^{11} F^1 + d_{k-1}^1 d_{k-1}^{1\prime} \tag{8.127}$$

for $j = N-1, \ldots N_u$ with $P_N^{11} = d_{k-1}^1 d_{k-1}^{1\prime}$, and

$$P_j^{11} = F^{1\prime}(P_{j+1}^{11} - \frac{P_{j+1}^{11} g^1 g^{1\prime} P_{j+1}^{11}}{\lambda + g^{1\prime} P_{j+1}^{11} g^1}) F^1 + d_{k-1}^1 d_{k-1}^{1\prime} \tag{8.128}$$

for $j = N_u - 1, \ldots, 0$.

Matrix P_j^{21} can be found from

$$P_j^{21} = F^{2\prime} P_{j+1}^{11} F^1 + d_{k-1}^2 d_{k-1}^{1\prime} \tag{8.129}$$

for $j = N-1, \ldots, N_u$, with $P_N^{21} = d_{k-1}^2 d_{k-1}^{1\prime}$ and

$$P_j^{21} = F^{2\prime} P_{j+1}^{21} B_j^1 + d_{k-1}^2 d_{k-1}^{1\prime} \tag{8.130}$$

for $j = N_u - 1, \ldots, 0$ with

$$B_j^1 = (I - \frac{g^1 g^{1\prime} P_{j+1}^{11}}{\lambda + g^{1\prime} P_{j+1}^{11} g^1}) F^1. \tag{8.131}$$

8.7.2.3 Infinite horizon problems. Assuming that matrix F^1 is stable and denotes

$$B^1 = (I - \frac{g^1 g^{1\prime} P^{11}}{\lambda + g^{1\prime} P^{11} g^1}) F^1. \tag{8.132}$$

Then to solve the infinite-horizon, $N \to \infty$, GPC problem, the following ALEs:

$$P^{11} = F^{1\prime} P^{11} F^1 + d_{k-1}^1 d_{k-1}^{1\prime} \tag{8.133}$$

$$P^{21} = F^{2\prime} P^{21} B^1 + d_{k-1}^2 d_{k-1}^{1\prime} \tag{8.134}$$

are to be solved and then N_u recursions of equations (8.128) and (8.130) are to be performed. In the case of LQG problem with $N_u = N$ and $N \to \infty$, only a system of algebraic equations:

(i) Riccati

$$P^{11} = F^{1\prime}(P^{11} - \frac{P^{11} g^1 g^{1\prime} P^{11}}{\lambda + g^{1\prime} P^{11} g^1}) F^1 + d_{k-1}^1 d_{k-1}^{1\prime} \tag{8.135}$$

(ii) Lyapunov of (8.134)

is to be solved.

8.8 Fast Algorithms

As it is well known (Morf, Sidhu & Kailath, 1973), for some special filtration problems matrix Riccati equations can be replaced by so called Chandrasekhar-type equations. Based on this theory, it will be shown that instead of updating n^2 entries of a Riccati matrix only $2n$ entries of two vectors plus one scalar variable are to be updated. For $n \geq 3$, this reduces the number of calculations.

8.8.1 Chandrasekhar-type equations for the controller

If ones aim is only to find a series of gain vectors k_i^c instead of matrices P_i, $i = 0, \ldots, N-1$, it can be found from a set of vector Chandrasekhar equations. Unfortunately, this is only valid for finite horizon LQG problems, i.e. when $N_u = N$.

Theorem 8.8.1. *The controller gain, k_i^c, reads*

$$k_i^c = \frac{q_i}{\lambda_i}, \tag{8.136}$$

where:

$$\lambda_i = \lambda_{i+1}(1 + \beta_{i+1}^2), \qquad \lambda_N = \lambda + b_0^2 \tag{8.137}$$

$$q_i = q_{i+1} + \beta_{i+1} F' p_{i+1}, \; q_N = b_0 d_k \tag{8.138}$$

$$p_i = F' p_{i+1} - \beta_{i+1} q_{i+1}, \; p_N = \sqrt{\lambda} d_k \tag{8.139}$$

and

$$\beta_i = (g' p_i)/\lambda_i. \tag{8.140}$$

The Riccati matrix P_i can then be calculated from

$$P_i = P_{i+1} + \frac{(p_i p_i')}{\lambda_i}, \quad P_N = d_{k-1}d_{k-1}'. \tag{8.141}$$

Proof. Let us introduce differences $\delta P_i = P_{i+1} - P_i$ of the Riccati matrix P_i and denote $F^*_{i+1} = F - gk^c_{i+1}$. Then λ_i, k^c_i and δP_i can be expressed by the differences δP_{i+1} as follows

$$\lambda_i = \lambda_{i+1} - g'\delta P_{i+1} g \tag{8.142}$$

$$k^c_i = k^c_{i+1} - \frac{F^{*\prime}_{i+1}\delta P_{i+1} g}{\lambda_i} \tag{8.143}$$

$$\delta P_i = F^{*\prime}_{i+1}[\delta P_{i+1} + \frac{\delta P_{i+1} g g' \delta P_{i+1}}{\lambda_i}]F^*_{i+1} \tag{8.144}$$

for $i = N-1, \ldots 0$ with the terminal condition for equation (8.144):

$$\delta P_{N-1} = P_N - P_{N-1} = d_k(-\frac{\lambda}{\lambda + b_0^2})d_k'. \tag{8.145}$$

From (8.145) it follows that for all $i = N-1, \ldots 0$

$$\delta P_i = w_i(-\phi_i)w_i', \tag{8.146}$$

and

$$w_i\phi_i w_i' = F^{*\prime}_{i+1} w_{i+1}[\phi_{i+1} - \frac{(w'_{i+1}g)^2}{\lambda_i}\phi^2_{i+1}]w'_{i+1}F^*_{i+1}. \tag{8.147}$$

The matrix equation in (8.147) is then factorized yielding the following system of equations:

$$w_i = (F - gk^{c\prime}_{i+1})' w_{i+1} \tag{8.148}$$

$$\phi_i = \phi_{i+1} - \frac{(w'_{i+1}g)^2}{\lambda_i}\phi^2_{i+1} \tag{8.149}$$

$$\lambda_i = \lambda_{i+1} + (w'_{i+1}g)^2\phi_{i+1}. \tag{8.150}$$

A transformation of equation (8.143) gives

$$k^c_i = k^c_{i+1} + \frac{(F - gk^{c\prime}_{i+1})' w_{i+1}\phi_{i+1}(w'_{i+1}g)}{\lambda_i}. \tag{8.151}$$

Introducing new variables, q_i and p_i, where

$$q_i = \lambda_i k^c_i, \; p_i = w_i(\phi_i\lambda_i)^{\frac{1}{2}} \tag{8.152}$$

with the terminal conditions $q_N = b_0 d_k$, and $p_N = \sqrt{\lambda}d_k$. From (8.151), when expressing k^c_i by $k^c_i = q_i/\lambda_i$ we have

$$q_i = q_{i+1} + \phi_{i+1}(w'_{i+1}g)F'w_{i+1}. \tag{8.153}$$

Finally, expressing w_{i+1} as

$$w_{i+1} = p_{i+1}(\phi_{i+1}\lambda_{i+1})^{-\frac{1}{2}} \tag{8.154}$$

gives

$$q_i = q_{i+1} + \frac{g'p_{i+1}}{\lambda_{i+1}}F'p_{i+1} \tag{8.155}$$

$$\lambda_i = \lambda_{i+1} + \frac{g'p_{i+1}}{\lambda_{i+1}} \tag{8.156}$$

Proceeding the same way, two first equations in (8.150) become

$$p_i = (F'p_{i+1} - \frac{g'p_{i+1}}{\lambda_{i+1}}q_{i+1})(\frac{\phi_i\lambda_i}{\phi_{i+1}\lambda_{i+1}})^{\frac{1}{2}} \tag{8.157}$$

$$\phi_i = \phi_{i+1}[1 - \frac{(g'p_{i+1})^2}{\lambda_i\lambda_{i+1}}]. \tag{8.158}$$

From (8.158) and (8.156) it follows, however, that $\phi_i\lambda_i = \phi_{i+1}\lambda_{i+1}$ and

$$p_i = F'p_{i+1} - \frac{g'p_{i+1}}{\lambda_{i+1}}q_{i+1}. \tag{8.159}$$

Finally, from (8.147) it results that $\delta P_i = -p_i p_i'/\lambda_i$. □

8.8.2 Chandrasekhar-type equations for the filter

It is now assumed that the process defined by (8.15)-(8.16) is a stationary one, i.e. that the covariance matrix $\sigma^2 Q_0$ is based on a solution of

$$Q_0 = FQ_0F' + hh'. \tag{8.160}$$

For such $Q_0 \geq 0$ to exist, the subsystem controllable from v_i must be stable.

Theorem 8.8.2. *The vector k^f_i is defined as*

$$k^f_i = \frac{h_i}{r_i}, \tag{8.161}$$

where

$$r_{i+1} = r_i(1 - \alpha_i), \; r_0 = 1 + d'Q_0 d \tag{8.162}$$

$$h_{i+1} = h_i - \alpha_i l_i, \; h_0 = Q_0 d \tag{8.163}$$

$$l_{i+1} = F^*(l_i - \alpha_i h_i), \; l_0 = r_0(F^*Q_0 d + h) \tag{8.164}$$

with

$$\alpha_i = (\boldsymbol{d}'\boldsymbol{l}_i)/r_i. \tag{8.165}$$

Matrix $\boldsymbol{\Sigma}_i$ is given by

$$\boldsymbol{\Sigma}_{i+1} = \boldsymbol{\Sigma}_i - \frac{(\boldsymbol{l}_i\boldsymbol{l}_i')}{r_i},\ \boldsymbol{\Sigma}_0 = \boldsymbol{Q}_0. \tag{8.166}$$

Proof. When defining

$$\delta\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}_{i+1} - \boldsymbol{\Sigma}_i \tag{8.167}$$

and

$$r_i = 1 + \boldsymbol{d}'\boldsymbol{\Sigma}_i\boldsymbol{d} \tag{8.168}$$

then the following equations hold:

$$r_{i+1} = r_i + \boldsymbol{d}'\delta\boldsymbol{\Sigma}_i\boldsymbol{d} \tag{8.169}$$

$$\boldsymbol{k}_{i+1}^f = \boldsymbol{k}_i^f + \frac{(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')\delta\boldsymbol{\Sigma}_i\boldsymbol{d}}{r_{i+1}} \tag{8.170}$$

$$\delta\boldsymbol{\Sigma}_{i+1} = \boldsymbol{F}^*(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')[\delta\boldsymbol{\Sigma}_i - \frac{\delta\boldsymbol{\Sigma}_i\boldsymbol{d}\boldsymbol{d}'\delta\boldsymbol{\Sigma}_i}{r_{i+1}}](\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')'\boldsymbol{F}^{*\prime} \tag{8.171}$$

We also have

$$\delta\boldsymbol{\Sigma}_0 = -r_0(\boldsymbol{F}\boldsymbol{k}_0^f + \boldsymbol{h})(\boldsymbol{F}\boldsymbol{k}_0^f + \boldsymbol{h})'. \tag{8.172}$$

The above formula can be rewritten in the form $\delta\boldsymbol{\Sigma}_0 = \boldsymbol{w}_0\varphi_0\boldsymbol{w}_0'$ with $\boldsymbol{w}_0 = \boldsymbol{F}\boldsymbol{k}_0^f + \boldsymbol{h}$, $\varphi_0 = -r_0$ which leads to the factorization $\delta\boldsymbol{\Sigma}_i = \boldsymbol{w}_i\varphi_i\boldsymbol{w}_i'$. Now, equation (8.171) takes the form

$$\boldsymbol{w}_{i+1}\varphi_{i+1}\boldsymbol{w}_{i+1}' = \boldsymbol{F}^*(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')\boldsymbol{w}_i[\varphi_i - \frac{(\varphi_i\boldsymbol{w}_i'\boldsymbol{d})^2}{r_{i+1}}]\boldsymbol{w}_i'(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')'\boldsymbol{F}^{*\prime} \tag{8.173}$$

which implies

$$\boldsymbol{w}_{i+1} = \boldsymbol{F}^*(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')\boldsymbol{w}_i \tag{8.174}$$

$$\varphi_{i+1} = \varphi_i - \frac{(\varphi_i\boldsymbol{w}_i'\boldsymbol{d})^2}{r_{i+1}}. \tag{8.175}$$

The remaining equations are

$$\boldsymbol{\Sigma}_{i+1} = \boldsymbol{\Sigma}_i + \varphi_i\boldsymbol{w}_i\boldsymbol{w}_i' \tag{8.176}$$

$$r_{i+1} = r_i + \varphi_i(\boldsymbol{d}'\boldsymbol{w}_i)^2 \tag{8.177}$$

$$\boldsymbol{k}_{i+1}^f = \boldsymbol{k}_i^f + \frac{(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')\varphi\boldsymbol{w}_i\boldsymbol{w}_i'\boldsymbol{d}}{r_{i+1}} \tag{8.178}$$

In the next step we introduce vectors $\boldsymbol{h}_i$ and $\boldsymbol{l}_i$ as follows

$$\boldsymbol{h}_i = r_i\boldsymbol{k}_i^f,\ \boldsymbol{l}_i = \boldsymbol{w}_i(-\varphi_ir_i)^{\frac{1}{2}}. \tag{8.179}$$

We are now able to eliminate variable φ_i and to transform equations (8.177)-(8.178) to the form defined in equations (8.162)-(8.164). Inserting $\boldsymbol{w}_i = \boldsymbol{l}_i(-\varphi_ir_i)^{-\frac{1}{2}}$ to (8.177) gives

$$r_{i+1} = r_i - \frac{(\boldsymbol{d}'\boldsymbol{l}_i)^2}{r_i}. \tag{8.180}$$

From (8.178) we have

$$\boldsymbol{h}_{i+1} = \boldsymbol{h}_i + \varphi_i\boldsymbol{w}_i\boldsymbol{w}_i'\boldsymbol{d}. \tag{8.181}$$

Finally, from (8.181) equation (8.163) is obtained. Similarly, from (8.175) and (8.177), we have

$$\varphi_{i+1}r_{i+1} = \varphi_ir_i \tag{8.182}$$

while from (8.175) and (8.179) it follows

$$\boldsymbol{l}_{i+1} = \boldsymbol{F}^*(\boldsymbol{I} - \frac{1}{r_i}\boldsymbol{h}_i\boldsymbol{d}')\boldsymbol{l}_i[\frac{\varphi_{i+1}r_{i+1}}{\varphi_ir_i}]^{\frac{1}{2}}. \tag{8.183}$$

As a result, combining (8.183) and (8.182) gives (8.164). □

8.9 Conclusion

In this chapter, a class of predictive control problems has been solved based on an explicit-delay 'innovations-type' state-space process model and a receding-horizon quadratic performance index. The solution consists of two parts.

The first part, which consists in finding the optimal controller gain is connected either with inverting a $N_u \times N_u$ matrix in (8.14) or with calculating the controller gain vector $\boldsymbol{k}_c$ from a combination of Lyapunov and Riccati equations. The computational complexity of the solution that bases on a Riccati equation depends both on the cost horizon N and the system order n and not on the control horizon N_u, and even infinite horizon problems can be solved within this approach.

The second part consists in finding the filtered state variable, which can be accomplished either optimally by using a full Kalman filter (8.29)-(8.33) or only asymptotically optimally by using the time invariant filter in (8.38). The above methods can be combined so that the predictions for n steps ahead are provided by the optimal Kalman filter or the asymptotically optimal filter and further predictions are calculated from the recursive equation (8.26).

It has been shown that Chandrasekhar equations can improve the computational efficiency as compared to Riccati equations because instead of updating n^2 entries of a Riccati matrix only $2n$ entries of two vectors plus one scalar variable are to be updated.

For $n \geq 3$, this reduces the number of calculations. The above savings are particularly important for systems with a large value of the delay/sampling period ratio, and for high-order step-response models.

Vector Chandrasekhar-type equations have been derived for both the controller and filter gain vectors.

9. Intersample Behavior of Controlled Systems

The chapter deals with discrete-time control of continuous-time systems driven by ZOH with pulse amplitude modulation and disturbed by a stationary Gaussian process with a rational spectral density. [1] The algorithms considered have the form of a linear feedback from the Kalman filter. We concentrate on some time functions that characterize the performance of the continuous-time system with discrete feedback. A methodology of their calculation is developed. Some results of the related works in the area are generalized and extended.

9.1 Introduction

The majority of the contemporary digital control algorithms are those designed in discrete-time. Usually the methods of the controller synthesis take the behaviour of a continuous-time system into account only at discrete-time instants, assuming that when the sampling period is small information about the system at sampling instants is sufficient to determine the inter-sample properties of the controlled system.

This assumption results in the controller synthesis based on the discrete model, usually in the form of ARIMAX model. Here the direction started by Åström should be mentioned that includes: minimum-variance controllers (MV), generalized minimum-variance controllers (GMV) (Åström & Wittenmark, 1997), and controllers that minimize one-step and multistep performance indices. LQG (Clarke et al., 1985) and GPC (Clarke et al., 1987; Clarke & Mohtadi, 1989) algorithms are typical members of that group.

Many methods of the controller synthesis (especially when the sampling frequency is high) lead to high-energy, sign changing controls. This results in a discrepancy between the system output behaviour at discrete time instants and its real, i.e. inter-sample behaviour. Also when the sampling period is long, inter-sample values of the continuous time system output can differ significantly from the values measured at sampling instants. As the algorithms are designed for stochastic models, variances of both the system output and the control signal are of great interest.

[1] The chapter is based on (Błachuta, 1996*a*) and (Błachuta & Polańska, 1995).

In (Błachuta & Ordys, 1987; Błachuta & Ordys, 1988; Błachuta & Polański, 1987) it was shown that transients in both GMV and GPC controlled systems can significantly differ, depending on the type of filter used.

The problem of intersample variances in sampled-data systems was addressed in (de Souza & Goodwin, 1984; Lennartson, Söderström & Sun, 1989; Lennartson & Söderström, 1989) and (Williamson, 1991). de Souza & Goodwin (1984) consider only cyclo-stationary output variances when single-stage minimum variance controllers are applied and do not generalize to transients and more advanced controllers. (Lennartson et al., 1989; Lennartson & Söderström, 1989) contain somewhat more general theory applicable to systems with arbitrary controllers based on from time-invariant state state estimator with emphasis put on the steady state. A methodology of the discrete systems design which takes into account the inter-sample phenomena was developed by Williamson (1991). He gave formulae in his book for the output variance of a continuous time system with a discrete-time controller. However, they are also valid in the steady state only.

The chapter extends the results obtained in (Błachuta & Ordys, 1987; Błachuta & Ordys, 1988; Błachuta & Polański, 1987; Ordys, 1993) for discrete-time ARMAX models to the continuous time. It also generalizes the formulae obtained in (Williamson, 1991) to the case of transient states in systems with optimal and asymptotically optimal filters when the state-space representations of GMV, LQG and GPC control laws are used.

The chapter is organized as follows. First, models of continuous time systems with stochastic disturbances are presented. Then a model of discrete-time observations and control as well as a resulting discrete model are discussed. Next a discrete-time controller is introduced as a linear function of the state estimate. Finally, a system of equation is derived which allows calculation of expected values and covariances of the state, control and output variables at arbitrary time instants.

The results are illustrated with examples calculated using algorithms implemented in MATLAB.

9.2 Continuous stochastic processes

A wide class of stochastic processes with a control input can be described by the following system of equations

$$d\boldsymbol{x}_t = (\boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{b}u_t)dt + \boldsymbol{c}d\xi_t \tag{9.1}$$
$$z_t = \boldsymbol{d}'\boldsymbol{x}_t. \tag{9.2}$$

Here z_t is a scalar process, $\boldsymbol{x}_t$ is an n-dimensional state vector, $\boldsymbol{A}$ is a matrix with constant coefficients, $\boldsymbol{c}$, $\boldsymbol{b}$ and $\boldsymbol{d}$ are vectors, u_t is a control input and ξ_t is a standard Wiener process (Gikhman & Skorokhod, 1969; Gikhman & Skorokhod, 1972; Kučera, 1972). The symbol

d stands for the differential. The initial condition $\boldsymbol{x}_0$ is a normally distributed random vector, $\boldsymbol{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{Q}_0)$.

Equations (9.1)-(9.2) can be treated as a compact notation of

$$d\boldsymbol{x}_t^1 = (\boldsymbol{A}^1\boldsymbol{x}_t^1 + \boldsymbol{b}^1u_t)dt \tag{9.3}$$
$$d\boldsymbol{x}_t^2 = \boldsymbol{A}^2\boldsymbol{x}_t^2dt + \boldsymbol{c}^2d\xi_t \tag{9.4}$$
$$z_t = \boldsymbol{d}_1'\boldsymbol{x}_t^1 + \boldsymbol{d}_2'\boldsymbol{x}_t^2. \tag{9.5}$$

Then the first equation is interpreted as a model of the control path while the second as a model of a disturbance.

The solution of equation (9.1) has the form

$$\boldsymbol{x}_t = e^{\boldsymbol{A}t}\boldsymbol{x}_0 + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{b}u_s ds + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{c}d\xi_s. \tag{9.6}$$

The process z_t given by equations (9.1)-(9.2) is completely characterized by its first two moments. The expected values are independent of vector c, and covariances are independent of mean values of both the initial state, $\boldsymbol{\mu}_0$, and control, u_t. This enables us to analyze equations for the first and second moments separately.

In the sequel we will assume that the following Lyapunov equation:

$$\boldsymbol{A}\boldsymbol{Q}_0 + \boldsymbol{Q}_0\boldsymbol{A}' = -\boldsymbol{c}\boldsymbol{c}' \tag{9.7}$$

has a symmetric solution $\boldsymbol{Q}_0 \geq 0$, which ensures the stationarity of the stochastic component. The spectral density $\Sigma(\omega)$ of the stochastic component can then be determined by

$$\Sigma(\omega) = \boldsymbol{d}'(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{c}\boldsymbol{c}'(-s\boldsymbol{I} - \boldsymbol{A}')^{-1}\boldsymbol{d}|_{s=j\omega}. \tag{9.8}$$

$\Sigma(\omega)$ is a real rational function and can be represented as:

$$\Sigma(\omega) = \frac{C(s)C(-s)}{A(s)A(-s)}|_{s=j\omega} = |\frac{C(j\omega)}{A(j\omega)}|^2, \tag{9.9}$$

where:

$$A(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) \tag{9.10}$$
$$C(s) = \boldsymbol{d}'[\operatorname{adj}(s\boldsymbol{I} - \boldsymbol{A})]\mathrm{c}. \tag{9.11}$$

From (9.8) and (9.11) it is seen that for a given spectral density function $\Sigma(\omega)$ of the process z_t (with $u_t = 0$) and the fixed polynomial $A(s)$ there exist polynomials $C(s)$, and thus vectors $\boldsymbol{c}$, for which the system (9.1)-(9.2) is a model of the process z_t. Among them we always can find such a vector $\boldsymbol{c}$, that all roots of the polynomial $C(s)$ lie in the left half-plane (Åström, 1970).

9.3 Sampling and discrete-time control

In this chapter we consider systems with constant inter-sample controls. Assuming that for $t < s < t+\tau$ there is $u_s = u_t =$ const we can write

$$\boldsymbol{x}_{t+\tau} = e^{A\tau}\boldsymbol{x}_t + u_t \int_0^\tau e^{Av}\boldsymbol{b}dv + \int_0^\tau e^{A(\tau-s)}\boldsymbol{c}d\xi_s. \tag{9.12}$$

The continuous-time signal which is a realization of the continuous-time process is sampled, that means it is measured at discrete, equally spaced time instants $t_i = hi$, where h is a sampling interval and i is an integer. Denote $y_i = y(t_i)$ and $\boldsymbol{x}_i = \boldsymbol{x}(t_i)$. The equation of a sampled process takes the form

$$y_i = \boldsymbol{d}'\boldsymbol{x}_i + r_i, \tag{9.13}$$

where r_i is a discrete, white Gaussian noise, i.e. $\mathrm{E}\,[r_i r_j] = 0$ for $i \neq j$ and $\mathrm{E}\,[r_i^2] = \rho^2$. The process r_i is a model of a measurement error. As the process y_i is discrete-time it can be described by a system of discrete-time stochastic equations

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{g}u_i + \boldsymbol{w}_i \tag{9.14}$$

$$y_i = \boldsymbol{d}'\boldsymbol{x}_i + r_i, \tag{9.15}$$

where $\boldsymbol{w}_i$ is a vector-valued white Gaussian noise with covariance matrix $\boldsymbol{W}$. Matrices $\boldsymbol{F}$, $\boldsymbol{W}$ and vector $\boldsymbol{g}$ are determined as follows

$$\boldsymbol{F} = e^{\boldsymbol{A}h},\ \boldsymbol{g} = \int_0^h e^{As}\boldsymbol{b}ds,\ \boldsymbol{W} = \int_0^h e^{As}\boldsymbol{c}\boldsymbol{c}'e^{A's}ds. \tag{9.16}$$

Let $\boldsymbol{Q}_0$ denote a covariance matrix of a stationary discrete process $\boldsymbol{x}_i$ with $u_i = 0$, $i = 0, 1, \ldots$ Then the matrix $\boldsymbol{Q}_0$ fulfills the discrete Lyapunov equation

$$\boldsymbol{Q}_0 = \boldsymbol{F}\boldsymbol{Q}_0\boldsymbol{F}' + \boldsymbol{W}. \tag{9.17}$$

Since the vectors $\boldsymbol{x}_t$ and $\boldsymbol{x}_i$ are the same at $t = ih$, their covariance matrices are equal at $t = 0$. From this property a method of computing the matrix $\boldsymbol{W}$ results in which we first calculate $\boldsymbol{Q}_0$ from the continuous-time Lyapunov equation in (9.7) and then from (9.17) we have

$$\boldsymbol{W} = \boldsymbol{Q}_0 - \boldsymbol{F}\boldsymbol{Q}_0\boldsymbol{F}'. \tag{9.18}$$

9.4 Control algorithms

In this section we discuss a basic control problem that leads to the state-space LQG or GPC algorithm in the form of a linear feedback from the state estimate:

$$u_i = -\boldsymbol{k}_i'\hat{\boldsymbol{x}}_{i|i}. \tag{9.19}$$

Let I_i be a performance index with the receding horizon N

$$I_i = \mathrm{E}\sum_{j=i}^{i+N-1} y_{j+1}^2 + \lambda \sum_{j=i}^{i+N_u-1} u_j^2, \tag{9.20}$$

where $N_u \leq N$, with the additional assumption that:

$$u_j = 0 \text{ for } j > N_u. \tag{9.21}$$

The horizon N is called a cost horizon while N_u - a control horizon. Depending on the values N, N_u and the relations between them various control laws can be obtained. For $N = N_u$ we get a general LQ problem with either one-step (Anderson & Moore, 1979; Åström, 1970; Błachuta, 1987; Błachuta & Ordys, 1987; Błachuta & Ordys, 1988), or multistep finite (Clarke et al., 1985) and infinite (Åström, 1970; Kučera, 1972) horizon. In the case of $N_u < N$ the GPC algorithm (Clarke et al., 1987; Clarke & Mohtadi, 1989) is obtained. We have the following theorem.

Theorem 9.4.1. *The optimal control u_i is determined by a linear relation*

$$u_i = -(\boldsymbol{k}_* + \frac{1}{r}\boldsymbol{d}_1)'\hat{\boldsymbol{x}}_{i|i}, \tag{9.22}$$

where

$$\boldsymbol{k}_* = \frac{\boldsymbol{F}_*'\boldsymbol{P}_0\boldsymbol{g}_,}{r + \boldsymbol{g}'\boldsymbol{P}_1\boldsymbol{g}}, \tag{9.23}$$

and $\boldsymbol{P}_0$ is obtained from the following recursive equations:

(i) Lyapunov for $j = N-1, \ldots N_u$:

$$\boldsymbol{P}_j = \boldsymbol{F}'\boldsymbol{P}_{j+1}\boldsymbol{F} + \boldsymbol{d}\boldsymbol{d}', \boldsymbol{P}_N = 0, \tag{9.24}$$

(ii) Riccati for $j = N_u - 1, \ldots, 0$:

$$\boldsymbol{P}_j = \boldsymbol{F}_*'(\boldsymbol{P}_{j+1} - \frac{\boldsymbol{P}_{j+1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_{j+1}}{r + \boldsymbol{g}'\boldsymbol{P}_{j+1}\boldsymbol{g}})\boldsymbol{F}_* + \boldsymbol{Q}_*. \tag{9.25}$$

Matrices $\boldsymbol{F}_*$, $\boldsymbol{Q}_*$ and scalar r are determined as follows

$$\boldsymbol{F}_* = \boldsymbol{F} - \frac{g_0}{r}\boldsymbol{g}\boldsymbol{d}_1,\ \boldsymbol{Q}_* = \frac{\lambda}{r}\boldsymbol{d}_1\boldsymbol{d}_1',\ r = g_0^2 + \lambda, \tag{9.26}$$

where $\boldsymbol{d}_1 = \boldsymbol{F}'\boldsymbol{d}$ and $g_0 = \boldsymbol{d}'\boldsymbol{g}$. For an infinite horizon problem matrix $\boldsymbol{P}$ is the positive definite solution of the algebraic Riccati equation

$$\boldsymbol{P} = \boldsymbol{F}_*'(\boldsymbol{P} - \frac{\boldsymbol{P}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}}{r + \boldsymbol{g}'\boldsymbol{P}\boldsymbol{g}})\boldsymbol{F}_* + \boldsymbol{Q}_*. \tag{9.27}$$

Remark 9.4.1. If $\lambda \neq 0$ a solution of equation (9.27) can be found using classical methods. In the case of $\lambda = 0$ we have $\boldsymbol{Q}_* = 0$ and then the solution $\boldsymbol{P} = 0$ is only stabilizing for minimum phase plants, i.e. plants with stable matrix $\boldsymbol{F}_*$. For discrete-time, nonminimum-phase plants specialized algorithms should be used (Kučera, 1972) to find the stabilizing $\boldsymbol{P} > 0$.

In the case of stable matrix $\boldsymbol{F}$ infinite cost horizon $N \to \infty$ can be assumed in the GPC algorithm with a finite value of N_u. Then, instead of recursive Lyapunov equation (9.24) the algebraic equation

$$\boldsymbol{P} = \boldsymbol{F}'\boldsymbol{P}\boldsymbol{F} + \boldsymbol{d}\boldsymbol{d}' \tag{9.28}$$

is to be solved and the solution is to be used for initialization of the recursive Riccati equation (9.25). It should be stressed that usually the GPC algorithm is derived in different way, where the ARIMAX model and a nonzero set-point are assumed.

9.4.1 Non-zero set-point

The above algorithms can also be used as a solution to a control problem with the reference signal changing in the step-wise manner when the moments of change are not known a'priori.

In the steady state the expected values fulfill the following system of equations:

$$\begin{aligned} \boldsymbol{A}\bar{\boldsymbol{x}}_\infty &= -\boldsymbol{b}\bar{u}_\infty && (9.29)\\ \boldsymbol{d}'\bar{\boldsymbol{x}}_\infty &= w && (9.30)\\ \boldsymbol{d}'\boldsymbol{A}\bar{\boldsymbol{x}}_\infty &= 0 && (9.31)\\ &\vdots \\ \boldsymbol{d}'\boldsymbol{A}^m\bar{\boldsymbol{x}}_\infty &= 0. && (9.32)\end{aligned}$$

Since for the systems with m poles at the origin the rank of matrix $\boldsymbol{A}$ is $n-m$, then to obtain the solution $\bar{\boldsymbol{x}}_\infty$ when the system is of Type m, $m-1$ equations of type (9.31) should be added and $\bar{u}_\infty = 0$ should be set. For systems of Type 0, $\bar{\boldsymbol{x}}_\infty$ and $\bar{u}_\infty$ can be found by solving the system of only two equations (9.29), (9.30).

Consider the following control problem:

$$\begin{aligned} \boldsymbol{x}_{i+1} &= \boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{g}u_i + \boldsymbol{w}_i,\ \boldsymbol{x}_0 \sim \mathcal{N}(0, \boldsymbol{Q}_0) && (9.33)\\ z_i &= \boldsymbol{d}'\boldsymbol{x}_i + r_i && (9.34)\\ I_i &= E\sum_{j=i}^{i+N-1}(y_{j+1} - w)^2 + \lambda\sum_{j=i}^{i+Nu-1}(u_j - \bar{u}_\infty)^2. && (9.35)\end{aligned}$$

Instead of solving this problem we can solve a problem in increments

$$\begin{aligned} \delta\boldsymbol{x}_{i+1} &= \boldsymbol{F}\delta\boldsymbol{x}_i + \boldsymbol{g}\delta u_i + \boldsymbol{w}_i,\ \delta\boldsymbol{x}_0 \sim \mathcal{N}(-\bar{\boldsymbol{x}}_\infty, \boldsymbol{Q}_0) && (9.36)\\ \delta z_i &= \boldsymbol{d}'\delta\boldsymbol{x}_i + r_i && (9.37)\\ I_i &= \mathrm{E}\sum_{j=i}^{i+N-1}\delta y_{j+1}^2 + \lambda\sum_{j=i}^{i+Nu-1}\delta u_j^2. && (9.38)\end{aligned}$$

It has exactly the same form as (9.14)-(9.15) and (9.20). The optimal control for the problem (9.33)-(9.35) can be obtained from the solution of the problem (9.36)-(9.38) as

$$\begin{aligned} u_i &= \bar{u}_\infty + \delta u_i && (9.39)\\ z_i &= w + \delta z_i. && (9.40)\end{aligned}$$

The optimal control of a system having at least one pole at the origin does not require any calculation of $\bar{u}_\infty$. Therefore it is more convenient and less sensitive.

In the case of Type 0 plants, the same effect can be obtained by introducing a discrete integral element to the controller. This corresponds to the approach to the LQG and GPC problems which is presented in (Clarke et al., 1985), (Clarke et al., 1987) and (Clarke & Mohtadi, 1989). The increments δu_i

$$\delta u_i = u_i - u_{i-1}. \tag{9.41}$$

should then be interpreted as a control variable for system (9.14)-(9.15) augmented, for the sake of the controller synthesis, by (9.41) with u_{i-1} regarded as an additional state variable.

9.5 State estimation

Denote by $\hat{\boldsymbol{x}}_{i|i-1}$ and $\hat{\boldsymbol{x}}_{i|i}$ linear state estimates that minimize mean square error of the state vector $\boldsymbol{x}_i$, given measurements up to the instants $i-1$ and i, respectively. They are produced by the Kalman filter defined by a set of equations, (Anderson & Moore, 1979)

$$\begin{aligned} \hat{\boldsymbol{x}}_{i|i} &= \hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{h}_i(y_i - \boldsymbol{d}'\hat{\boldsymbol{x}}_{i|i-1}),\ \hat{\boldsymbol{x}}_{0|-1} = \boldsymbol{\mu}_0 && (9.42)\\ \hat{\boldsymbol{x}}_{i+1|i} &= \boldsymbol{F}\hat{\boldsymbol{x}}_{i|i} + \boldsymbol{g}u_i. && (9.43)\end{aligned}$$

Vector $\boldsymbol{h}_i$ is determined by covariances $\boldsymbol{\Sigma}_{i|i}$ and $\boldsymbol{\Sigma}_{i|i-1}$ of the state estimate errors as follows

$$\boldsymbol{h}_i = \frac{\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}}{\rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}} \tag{9.44}$$

$$\begin{aligned} \boldsymbol{\Sigma}_{i|i} &= \boldsymbol{\Sigma}_{i|i-1} - \frac{\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}\boldsymbol{d}'\boldsymbol{\Sigma}_{i|i-1}}{\rho^2 + \boldsymbol{d}'\boldsymbol{\Sigma}_{i|i-1}\boldsymbol{d}},\ \boldsymbol{\Sigma}_{0|-1} = \boldsymbol{Q}_0 && (9.45)\\ \boldsymbol{\Sigma}_{i+1|i} &= \boldsymbol{W} + \boldsymbol{F}\boldsymbol{\Sigma}_{i|i}\boldsymbol{F}'. && (9.46)\end{aligned}$$

9.5.1 Time-independent estimator

In applications (Kwakernaak & Sivan, 1972) recursive equations (9.44)-(9.46) are often replaced by the following algebraic Riccati equation

$$\boldsymbol{\Sigma} = \boldsymbol{W} + \boldsymbol{F}(\boldsymbol{\Sigma} - \frac{\boldsymbol{\Sigma dd'\Sigma}}{\rho^2 + \boldsymbol{d'\Sigma d}})\boldsymbol{F}'. \tag{9.47}$$

Filter parameters are then time-independent and are given by

$$\boldsymbol{h} = \boldsymbol{F\Sigma d}/\sigma^2,\ \sigma^2 = \rho^2 + \boldsymbol{d'\Sigma d}, \tag{9.48}$$

where $\boldsymbol{\Sigma}$ is the non-negative definite symmetric solution of the equation in (9.47). The filter is only asymptotically optimal. It is, however, of a great practical importance because, when used for control, it gives controllers with time-independent parameters. On the basis of (9.42)-(9.43), the predictor equations take the following form

$$\hat{\boldsymbol{x}}_{i+1|i} = \boldsymbol{F}\hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{g}u_i + \boldsymbol{h}e_i,\ \hat{\boldsymbol{x}}_{0|-1} = \boldsymbol{\mu}_0 \tag{9.49}$$

$$y_i = \boldsymbol{d}'\boldsymbol{x}_i + e_i. \tag{9.50}$$

9.6 Characteristics at sampling instants

In this section we will give the formulae that enable us to calculate the evolution of the mean value and covariance matrix of the state vector of the closed loop system at sampling instants. It is also a starting point to the calculation of the inter-sample mean and variance of the output and control signals.

Denote

$$\bar{\boldsymbol{x}}_i = \mathrm{E}(\boldsymbol{x}_i),\ \bar{z}_i = \mathrm{E}(z_i),\ \bar{u}_i = -\boldsymbol{k}_i'\bar{\boldsymbol{x}}_i \tag{9.51}$$

$$\boldsymbol{\theta}_i = \mathrm{E}(\boldsymbol{x}_i - \bar{\boldsymbol{x}}_i)(\boldsymbol{x}_i - \bar{\boldsymbol{x}}_i)',\ \nu_i^2 = \mathrm{E}(z_i - \bar{z}_i)^2,\ \mu_i^2 = \mathrm{E}(u_i - \bar{u}_i)^2. \tag{9.52}$$

Theorem 9.6.1. *Assuming that a control system contains the true Kalman filter (9.42)-(9.46), then the state covariance $\boldsymbol{\theta}_i$, output variances ν_i^2 and control variances μ_i^2 can be written as follows*

$$\boldsymbol{\theta}_{i+1} = (\boldsymbol{F} - \boldsymbol{gk}_i')\boldsymbol{\theta}_i(\boldsymbol{F} - \boldsymbol{gk}_i') + \boldsymbol{gk}_i'\boldsymbol{\Sigma}_{i|i}\boldsymbol{k}_i\boldsymbol{g}' + \boldsymbol{W} \tag{9.53}$$

$$\nu_i^2 = \boldsymbol{d}'\boldsymbol{\theta}_i\boldsymbol{d} \tag{9.54}$$

$$\mu_i^2 = \boldsymbol{k}_i'(\boldsymbol{\theta}_i + \boldsymbol{\Sigma}_{i|i})\boldsymbol{k}_i. \tag{9.55}$$

A variant of the above formula of Theorem 9.6.1, valid for the steady state, is given in (Williamson, 1991). The formulae in (9.53)-(9.55) are true under the assumption that $\mathrm{E}(\boldsymbol{x}_i\tilde{\boldsymbol{x}}_{i|i}) = 0$. Unfortunately, this assumption is not satisfied in transient process when the time-independent state estimator (9.49)-(9.50) is used. Now we will derive slightly more complicated formulae which do not require that $\mathrm{E}(\boldsymbol{x}_i\tilde{\boldsymbol{x}}_{i|i}) = 0$ and do not use the matrices $\boldsymbol{\Sigma}_{i|i}$. Proceeding this way we will obtain correct results also in the case when the steady-state filter (9.49)-(9.50) is used.

Theorem 9.6.2. *For the expected values there is:*

$$E(\bar{\boldsymbol{x}}_{i|i-i}) = 0 \tag{9.56}$$

$$\bar{\boldsymbol{x}}_{i+1} = (\boldsymbol{F} - \boldsymbol{gk}_i{}')\bar{\boldsymbol{x}}_i,\ \bar{\boldsymbol{x}}_0 = \boldsymbol{\mu}_0. \tag{9.57}$$

Covariance matrix $\boldsymbol{Q}_i$ results from the recursive Lyapunov equation

$$\boldsymbol{\Theta}_{i+1} = \boldsymbol{\Omega}_i\boldsymbol{\Theta}_i\boldsymbol{\Omega}_i' + \boldsymbol{\Gamma}_i\boldsymbol{V}_i\boldsymbol{\Gamma}_i'. \tag{9.58}$$

The corresponding relations for the output z_i and control u_i are:

$$\bar{z}_i = \boldsymbol{d}'\bar{\boldsymbol{x}}_i,\ \bar{u}_i = -\boldsymbol{k}_i'\bar{\boldsymbol{x}}_i \tag{9.59}$$

$$\nu_i^2 = \boldsymbol{d}'\boldsymbol{\Theta}_i^{11}\boldsymbol{d} \tag{9.60}$$

$$\mu_i^2 = \boldsymbol{k}_i'\boldsymbol{\Theta}_i^{11}\boldsymbol{k}_i + \boldsymbol{l}_i'\boldsymbol{\Theta}_i^{22}\boldsymbol{l}_i + 2\boldsymbol{k}_i'\boldsymbol{\Theta}_{12}\boldsymbol{l}_i + \alpha_i^2\rho^2, \tag{9.61}$$

where

$$\boldsymbol{l}_i = \boldsymbol{k}_i - \alpha_i\boldsymbol{d},\ \alpha_i = \boldsymbol{k}_i'\boldsymbol{h}_i. \tag{9.62}$$

Matrices $\boldsymbol{\Omega}_i$, $\boldsymbol{\Gamma}_i$ and $\boldsymbol{V}_i$ are block matrices with the following pattern

$$\boldsymbol{\Omega}_i = \begin{bmatrix} \boldsymbol{F} - \boldsymbol{gk}_i' & \boldsymbol{gk}_i'(\boldsymbol{I} - \boldsymbol{h}_i\boldsymbol{d}') \\ 0 & \boldsymbol{F}(\boldsymbol{I} - \boldsymbol{h}_i\boldsymbol{d}') \end{bmatrix} \tag{9.63}$$

$$\boldsymbol{\Gamma}_i = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{gk}_i' \\ \boldsymbol{I} & -\boldsymbol{F} \end{bmatrix},\ \boldsymbol{V}_i = \begin{bmatrix} \boldsymbol{W} & 0 \\ 0 & \boldsymbol{h}_i\boldsymbol{h}_i'\rho \end{bmatrix}. \tag{9.64}$$

Matrix $\boldsymbol{Q}_0$ is also a block matrix with

$$\boldsymbol{\Theta}_0^{11} = \boldsymbol{\Theta}_0^{22} = \boldsymbol{\Theta}_0^{12} = \boldsymbol{\Theta}_0^{21} = \boldsymbol{Q}_0, \tag{9.65}$$

where $\boldsymbol{Q}_0$ is the solution to the Lyapunov equation (9.7).

Proof. Denote

$$\boldsymbol{X} = [\boldsymbol{x}_i', \bar{\boldsymbol{x}}_{i|i-1}]' \tag{9.66}$$

$$\delta\boldsymbol{X} = [(\boldsymbol{x}_i - \bar{\boldsymbol{x}}_i)',\ \tilde{\boldsymbol{x}}_{i|i-1}]' \tag{9.67}$$

and

$$\boldsymbol{\Theta}_i = \mathrm{E}(\delta \boldsymbol{X}_i \delta \boldsymbol{X}_i'). \tag{9.68}$$

Then the equations of the system (9.14)-(9.15) controlled using algorithm (9.19) and the filter (9.42)-(9.43) or (9.49)-(9.50) can be written in the compact form

$$\boldsymbol{X}_{i+1} = \begin{bmatrix} \boldsymbol{F} - \boldsymbol{g}\boldsymbol{k}_i' & \boldsymbol{g}\boldsymbol{k}_i'(\boldsymbol{I} - \boldsymbol{h}_i\boldsymbol{d}') \\ 0 & \boldsymbol{F}(\boldsymbol{I} - \boldsymbol{h}_i\boldsymbol{d}') \end{bmatrix} \boldsymbol{X}_i + \begin{bmatrix} \boldsymbol{I} - \boldsymbol{g}\boldsymbol{k}_i' \\ \boldsymbol{I} - \boldsymbol{F} \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_i \\ \boldsymbol{h}_i r_i \end{bmatrix} \tag{9.69}$$

or

$$\boldsymbol{X}_{i+1} = \boldsymbol{\Omega}_i \boldsymbol{X}_i + \boldsymbol{\Gamma}_i \boldsymbol{v}_i \tag{9.70}$$

and

$$u_i = -\boldsymbol{k}_i'\boldsymbol{x}_i + \boldsymbol{k}_i'(\boldsymbol{I} - \boldsymbol{h}_i\boldsymbol{d}')\tilde{\boldsymbol{x}}_{i|i-1} - \boldsymbol{k}_i'\boldsymbol{h}_i r_i. \tag{9.71}$$

Now taking into account that $\mathrm{E}\{\boldsymbol{w}_i\} = 0$, $\mathrm{E}\{r_i\} = 0$ and $\mathrm{E}\{\tilde{\boldsymbol{x}}_{0|-1}\} = 0$, we get (9.57)-(9.58) and

$$\delta \boldsymbol{X}_{i+1} = \boldsymbol{\Omega}_i \delta \boldsymbol{X}_i + \boldsymbol{\Gamma}_i \boldsymbol{v}_i. \tag{9.72}$$

In this way equations (9.58) and (9.60)-(9.64) are obtained. Analogously, for variances we have

$$\nu_i^2 = \boldsymbol{d}'\boldsymbol{\Theta}_i^{11}\boldsymbol{d} \tag{9.73}$$

$$\mu_i^2 = \boldsymbol{k}_i'[\boldsymbol{\Theta}_i^{11} + (\boldsymbol{I} - \boldsymbol{h}_i\boldsymbol{d}')\boldsymbol{\Theta}_i^{22}(\boldsymbol{I} - \boldsymbol{d}\boldsymbol{h}_i') + 2\boldsymbol{\Theta}_i^{12}(\boldsymbol{I} - \boldsymbol{d}\boldsymbol{h}_i') + \boldsymbol{h}_i\boldsymbol{h}_i'\rho^2]\boldsymbol{k}_i. \tag{9.74}$$

□

Formula (9.74) can be transformed into the numerically more efficient form of (9.61).

9.7 Inter-sample characteristics

From the relation (9.51) it is seen that the mean values and covariances at inter-sample time instants can be determined by the appropriate values at sampling instants. As a conclusion we get the following result.

Theorem 9.7.1. *The mean value and covariance of the output $z_i(\tau)$ are given by the formulae:*

$$\bar{z}_i(\tau) = \boldsymbol{f}_i^1(\tau)'\bar{\boldsymbol{x}}_i \tag{9.75}$$

$$\begin{aligned} \nu_i^2(\tau) &= \boldsymbol{f}_i^{1\prime}(\tau)\boldsymbol{\Theta}_i^{11}\boldsymbol{f}_i^1(\tau) + \boldsymbol{f}_i^{2\prime}(\tau)\boldsymbol{\Theta}_i^{22}\boldsymbol{f}_i^2(\tau) + 2\boldsymbol{f}_i^{1\prime}(\tau)\boldsymbol{\Theta}_i^{12}\boldsymbol{f}_i^2(\tau) \\ &+ \boldsymbol{d}'\boldsymbol{W}(\tau)\boldsymbol{d} + \rho^2[\gamma_i^{12}(\tau)]^2, \end{aligned} \tag{9.76}$$

where $\bar{\boldsymbol{x}}_i$ and $\boldsymbol{\Theta}_i$ are given by the equations (9.57)-(9.58). The remaining values are determined as follows

$$\boldsymbol{f}_i^1(\tau) = \boldsymbol{f}(\tau) - \gamma(\tau)\boldsymbol{k}_i,\ \boldsymbol{f}_i^2(\tau) = \gamma(\tau)\boldsymbol{l}_i,\ \gamma_i^{12}(\tau) = \gamma(\tau)\alpha_i \tag{9.77}$$

$$\boldsymbol{W}(\tau) = \int_0^\tau e^{\boldsymbol{A}s}\boldsymbol{c}\boldsymbol{c}'e^{\boldsymbol{A}'s}ds \tag{9.78}$$

and

$$\boldsymbol{f}(\tau) = \boldsymbol{d}'\boldsymbol{F}(\tau),\ \gamma(\tau) = \boldsymbol{d}'\boldsymbol{g}(\tau),\ \alpha_i = \boldsymbol{k}_i'\boldsymbol{h}_i,\ \boldsymbol{l}_i = \boldsymbol{k}_i - \alpha_i\boldsymbol{d}. \tag{9.79}$$

9.8 Remarks on Simulation

Simulation of the continuous-time system (9.1)-(9.2) with sampled measurements (9.13) and a discrete-time controller in (9.19) which uses the state estimator (9.43)-(9.47) or (9.50)-(9.51) can be performed in two different ways.

The first method uses the intersample solution of the equation of the controlled process. Assume that the discrete time instants i are assigned the time instants $t_i = ih$. In that case (9.12) can be written for $0 \le \tau \le h$ as follows:

$$\boldsymbol{x}_i(\tau) = \boldsymbol{F}(\tau)\boldsymbol{x}_i + \boldsymbol{g}(\tau)u_i + \boldsymbol{w}(\tau), \tag{9.80}$$

where

$$\boldsymbol{F}(\tau) = e^{\boldsymbol{A}\tau},\ \boldsymbol{g}(\tau) = \int_0^\tau e^{\boldsymbol{A}v}\boldsymbol{b}dv,\ \boldsymbol{w}(\tau) = \int_0^\tau e^{\boldsymbol{A}(\tau-s)}\boldsymbol{c}d\xi(s). \tag{9.81}$$

From (9.80) it follows that for $0 \le \tau \le h$ there is

$$z_i(\tau) = \boldsymbol{f}'(\tau)\boldsymbol{x}_i + \gamma(\tau)u_i + \psi(\tau), \tag{9.82}$$

where

$$\boldsymbol{f}(\tau) = \boldsymbol{d}'\boldsymbol{F}(\tau),\ \gamma(\tau) = \boldsymbol{d}'\boldsymbol{g}(\tau), \tag{9.83}$$

and $\psi(\tau)$ is a white noise with zero mean and variance

$$\sigma_\psi^2(\tau) = \boldsymbol{d}'\boldsymbol{W}(\tau)\boldsymbol{d}. \tag{9.84}$$

Then having the values of the state vector $\boldsymbol{x}_i$ and control $u_i = -\boldsymbol{k}'\hat{\boldsymbol{x}}_{i|i}$ at sampling instants $t_i = ih$ we can find the values of the output $z_i(\tau)$ at an arbitrary time instant $t = ih + \tau$.

The second method can be applied when we are only interested in the output values at N_d equally spaced time instant within the sampling period. It bases directly on equations (9.14)-(9.15). Assuming that the discretization was performed with the period $h_N = h/N_d$ then the closed loop system is simulated with the assumption that during N_d steps the control u_i does not change:

$$u_i = -\boldsymbol{k}_j\hat{\boldsymbol{x}}_{j|j},\ j = i \operatorname{div} N_d. \tag{9.85}$$

In both methods the initial condition $\boldsymbol{x}_0$ and the vector $\boldsymbol{w}_i$ are defined as follows:

$$\begin{aligned} \boldsymbol{x}_0 &= \boldsymbol{\mu}_0 + \boldsymbol{L}\boldsymbol{n}, \; \boldsymbol{w}_i = \boldsymbol{M}\boldsymbol{m}_i \\ \boldsymbol{Q}_0 &= \boldsymbol{L}\boldsymbol{L}', \; \boldsymbol{W} = \boldsymbol{M}\boldsymbol{M}'. \end{aligned} \tag{9.86}$$

Matrices $\boldsymbol{L}$ and $\boldsymbol{M}$ can be calculated using the Cholesky decomposition. Vectors $\boldsymbol{m}_i$ and $\boldsymbol{n}$ are independent zero mean Gaussian variables with unit variances, i.e.

$$\mathrm{E}\{\boldsymbol{m}_i\boldsymbol{m}_i'\} = \boldsymbol{I}, \; i = 0, 1, \ldots, \; \mathrm{E}\{\boldsymbol{n}\boldsymbol{n}'\} = \boldsymbol{I}, \; \mathrm{E}\{\boldsymbol{m}\boldsymbol{n}'\} = 0. \tag{9.87}$$

9.9 Example

A control system was investigated in which the control path is given by the following transfer function:

$$\frac{0.9s + 1}{(s+1)^3}, \tag{9.88}$$

while the disturbance channel is represented by

$$\frac{0.2}{(s+0.1)^2} \tag{9.89}$$

driven by white noise with unit variance. The variance of the measurement error was $\rho^2 = 0.2$. A hybrid LQG control algorithm was used minimizing

$$I = \lim_{T\to\infty} \mathrm{E}\{\frac{1}{T}\int_0^T [w(t) - y(t)]^2 dt\}. \tag{9.90}$$

The sampling period was $h = 0.3$. Both the expected values of output and control signals and variances of these signals were calculated when the reference value $w(t)$ changed at $t = 0$ from 0 to 0.1. Two types of filters were examined: the optimal Kalman filter and its asymptotic, time invariant version. In figures 9.1–9.4 transients obtained with a controller based on optimal filter are presented whereas in figures 9.5–9.6 results obtained with the time invariant filter are displayed.

9.10 Conclusions

Studying the inter-sample behaviour of the continuous-time stochastic systems with discrete-time controllers gives much more insight into the properties of a control process than only restricting attention to the sampling instants. In the chapter tools are derived, which are independent of the type of the filter used, for calculating important characteristics of control systems valid for arbitrary time instants in both transient and stationary states. State-space representations of continuous-time stochastic processes proved to be efficient for controller synthesis, giving unified approach to the commonly used control algorithms, and for calculation of important characteristics.

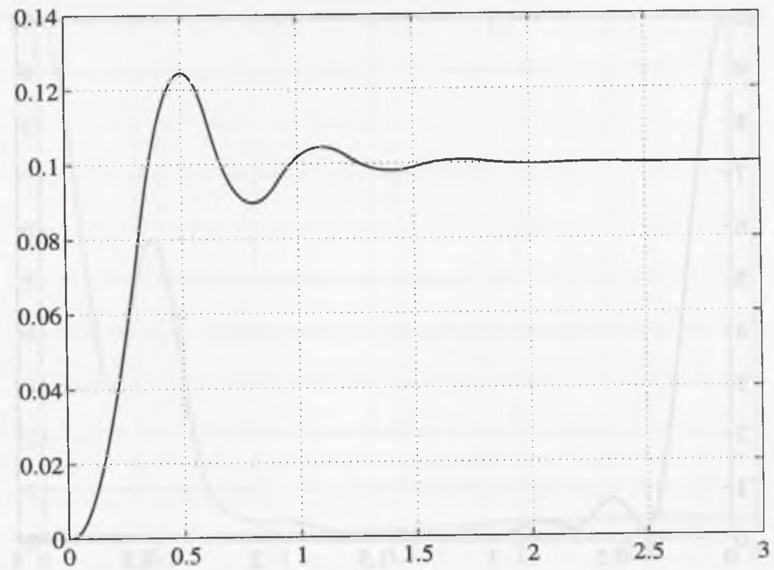

Fig. 9.1. Output mean

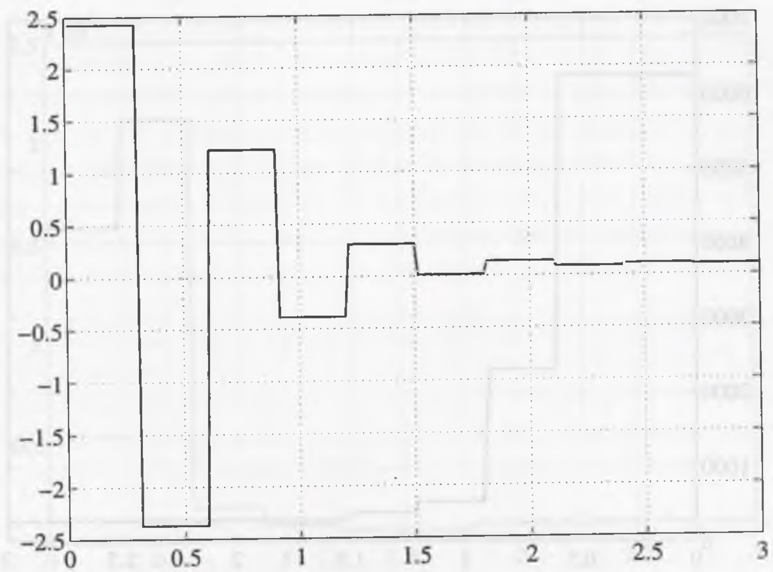

Fig. 9.2. Control mean

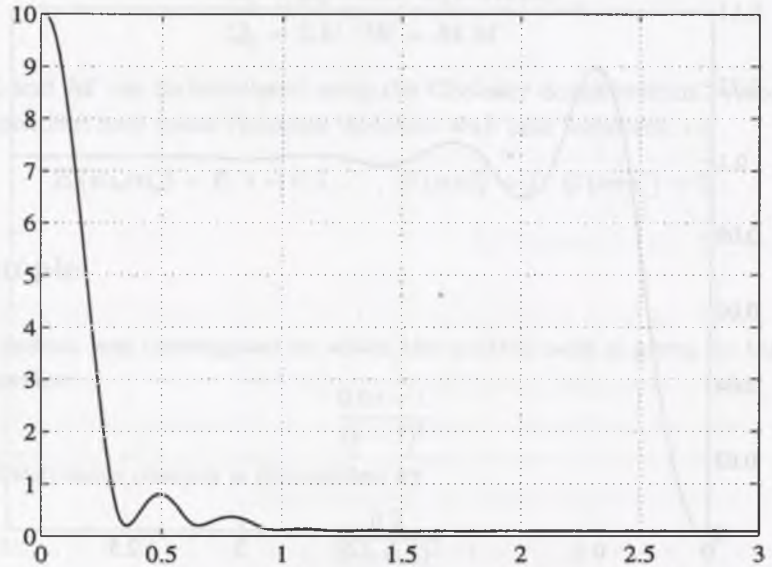

Fig. 9.3. Output variance

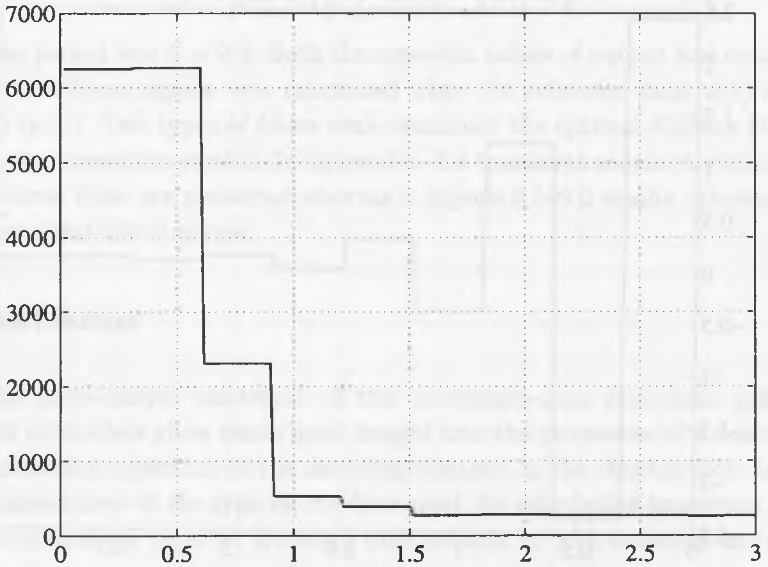

Fig. 9.4. Control variance

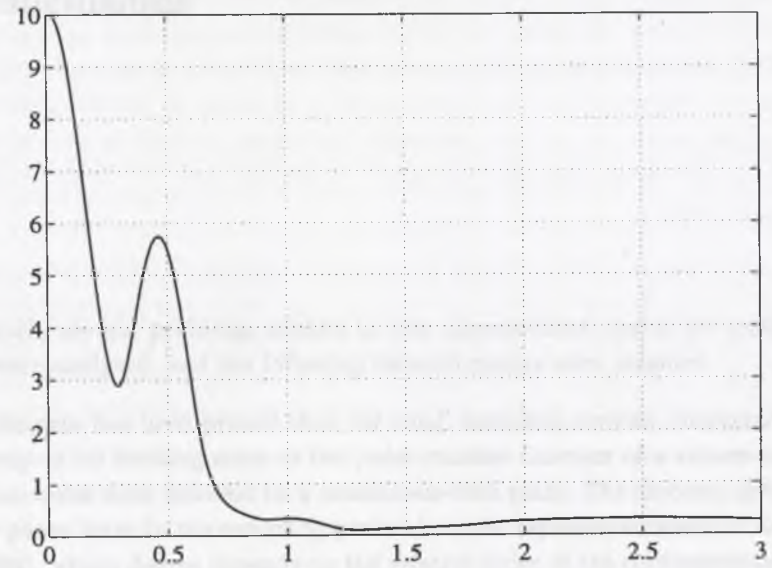

Fig. 9.5. Output variance

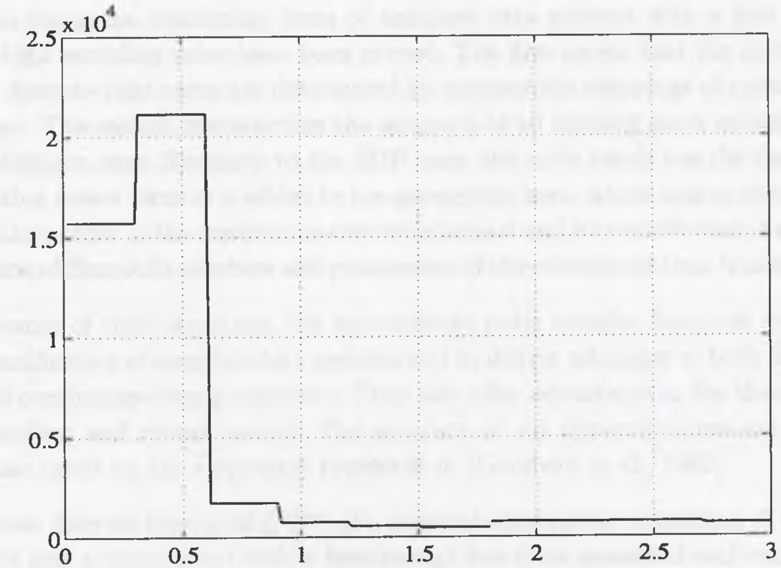

Fig. 9.6. Control variance

10. Conclusions

In this work, several problems related to the discrete-time control of continuous-time systems were analysed, and the following detailed results were attained:

1. A theorem has been proved that, for small sampling periods, characterizes the accuracy of all limiting zeros of the pulse transfer function of a system composed of a zero-order hold followed by a continuous-time plant. The theorem gives a correcting power term in the sampling period h to the asymptotic result of Åström et al. (1984), whose degree depends on the relative order of the continuous-time counterpart and its contribution is expressed in terms of Bernoulli numbers and the poles and zeros of the continuous-time transfer function. The result allows the accuracy of approximate pulse-transfer functions to be determined.
2. Two theorems concerning zeros of sampled data systems with a first order hold at high sampling rates have been proved. The first shows that the limiting intrinsic discrete-time zeros are determined by exponential mappings of continuous-time zeros. The second characterizes the accuracy of all limiting zeros including the discretization ones. Similarly to the ZOH case, the main result has the form of a correcting power term in h added to the asymptotic zero, whose degree depends on the relative order of the continuous-time counterpart and its contribution is expressed in terms of Bernoulli numbers and parameters of the continuous-time transfer function.
3. Because of their structure, the approximate pulse transfer functions are useful for identification of sampled-data systems and to deliver estimates of both discrete-time and continuous-time parameters. They also offer advantages in the theory of model matching and robust control. The accuracy of our approximations are superior to those based on the δ-operator presented in (Goodwin et al., 1986).
4. A new discrete-time model, $\mathcal{D}^-$, of a sampled-data system consisting of a zero-order hold and a linear plant with a feedthrough has been presented and compared with the classical model $\mathcal{D}^+$. It has been shown that because of violation of the closed-loop causality the classical model $\mathcal{D}^+$ related to the right-side limit of the output signal with the transfer function $H^+(z)$ is not feasible for feedback modeling if there

is a feedthrough in both the plant and controller. The new model, $\mathcal{D}^-$, related to the left-side limit of a discontinuous output signal has been shown to be appropriate for modeling feedback systems. Its transfer function $H^-(z)$ appears to be vital for both the return difference and the characteristic polynomial of the closed-loop system. $\mathcal{D}^-$, whose sensitivity to the unmodeled dynamics is small is also better suited for state estimation and observer-based controllers than $\mathcal{D}^+$, whose sensitivity is extremely high.

5. It has been shown that the purely discrete-time approach to the LQR based control systems design suffers from severe disadvantages when the sampling rate becomes high. They demonstrate as 'ringing' and high magnitudes of the control signal. These phenomena are caused by the properties of the sampling zeros of pulse transfer functions at high sampling rates. The proposed method of a hybrid discrete-time controller design does not exhibit these disadvantages. Provided that the order of the desired output model equals to the relative order of the continuous-time system, the control signal tends to a smooth continuous-time function when the sampling rate increases.

6. A class of second-order continuous-time stochastic processes was investigated and the issue of their sampling was discussed. As a result of sampling discrete second-order random processes, described by linear time-invariant state-space models with a vector input were obtained. Furthermore, a set of simple representations covariance equivalent with a vector driven model was proposed. They rely on two sources of randomness. The first is a scalar driving noise v_i, and the second is the n-dimensional initial random vector $\boldsymbol{x}_0^*$. These representations are distinct from the innovations representation. Moreover, they are time invariant, which is an advantage when using them in simulation, prediction, and parameter estimation.

7. A class of predictive control problems has been solved based on an explicit-delay 'innovations-type' state-space process model and a receding-horizon quadratic performance index. The solution consists of two parts. The first part, which consists in finding the optimal controller gain is connected either with inverting a $N_u \times N_u$ matrix or with calculating the controller gain vector $\boldsymbol{k}_c$ from a combination of Lyapunov and Riccati equations. The computational complexity of the solution that bases on a Riccati equation depends both on the cost horizon N and the system order n and not on the control horizon N_u, and even infinite horizon problems can be solved within this approach. The second part consists in finding the filtered state variable, which can be accomplished either optimally by using a full Kalman filter or only asymptotically optimally by using a time invariant filter. It has been shown that for $n \geq 3$ Chandrasekhar equations improve the computational efficiency as compared to Riccati equations because instead of updating n^2 entries of a Riccati matrix only $2n$ entries of two vectors plus one scalar variable are to be updated. Vector Chandrasekhar-type equations have been derived for both the controller and filter gain vectors.

8. A study of the inter-sample behavior of the continuous-time stochastic systems with discrete-time controllers gives much more insight into the properties of a control process than only restricting attention to the sampling instants. Tools are derived, which are independent of the type of the filter used, for calculating important characteristics of control systems valid for arbitrary time instants in both transient and stationary states. State-space representations of continuous-time stochastic processes proved to be efficient for controller synthesis, giving unified approach to the commonly used control algorithms, and for calculation of important characteristics.

A. Proofs

A.1 Proofs for Chapter 2

For small h and any finite s, s_1, $s_2 \in \mathbb{C}$ the following obvious identities hold:

$$e^{s_1 h} - e^{s_2 h} = (s_1 - s_2)h + o(h) \tag{A.1}$$

$$1 - e^{-sh} = sh + o(h), \tag{A.2}$$

which will be used in what follows.

Proof of Lemma 2.2.1 From (2.6)-(2.8) we have

$$\Delta_h(s) = \frac{1 - e^{-sh}}{h} \sum_{l=1}^{\infty} \gamma_l(s, h), \tag{A.3}$$

where

$$\gamma_l(s, h) = \frac{G(s - jl\omega_s)}{s - jl\omega_s} + \frac{G(s + jl\omega_s)}{s + jl\omega_s}. \tag{A.4}$$

Inserting (2.22) into (A.4) yields:

$$\begin{aligned} \gamma_l(s, h) = {} & g_k h^{k+1} \frac{(sh + j2\pi l)^{k+1} + (sh - j2\pi l)^{k+1}}{[(sh)^2 + (2\pi l)^2]^{k+1}} \\ & + g_{k+1} h^{k+2} \frac{(sh + j2\pi l)^{k+2} + (sh - j2\pi l)^{k+2}}{[(sh)^2 + (2\pi l)^2]^{k+2}} + o(h^{k+2}). \end{aligned} \tag{A.5}$$

On expanding the binomials in (A.5), reducing common terms due to $(j)^q = (-j)^q = (-1)^{\frac{q}{2}}$ for q even and $(j)^q = -(-j)^q$ for q odd, and taking only the smallest powers of h one gets:

$$\gamma_l(s, h) = \begin{cases} (-1)^{\frac{k+1}{2}} 2 g_k (\frac{h}{2\pi l})^{k+1} + o(h^{k+1}), & k \text{ odd} \\ (-1)^{\frac{k}{2}} 2[(k+1)s g_k - g_{k+1}](\frac{h}{2\pi l})^{k+2} + o(h^{k+2}), & k \text{ even.} \end{cases} \tag{A.6}$$

Calculating the sum of the infinite series in (A.3) and employing (A.2) yields:

$$\phi_1(k,s) = (-1)^{\frac{k+1}{2}} 2 s g_k \frac{\zeta(k+1)}{(2\pi)^{k+1}} + o(h^{k+1}) \tag{A.7}$$

$$\phi_2(k,s) = (-1)^{\frac{k}{2}} 2s[(k+1)s g_k - g_{k+1}]\frac{\zeta(k+2)}{(2\pi)^{k+2}} + o(h^{k+2}), \tag{A.8}$$

where $\zeta(x) = \sum_{n=0}^{\infty} 1/n^x$ is the Riemann zeta function (Edwards, 1974; Titchmarsh, 1986). For even x, i.e. $x = 2i$, there is:

$$\zeta(2i) = \frac{(2\pi)^{2i}|B_{2i}|}{2(2i)!}. \tag{A.9}$$

The proof is completed by taking (2.23) and properties of B_{2i} collected in Remark 2.2.1 into account. □

Proof of Lemma 2.2.2 Equation (2.22) can be seen as a result of the expansion

$$G(s) = \sum_{i=k}^{\infty} \frac{g_i}{s^i} \tag{A.10}$$

of $G(s)$ about $s = \infty$, where g_i are the Markov parameters of $G(s)$, which can be calculated for any i recursively from:

$$g_k = \frac{\beta_m}{\alpha_n}, \; g_{i+k} = \frac{1}{\alpha_n}(\beta_{m-i} - \sum_{j=1}^{i} \alpha_{n-j} g_{j+k-1}), \tag{A.11}$$

or, for $i \geq n$, from:

$$\alpha_n g_{i+k} + \alpha_{n-1} g_{i+k-1} \cdots + \alpha_0 g_{i-m} = 0. \tag{A.12}$$

Since the characteristic polynomial of the difference equation (A.12) is the same as that of the continuous-time system $G(s)$, the Markov parameters g_i are related to the poles π_i by the formula

$$g_{k+i} = \sum_{j=1}^{n} c_j \pi_j^i, \tag{A.13}$$

where c_j are choosen so as to match the first $n-1$ values of the Markov parameters defined in (A.11). For $|s| > \max|\mathrm{Re}\, \pi_j|, j = 1, 2, \ldots n$ the series in (A.10) is convergent and the rest of the proof is immediate. □

Proof of Lemma 2.2.3 Equation (2.24) follows directly from (2.22) with

$$\Delta H(z) = (1 - z^{-1}) \mathcal{Z}_h(\frac{\Delta G(s)}{s}). \tag{A.14}$$

On the other hand, from (A.10) it follows that

$$\Delta H(z) = h^{k+1} \delta(z, h), \tag{A.15}$$

where

$$\delta(z,h) = \sum_{i=2}^{\infty} \frac{g_{k+i}}{\mathcal{E}_{k+1}(1)} \frac{\mathcal{E}_{k+i}(z)}{(z-1)^{k+i}} h^{i-1}. \tag{A.16}$$

As a result, $\lim_{h \to 0} \delta(z,h) = 0$, which proves (2.25). □

Proof of Theorem 2.3.1 To prove item (i) assume that $G(s)$ is of Type $l \geq 0$. From the fact that the steady-state properties are preserved in the sampled-data system, i.e.:

$$\lim_{z \to 1} (\frac{z-1}{h})^l H(z) = \lim_{s \to 0} s^l G(s) \tag{A.17}$$

it results that the coefficient b_{n-1} in eqn. (2.4) has the form

$$b_{n-1} = h^l \frac{\prod_{i=1}^{n-l}(1 - p_i)}{\prod_{i=1}^{n-1}(1 - z_i)} \tag{A.18}$$

or, taking (2.11) into account:

$$b_{n-1} = \frac{h^l}{E_k(1)} \frac{\prod_{i=1}^{n-l}(1 - p_i)}{\prod_{i=1}^{m}(1 - z_i)}. \tag{A.19}$$

Applying (A.1)-(A.2), equations (2.2), (2.15) and the relationship $\lim_{h \to 0} E_k(z) = \mathcal{E}_k(z)$ yields eqn. (2.27) proving item (i).

Assume that the continuous-time transfer function has a multiple zero with multiplicity μ. Denote $\mathcal{J} = \{j, j+1, \cdots j + \mu - 1\}$ a set of integers indicating those zeros. Insert the asymptotic zero $z_j' = e^{\sigma_j h}$ into $H(z)$. Then, according to eqn. (2.7), one gets:

$$H(z_j') = b_{n-1} E_k(z_j') \frac{\prod_{i=1}^{m}(z_j' - z_i)}{\prod_{i=1}^{n}(z_j' - p_i)} = \Delta_h(\sigma_j), \tag{A.20}$$

from which

$$\prod_{i=0}^{\mu-1}(z_j' - z_{j+i}) = \frac{\prod_{i=1}^{n}(z_j' - p_i)}{\prod_{i=1, i \notin \mathcal{J}}^{m}(z_j' - z_i)} \frac{\Delta_h(\sigma_j)}{b_{n-1} E_k(z_j')}. \tag{A.21}$$

Applying eq. A.2 to $z_j' - p_i$ and $z_j' - z_i$ gives

$$\prod_{i=0}^{\mu-1}(z_j' - z_{j+i}) = [\frac{\prod_{i=1}^{n}(\sigma_j - \pi_i)}{\prod_{i=1, i \notin \mathcal{J}}^{m}(\sigma_j - \sigma_i)} h^{k+\mu} + o(h^{k+\mu})] \frac{\Delta_h(\sigma_j)}{b_{n-1} E_k(z_j')}. \tag{A.22}$$

Finally, employing (2.33) and inserting b_{n-1} from (2.27) yields (2.28). This proves item (*ii*) of Theorem 2.3.1.

Inserting ζ_j' into (2.4) yields:

$$H(\zeta_j') = b_{n-1} E_k(\zeta_j') \frac{\prod_{i=1}^{m} (\zeta_j' - z_i)}{\prod_{i=1}^{n} (\zeta_j' - p_i)}, \tag{A.23}$$

while inserting ζ_j' into (2.24) of Lemma 2.2.3 returns:

$$H(\zeta_j') = g_{k+1} \frac{h^{k+1}}{\mathcal{E}_{k+1}(1)} \frac{\mathcal{E}_{k+1}(\zeta_j')}{(\zeta_j' - 1)^{k+1}} + o(h^{k+1}). \tag{A.24}$$

For small h, equation (A.23) can be written in the form:

$$H(\zeta_j') = \left\{ \frac{g_k}{k!} h^k + o(h^k) \right\} \frac{E_k(\zeta_j')}{(\zeta_j' - 1)^k} [1 + o(h)]. \tag{A.25}$$

A comparison of (A.24) and (A.25) leads to

$$E_k(\zeta_j') = \frac{g_{k+1}}{g_k} \frac{\mathcal{E}_{k+1}(\zeta_j')}{k+1} \frac{1}{\zeta_j' - 1} h + o(h). \tag{A.26}$$

Finally, taking (2.11), (2.23) and $\zeta_j - \zeta_i' \to \zeta_j' - \zeta_i' \neq 0$ for $i \neq j$ into account yields (2.29) with (2.30) which proves item (*iii*) of the Theorem 2.3.1. □

A.2 Proofs for Chapter 3

Proof of Lemma 3.3.1 From (3.8)-(3.10) we have

$$\Delta_h(s) = \frac{(1 - e^{-sh})^2}{h^2} \sum_{l=1}^{\infty} \gamma_l(s, h) \tag{A.27}$$

with

$$\gamma_l(s, h) = \delta_l(s, h) + h\epsilon_l(s, h), \tag{A.28}$$

where

$$\delta_l(s, h) = \frac{G(s - jl\omega_s)}{(s - jl\omega_s)^2} + \frac{G(s + jl\omega_s)}{(s + jl\omega_s)^2} \tag{A.29}$$

$$\epsilon_l(s, h) = \frac{G(s - jl\omega_s)}{s - jl\omega_s} + \frac{G(s + jl\omega_s)}{s + jl\omega_s}. \tag{A.30}$$

Using expansion (2.22) and performing some calculations yields:

$$\gamma_l(s, h) = \begin{cases} (-1)^{\frac{k+1}{2}} 2 g_k \dfrac{h^{k+2}}{(2\pi l)^{k+1}} + o(h^{k+2}), & k \text{ odd} \\ (-1)^{\frac{k+2}{2}} 2 g_k \dfrac{h^{k+2}}{(2\pi l)^{k+2}} + o(h^{k+2}), & k \text{ even.} \end{cases} \tag{A.31}$$

Calculating the sum of the infinite series in (A.27) and employing (A.2) yields:

$$\Delta_h(s) = \begin{cases} (-1)^{\frac{k+1}{2}} \zeta(k+1) 2 s^2 g_k \dfrac{h^{k+2}}{(2\pi)^{k+1}} + o(h^{k+2}) \\ (-1)^{\frac{k+2}{2}} \zeta(k+2) 2 s^2 g_k \dfrac{h^{k+2}}{(2\pi)^{k+2}} + o(h^{k+2}) \end{cases} \tag{A.32}$$

with the first row for k odd and the second for k even, where $\zeta(x) = \sum_{n=0}^{\infty} 1/n^x$ is the Riemann zeta function (Edwards, 1974; Titchmarsh, 1986).

For even x, i.e. $x = 2i$, there is:

$$\zeta(2i) = \frac{(2\pi)^{2i} |B_{2i}|}{2(2i)!}, \tag{A.33}$$

upon which

$$\Delta_h(s) = \begin{cases} (-1)^{\frac{k+1}{2}} \dfrac{|B_{k+1}|}{(k+1)!} s^2 g_k h^{k+2} + o(h^{k+2}) \\ (-1)^{\frac{k+2}{2}} \dfrac{|B_{k+2}|}{(k+2)!} s^2 g_k h^{k+2} + o(h^{k+2}) \end{cases} \tag{A.34}$$

is arrived at. The proof is completed by taking (2.23) and properties of B_{2i} collected in Remark 2.2.1 into account. □

Proof of Lemma 3.3.2 From (2.22) it follows that

$$H(z) = c_k(z) h^k + c_{k+1}(z) h^{k+1} + \Delta H(z), \tag{A.35}$$

with

$$\Delta H(z) = (1 - z^{-1})^2 \mathcal{Z}\{\frac{1 + sh}{s^2 h} \Delta G(s)\}. \tag{A.36}$$

On the other hand, from (A.10) it follows that

$$\Delta H(z) = h^{k+1} \delta(z, h), \tag{A.37}$$

where

$$\delta(z, h) = \sum_{i=2}^{\infty} \frac{g_{k+i}}{\mathcal{F}_{k+1}(1)} \frac{\mathcal{F}_{k+i}(z)}{z(z-1)^{k+i}} h^{i-1}. \tag{A.38}$$

As a result of (A.38), $\lim_{h \to 0} \delta(z, h) = 0$, which proves (3.13). □

Proof of Theorem 3.4.2 Denote z_i, $i = 1, 2 \dots m$ the intrinsic zeros of $H(z)$, which due to Theorem 3.4.2 are related to the zeros σ_i of $G(s)$, while $\zeta_i = z_{m+i}$, $i = 1, 2, \dots k$ denote the discretization zeros. Then for h small enough $H(z)$ admits the following factorization:

$$H(z) = b_n \frac{F_k(z) \prod_{i=1}^{m} (z - z_i)}{z \prod_{i=1}^{n} (z - p_i)} \tag{A.39}$$

with a polynomial

$$F_k(z) = \prod_{i=1}^{k} (z - \zeta_i). \tag{A.40}$$

To prove item (i) assume that $G(s)$ is of Type l, $l \geq 0$. From the fact that the steady-state properties are preserved in the sampled-data system, i.e.:

$$\lim_{z \to 1} \left(\frac{z-1}{h}\right)^l H(z) = \lim_{s \to 0} s^l G(s) \tag{A.41}$$

it results that the coefficient b_n in eqn. (3.4) has the form

$$b_n = h^l \frac{\prod_{i=1}^{n-l} (1 - p_i)}{\prod_{i=1}^{n} (1 - z_i)} \tag{A.42}$$

or, taking (A.39) into account:

$$b_n = \frac{h^l}{F_k(1)} \frac{\prod_{i=1}^{n-l} (1 - p_i)}{\prod_{i=1}^{m} (1 - z_i)} \tag{A.43}$$

Applying equations (3.2), (A.1) - (A.2) and the relationship $\lim_{h \to 0} F_k(z) = \mathcal{F}_k(z)/(k+2)$ yields eqn. (3.16) proving item (i).

Assume that the continuous-time transfer function has a multiple zero with the multiplicity μ. Denote $\mathcal{J} = \{j, j+1, \cdots j+\mu-1\}$ a set of integers indicating those zeros. Insert the asymptotic zero $z'_j = e^{\sigma_j h}$ into $H(z)$. Then, according to eqn. (3.9), one gets:

$$H(z'_j) = b_n F_k(z'_j) \frac{\prod_{i=1}^{m} (z'_j - z_i)}{z'_j \prod_{i=1}^{n} (z'_j - p_i)} \doteq \Delta_h(\sigma_j), \tag{A.44}$$

from which

$$\prod_{i=0}^{\mu-1} (z'_j - z_{j+i}) = \frac{\prod_{i=1}^{n} (z'_j - p_i)}{\prod_{\substack{i=1 \\ i \notin \mathcal{J}}}^{m} (z'_j - z_i)} \frac{z'_j \Delta_h(\sigma_j)}{b_n F_k(z'_j)}. \tag{A.45}$$

Applying (A.1) - (A.2) to $z'_j - p_i$ and $z'_j - z_i$ gives

$$\prod_{i=0}^{\mu-1} (z'_j - z_{j+i}) = \left[\frac{\prod_{i=1}^{n} (\sigma_j - \pi_i)}{\prod_{\substack{i=1 \\ i \notin \mathcal{J}}}^{m} (\sigma'_j - \sigma_i)} h^{k+\mu} + o(h^{k+\mu})\right] \frac{\Delta_h(\sigma_j)}{b_n F_k(z'_j)}. \tag{A.46}$$

Finally, employing (3.22) and inserting b_n from (3.16) yields (3.17). This proves item (ii) of Theorem 3.4.2.

Inserting ζ'_j into (3.4) yields:

$$H(\zeta'_j) = b_n F_k(\zeta'_j) \frac{\prod_{i=1}^{m} (\zeta'_j - z_i)}{\zeta'_j \prod_{i=1}^{n} (\zeta'_j - p_i)}, \tag{A.47}$$

while inserting ζ'_j into (3.13) of Lemma 3.3.2 returns:

$$H(\zeta'_j) = g_{k+1} \frac{h^{k+1}}{\mathcal{F}_{k+1}(1)} \frac{\mathcal{F}_{k+1}(\zeta'_j)}{\zeta'_j (\zeta'_j - 1)^{k+1}} + o(h^{k+1}). \tag{A.48}$$

For small h, equation (A.47) can be written in the form:

$$H(\zeta'_j) = \left\{\frac{k+2}{(k+1)!} g_k h^k + o(h^k)\right\} \frac{F_k(\zeta'_j)}{\zeta'_j (\zeta'_j - 1)^k} [1 + o(h)]. \tag{A.49}$$

A comparison of (A.48) and (A.49) leads to

$$F_k(\zeta'_j) = \frac{g_{k+1}}{g_k} \frac{\mathcal{F}_{k+1}(\zeta'_j)}{(k+2)^2} \frac{1}{\zeta'_j - 1} h + o(h). \tag{A.50}$$

Finally, taking (A.40), (2.23) and $\zeta_j - \zeta'_i \to \zeta'_j - \zeta'_i \neq 0$ for $i \neq j$ into account yields (3.18) with (3.19) which proves item (iii) of the Theorem 3.4.2. □

A.3 Proofs for Chapter 7

Proof of Lemma 7.5.1 Observe that

$$\boldsymbol{F \Sigma d} = \boldsymbol{F}(\boldsymbol{\Sigma}_+ + \boldsymbol{S})\boldsymbol{d} = \boldsymbol{FSd} + \sigma_+^2 \boldsymbol{h}_+ \tag{A.51}$$

and

$$\rho^2 + \boldsymbol{d}' \boldsymbol{\Sigma d} = \sigma_+^2 + \boldsymbol{d}' \boldsymbol{Sd}. \tag{A.52}$$

Subtracting

$$\boldsymbol{\Sigma}_+ = \boldsymbol{W} + \boldsymbol{F \Sigma}_+ \boldsymbol{F}' - \frac{\boldsymbol{F \Sigma}_+ \boldsymbol{dd}' \boldsymbol{\Sigma}_+ \boldsymbol{F}'}{\rho^2 + \boldsymbol{d}' \boldsymbol{\Sigma}_+ \boldsymbol{d}} \tag{A.53}$$

from (7.80) and employing (7.102)-(A.52) yields:

$$S = FSF' + \sigma_+^2 h_+ h_+' - \frac{(FSd + \sigma_+^2 h_+)(FSd + \sigma_+^2 h_+)'}{\sigma_+^2 + d'Sd}. \tag{A.54}$$

It is now easy to check that equations (A.54) and (7.101) are equivalent. □

Proof of Lemma 7.5.2 Inserting (7.105) into (7.101) leads to

$$X = \Lambda X \Lambda' - \frac{\Lambda X \delta\delta' X \Lambda'}{\sigma_+^2 + \delta' X \delta}. \tag{A.55}$$

Equation (A.55) is fulfilled by X of (7.109) if X_1 fulfills

$$X_1 = \Lambda_1 X_1 \Lambda_1' - \frac{\Lambda_1 X_1 \delta_1 \delta_1' X_1 \Lambda_1'}{\sigma_+^2 + \delta_1' X_1 \delta_1}. \tag{A.56}$$

Now, consider the following Lyapunov equation:

$$\Lambda_1' X_1^{-1} \Lambda_1 = X_1^{-1} + \frac{\delta_1 \delta_1'}{\sigma^2}. \tag{A.57}$$

Matrix X_1^{-1} which fulfills (A.57) is unique and nonsingular, which follows from the observability of (F, d). X_1^{-1} also fulfills (A.56) which follows from a matrix identity

$$(F^{-1} + BC^{-1}B')^{-1} = F - FB(B'FB + C)^{-1}B'F. \tag{A.58}$$

From (A.57) and (7.107) follows (7.110). □

Proof of Theorem 7.5.1

$$\sigma^2 = \rho^2 + d'\Sigma d = \rho^2 + d'\Sigma_+ d + d'Sd = \sigma_+^2 + d'Sd, \tag{A.59}$$

which proves (7.112). Observe that

$$F_+^* Sd = FSd - h_+ d'Sd = F(S - S_+)d - h_+(\sigma^2 - \sigma_+^2) = \sigma(h - h_+), \tag{A.60}$$

which proves (7.113).

Observe that

$$F^* = F_+^* - (h - h_+)d'. \tag{A.61}$$

On inserting (A.60) to (A.61) equation (7.114) is arrived at. □

Proof of Theorem 7.5.2 Inserting matrix S determined by equation (7.108) and matrix F_+^* determined by (7.105) to (7.114) one gets:

$$F^* = T^{-1}[\Lambda(I - \frac{X\delta\delta'}{\sigma^2})]T. \tag{A.62}$$

From (7.106) and (7.109) it follows:

$$F^* = T^{-1} \begin{bmatrix} H & 0 \\ 0 & \Lambda_2 \end{bmatrix} T, \tag{A.63}$$

where

$$H = \Lambda_1 - \Lambda_1 X_1 \frac{\delta_1 \delta_1'}{\sigma^2}. \tag{A.64}$$

Observe that (A.56) can be rewritten in the form of

$$X_1 = H X_1 \Lambda_1' \tag{A.65}$$

so that

$$H = X_1 (\Lambda_1')^{-1} X_1^{-1} \tag{A.66}$$

Finally, upon inserting (A.65) into (A.62) one gets

$$F^* = M^{-1} \begin{bmatrix} (\Lambda_1')^{-1} & 0 \\ 0 & \Lambda_2 \end{bmatrix} M, \; M = \begin{bmatrix} X_1^{-1} & 0 \\ 0 & I \end{bmatrix} T. \tag{A.67}$$

This proves the first part of the theorem. To prove the second part observe that (A.56) can be rewritten using

$$\sigma^2 = \sigma_+^2 + \delta_1' X_1 \delta_1 \tag{A.68}$$

as follows:

$$X_1 = \Lambda_1 X_1 \Lambda_1' - \frac{\Lambda_1 X_1 \delta_1 \delta_1' X_1 \Lambda_1'}{\sigma^2}. \tag{A.69}$$

Now, applying the identity:

$$\det(F - BCD) = \det F \det B \det(D^{-1} - CF^{-1}B) \tag{A.70}$$

with

$$F = \Lambda_1 X_1 \Lambda_1', \; B = \Lambda_1 X_1 \delta_1', \; C = \delta_1' X_1 \Lambda_1', \; D = 1/\sigma^2 \tag{A.71}$$

results in

$$\det X_1 = \frac{1}{\sigma^2} \det(\Lambda_1 X_1 \Lambda_1')(\sigma^2 - \delta_1' X_1 \delta_1) = \frac{\sigma_+^2}{\sigma^2} (\det \Lambda_1)^2 \det X_1. \tag{A.72}$$

□

Bibliography

Ackermann, J. (1985), *Sampled-Data Control Systems*, Springer-Verlag.

Ackermann, J. (1993), *Robust Control*, Springer-Verlag.

Ackermann, J. & Hu, H. (1991), 'Robustness of sampled-data control systems with uncertain physical plant parameters', *Automatica* **27**, 705–710.

Akaike, H. (1975), 'Markovian representation of stochastic processes by canonical variables', *SIAM J. Contr.* **13**, 162–173.

Anderson, B. & Moore, J. (1979), *Optimal Filtering*, Prentice Hall.

Åström, K. (1970), *Introduction to Stochastic Control Theory*, Academic.

Åström, K. & Wittenmark, B. (1997), *Computer Controlled Systems*, Prentice-Hall.

Åström, K., Hagander, P. & Sternby, J. (1984), 'Zeros of Sampled Systems', *Automatica* **20**, 31–38.

Badawi, A., Lindquist, A. & Pavon, M. (1979), 'A stochastic realization approach to the smoothing problem', *IEEE Trans. Automat. Contr.* **AC-24**, 878–888.

Bamieh, B. & Pearson, J. (1992), 'The $\mathcal{H}_2$ problem for sampled-data systems', *Systems and Control Letters* **19**, 1–12.

Bar-Shalom, Y. & Tse, E. (1973), 'Dual effect, certainty equivalence and separation in stochastic control', *IEEE Trans. Auto. Control* **AC-19**, 494–500.

Baron, E. (1989), A software package for digital control systems design, Master's thesis, Department of Automatic Control, Silesian Technical University, Gliwice. Written under supervision of M. Blachuta. (in Polish).

Bergstrom, A. (1976), *Statistical Inference in Continuous Time Economic Models*, North-Holland.

Bitmead, R., Gevers, M. & Wertz, V. (1990), *Adaptive Optimal Control: The Thinking Man's GPC*, Prentice Hall.

Błachuta, M. (1982), Singular linear-quadratic control problems, PhD thesis, Silesian Technical University, Gliwice, Poland. (in Polish).

Błachuta, M. (1986), On finite sample prediction of time series, *in* 'Contributions to the theory of optimal prediction and control', Vol. 235, Institute for Advanced Studies, Vienna.

Błachuta, M. (1987), 'Comments on 'Design of stochastic discrete-time linear optimal regulators'', *Int. J. Systems. Sci* **18**, 1387–1390.

Błachuta, M. (1992), A method for synthesis of discrete-time controller for continuous-time plant, *in* 'Preprints of the XXI Polish Conference on Applications of Mathematics', Polish Academy of Science, Committee of Mathematics, Zakopane-Kościelisko, Poland. (in Polish).

Błachuta, M. (1993*a*), A critical analysis of digital control systems description methods with regard to design, realization and simulation, *in* 'Preprints of the XXII Polish Conference on Applications of Mathematics', Polish Academy of Science, Committee of Mathematics, Zakopane-Kościelisko, Poland. (in Polish).

Błachuta, M. (1993*b*), 'Identification and prediction methods of methane emanation process for automatic control of ventilation', *Mechanizacja i Automatyzacja Górnictwa* **31**, 39–42. (in Polish).

Błachuta, M. (1994), Parameters estimation for continuous-time stochastic models based on sampled data, *in* 'Preprints of the XXIII Polish Conference on Applications of Mathematics', Polish Academy of Science, Committee of Mathematics, Zakopane-Kościelisko, Poland. (in Polish).

Błachuta, M. (1996*a*), 'Characteristics of continuous-time stochastic dynamical systems with a discrete-time feedback', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **120**, 83–98. (in Polish).

Błachuta, M. (1996*b*), 'Continuos-time methods in digital control systems design, Part 1: Discrete-time operators and transfer functions', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **120**, 99–118. (in Polish).

Błachuta, M. (1996*c*), 'Continuos-time methods in digital control systems design, Part 2: Approximations and state-space methods', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **120**, 119–140. (in Polish).

Błachuta, M. (1996*d*), 'Estimation of continuous-time stochastic dynamical systems based on sampled data', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **120**, 61–82. (in Polish).

Błachuta, M. (1996*e*), 'Identification methods of stationary and nonstationary time series', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **120**, 15–38. (in Polish).

Błachuta, M. (1996*f*), Optimal LQG and GPC controllers: Covariance characterization, structured models and Chandrasekhar equations, *in* 'Proc. of the 13th IFAC World Congress', San Francisco, pp. 253–258.

Błachuta, M. (1996*g*), 'PIPS: A didactical version of a program for identification and prediction of time series', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **120**, 39–49. (in Polish).

Błachuta, M. (1996*h*), State-space approach to LQG and GPC control, *in* 'Proc. of the 2nd Portuguese Conference on Automatic Control, Controlo'96', Vol. II, Porto, Portugal, pp. 789–794.

Błachuta, M. (1996*i*), A unified state-space approach to LQG and GPC controllers, *in* 'Proc. of the UKACC Conference Control'96', Exeter, UK, pp. 66–71.

Błachuta, M. (1997*a*), Asymptotic approximations of pulse transfer functions, *in* 'Proceedings of the Fourth International Symposium on Methods and Models in Automation and Robotics MMAR'97', Vol. 1, Institute of Control Engineering, Technical University of Szczecin, Miedzyzdroje, Poland, pp. 221–226.

Błachuta, M. (1997*b*), Continuous-time design of discrete-time control systems, *in* 'CD-ROM Proc. of the 1997 European Control Conference, ECC'97', Eur. Union Control Assoc., Brussels, Belgium.

Błachuta, M. (1997*c*), Hybrid design of of sampled-data control systems, *in* 'Preprints of the 4th DYCOMANS Workshop: 'Control and Menagement in Computer Integrated Systems', Zakopane, Poland, pp. 29–34.

Błachuta, M. (1997*d*), On approximate pulse transfer functions, *in* 'Proc. of the 36th Conference on Decision and Control', IEEE CSS, San Diego, CA, pp. 357–362.

Błachuta, M. (1997*e*), On sampled-data control systems with discontinuous output, *in* 'CD-ROM Proceedings of the 5th IEEE Mediterranean Conference on Control and Systems', IEEE CSS, Paphos, Cyprus.

Błachuta, M. (1997*f*), On zeros of sampled systems, *in* 'Proc. of the 1997 American Control Conference', American Automatic Control Council, Albuquerque, NM, pp. 3205–3209.

Błachuta, M. (1997*g*), Sampled-data control with discontinuous output, *in* 'Proceedings of the Fourth International Symposium on Methods and Models in Automation and Robotics MMAR'97', Vol. 1, Institute of Control Engineering, Technical University of Szczecin, Miedzyzdroje, Poland, pp. 215–220.

Błachuta, M. (1998*a*), 'On approximate pulse transfer functions', *IEEE Trans. on Auto. Control.* accepted for publication.

Błachuta, M. (1998*b*), On fast state-space algorithms for predictive control, *in* 'Proceedings of the Fifth International Symposium on Methods and Models in Automation and Robotics MMAR'98', Vol. 2, Institute of Control Engineering, Technical University of Szczecin, Miedzyzdroje, Poland, pp. 455–460.

Błachuta, M. (1998*c*), On zeros of pulse transfer functions of systems with first-order hold, *in* 'Proc. of the 37th Conference on Decision and Control', IEEE CSS, Tampa, FL.

Błachuta, M. (1998*d*), 'On zeros of sampled systems', *IEEE Trans. on Auto. Control*. accepted for publication.

Błachuta, M. (1999*a*), 'Discrete-time modeling of sampled-data control systems with direct feedthrough', *IEEE Trans. on Auto. Control* **AC-44**, 134–139.

Błachuta, M. (1999*b*), 'On fast state-space algorithms for predictive control', *Int. J. Appl. Math. and Comp. Sci.* **9**, 149–160.

Błachuta, M. & Ordys, A. (1984), 'Relationship between Åström and Kalman regulators in the minimum-variance control problem', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **74**, 25–37. (in Polish).

Błachuta, M. & Ordys, A. (1987), 'Optimal and asymptotically optimal linear regulators resulting from a one-stage performance index', *Int. J. Systems Sci.* **18**, 1377–1385.

Błachuta, M. & Ordys, A. (1988), 'Comparison of Clarke, Hastings-James and Kalman algorithms', *Podstawy Sterowania* **18**, 251–278. (in Polish).

Błachuta, M. & Ordys, A. (1989), 'Relationship between Clarke, Hastings-James and Kalman algorithms for the minimum-variance control problem', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **93**, 25–38. (in Polish).

Błachuta, M. & Ordys, A. (1992), 'Dynamical properties of Åström and Kalman algorithms in the minimum variance control problem', *Scientific Bulletins of the Silesian Technical University, s. Automatyka* **103**, 13–25. (in Polish).

Błachuta, M. & Polańska, J. (1995), Covariance characterization of stochastic continuous-time dynamic systems with optimal discrete-time feedback, *in* 'Proc. of the European Control Conference ECC'95', Vol. 3, Eur. Union Control Assoc., Rome, Italy, pp. 1708–1713.

Błachuta, M. & Polański, A. (1986*a*), On state-space and time-series representations of a Gauss-Markov process, *in* 'Contributions to the theory of optimal prediction and control', Vol. 235, Institute for Advanced Studies, Vienna.

Błachuta, M. & Polański, A. (1986*b*), Symmetric solutions of a discrete algebraic Riccati equation, *in* 'Contributions to the theory of optimal prediction and control', Vol. 235, Institute for Advanced Studies, Vienna.

Błachuta, M. & Polański, A. (1987), 'On time invariant representations of discrete random processes', *IEEE Trans. Auto. Control* **AC-32**, 1125–1127.

Błachuta, M. & Polański, A. (1990), 'Representations of discrete-time Gauss-Markov processes and symmetric solutions of Riccati equation', *Archiwum Automatyki i Telemechaniki* **35**, 177–189. (in Polish).

Bose, N. (1983), 'Properties of the Q_n matrix in bilinear transformation', *Proc. IEEE* **71**, 1110–1111.

Burshtein, D. (1993), 'An efficient algorithm for calculating the likelihood and likelihood gradient of ARMA models', *IEEE Trans. Auto. Control* **AC-38**, 336–340.

Bush, A. & Fielder, D. (1973), 'Simplified algebra for the bilinear and related transformations', *IEEE Trans. Audio Electroaccoust.* **AU-21**, 127–128.

Byun, G., Kwon, W. & Choi, H. (1990), Recursive solution of generalized predictive control and its relation with receding horizon tracking control, *in* 'Proc. of the 11th IFAC World Congress', Tallin, USSR, pp. 64–69.

Caines, P. (1972), 'Relationship between Box-Jenkins-Åström control and Kalman linear regulator', *Proc. IEE* **119**, 615–620.

Chen, T. & Francis, B. (1991), '$\mathcal{H}_2$-optimal sampled-data control', *IEEE Trans. Auto. Control* **AC-36**, 387–397.

Chen, T. & Francis, B. (1995), *Optimal Control of Sampled-Data Systems*, Springer-Verlag.

Chisci, L., Lombardi, A., Mosca, E. & Rossiter, A. (1996), 'State-space approach to stabilizing stochastic predictive control', *Int. J. Control* pp. 619–637.

Clarke, D. (1984), 'Self-tuning control of nonminimum-phase systems', *Automatica* **20**, 501–517.

Clarke, D. & Hastings-James, R. (1971), 'Design of digital controllers for randomly disturbed systems', *Proc. IEE* **118**, 1503–1506.

Clarke, D. & Mohtadi, C. (1989), 'Properties of Generalized Predictive Control', *Automatica* **25**, 137–148.

Clarke, D., Kanjilal, P. & Mohtadi, C. (1985), 'A generalized LQG approach to self-tunning control, Part I and II', *Int. J. Control* **41**, 1509–1564.

Clarke, D., Mohtadi, C. & Tuffs, P. (1987), 'Generalized Predictive Control, Part I and II', *Automatica* **23**, 137–160.

Clements, D. & Anderson, B. (1978), *Singular Optimal Control - The Linear-Quadratic Problem*, Lecture Notes in Control and Information Sciences, Springer-Verlag.

Cutler, C. & Ramaker, B. (1980), Dynamic Matrix Control - A computer control algorithm, *in* 'Proc. JACC', San Francisco.

Dahleh, M. & Diaz-Bobillo, I. (1995), *Control of Uncertain Systems*, Prentice-Hall.

Davies, A. (1974), 'Bilinear transformation of polynomials', *IEEE Trans. Circuits and Systems* **CAS-21**, 792–794.

de Nicolao, G. & Strada, S. (1997), 'On the stability of receding horizon LQ control with zero-state terminal constraint', *Trans. Auto. Control* **AC-42**, 257–260.

de Nicolao, G., Magni, L. & Scattolini, R. (1996), 'On the robustness of Receding-Horizon Control with terminal constraints', *IEEE Trans. Auto. Control* **AC-41**, 451–453.

de Souza, C. & Goodwin, G. (1984), 'Intersample variances in discrete minimum-variance control', *IEEE Trans. Auto. Control* **AC-29**, 759–761.

Doob, J. (1953), *Stochastic Processes*, Wiley.

Dorato, P. & Levis, A. (1971), 'Optimal linear regulators: The discrete-time case', *IEEE Trans. Auto. Control* **AC-16**, 613–620.

Edwards, M. (1974), *Riemann's Zeta Function*, Academic Press.

Esfandiari, F. & Khalil, H. (1989), 'On the robustness of sampled-data control to unmodeled high-frequency dynamics', *IEEE Trans. Auto. Control* **AC-34**, 900–903.

Feuer, A. & Goodwin, G. (1996), *Sampling in Digital Signal Processing and Control*, Birkhäuser.

Forsythe, W. & Goodall, R. (1991), *Digital Control*, MacMillan Education.

Franklin, G., Powell, J. & Emani-Naeini, A. (1986), *Feedback Control of Dynamic Systems*, Addison-Wesley.

Franklin, G., Powell, J. & Workman, M. (1990), *Digital Control of Dynamic Systems*, Addison-Wesley.

Friedland, B. (1967), 'On solutions of the Riccati equation in optimization problems', *IEEE Trans. Automat. Contr.* **AC-19**, 303–304.

Friedlander, B., Kailath, T. & Ljung, L. (1976), 'Scattering theory and linear least squares estimation', *J. Franklin Instit.* **301**, 71–82.

Frobenius, G. (1910), 'On Bernoulli numbers and Euler polynomials', *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* pp. 809–847. (in German).

Fu, Y. & Dumont, G. (1989), 'Choice of sampling to ensure minimum-phase behavior', *IEEE Trans. Auto. Control* **AC-34**, 560–563.

Gambier, A. & Unbehauen, H. (1993), A state-space generalized model-based predictive control for linear multivariable systems and its interrelation with the receding horizon LQG-control, *in* 'Proc. of the 32nd Conference on Decision and Control', San Antonio, TX.

Gardner, G., Harvey, A. & Phillips, G. (1980), 'An algorithm for exact maximum likelihood estimation of autoregressive-moving average models by means of Kalman filtering', *Appl. Statist.*

Gessing, R. (1993), About some properties of discrete-time transfer functions for small sampling periods, *in* 'Proc. of the 1993 European Control Conference, ECC'93', Eur. Union Control Assoc., Groningen, The Netherlands, pp. 1699–1702.

Gessing, R. (1995), 'Comments on 'A modification and the Tustin approximation'', *IEEE Trans. Auto. Control* **AC-40**, 942–944.

Gessing, R. (1996), 'Causal and non-causal discrete-time transfer functions and their applications', *Int. J. Control* **65**, 195–205.

Gikhman, I. & Skorokhod, A. (1969), *Introduction to the Theory of Random Processes*, Saunders College Publishing.

Gikhman, I. & Skorokhod, A. (1972), *Stochastic Differential Equations*, Springer.

Goodwin, G., Leal, R., Mayne, D. & Middleton, R. (1986), 'A rapprochement between continuous and discrete model reference adaptive control', *Automatica* **22**, 199–207.

Grimble, M. (1990), 'LQG predictive optimal control for adaptive applications', *Automatica* **26**, 949–961.

Grimble, M. (1992), 'Generalized predictive optimal control: an introduction to the advantages and limitations', *Int. J. Systems Sci.* **23**, 85–98.

Grimble, M. (1994), 'State-space approach to LQG multivariable prediction and feedforward optimal control', *Trans. ASME, Int. J. Dynamic Systems, Measurement and Control* **116**, 610–617.

Grimble, M. (1995), Multivariable linear quadratic Generalized Predictive Control, *in* 'Proc. of the 34th Conference on Decision and Control', New Orleans, LA, pp. 1060–1065.

Hagander, P. (1993), 'Comments on 'Conditions for stable zeros of sampled systems'', *IEEE Trans. Auto. Control* **AC-38**, 830–831.

Hagiwara, T. (1996), 'Analytic study on the intrinsic zeros of sampled data systems', *IEEE Trans. Auto. Control* **AC-41**, 261–263.

Hagiwara, T. & Araki, M. (1995), 'FR-operator approach to the $\mathcal{H}_2$-analysis and synthesis of sampled-data systems', *IEEE Trans. Auto. Control* **AC-40**, 1411–1421.

Hagiwara, T., Yuasa, T. & Araki, M. (1993), 'Stability of the limiting zeros of sampled-data systems with zero- and first-order holds', *Int. J. Control* **58**, 1325–1346.

Hangstrup, M., Ordys, A. & Grimble, M. (1997), Dynamic algorithm for LQGPC predictive control, *in* 'Proceedings of the Fourth International Symposium on Methods and Models in Automation and Robotics MMAR'97', Vol. 2, Institute of Control Engineering, Technical University of Szczecin, Miedzyzdroje, Poland, pp. 453–460.

Hannan, E. (1988), 'The estimation of the order of an ARMA process', *Ann. Statist* **8**, 1071–1081.

Hannan, E. & Kavalieris, L. (1983), 'Linear estimation of ARMA processes', *Automatica* **19**, 447–448.

Hannan, E. & Rissanen, J. (1982), 'Recursive estimation of mixed autoregressive-moving average order', *Biometrika* **69**, 81–94.

Hara, S., Fujioka, H. & Kabamba, P. (1994), 'A hybrid state-space approach to sampled-data feedback control', *Lin. Alg. Applic.* **205-206**, 675–712.

Hara, S., Katori, H. & Kondo, R. (1989), 'The relation between real poles and real zeros in SISO sampled systems', *IEEE Trans. Auto. Control* **AC-34**, 632–635.

Hara, S., Yamamoto, Y. & Fujioka, H. (1996), Modern and classical analysis/synthesis methods in sampled-data control - a brief overview with numerical examples, *in* 'Proc. of the 32nd Conference on Decision and Control', Kobe, Japan, pp. 1251–1256.

Hara, S., Yamamoto, Y. & Fujioka, H. (1997), $\mathcal{H}_{sys}$ module: a software package for analysis and synthesis of sampled-data control systems, *in* 'CD-ROM Proc. of the 1997 European Control Conference, ECC'97', Eur. Union Control Assoc., Brussels, Belgium.

Hashimoto, Y., Yoneya, A. & Togari, Y. (1989), 'Design method of a quadratic performance index using a reference model', *Int. J. Control* **50**, 1169–1184.

Hayakawa, Y., Hosoe, S. & Ito, M. (1983), 'On the limiting zeros of sampled multivariable systems', *Systems and Control Letters* **2**, 292–300.

Heinen, J. & Siddique, B. (1988), 'A simple algorithm for arbitrary polynomial transformation', *IEEE Trans. Accoust., Speech, Signal Processing* **ASSP-36**, 77–80.

Houpis, C. & Lamont, G. (1985), *Digital Control Systems*, McGraw-Hill.

Hughes, D. (1973), 'Equivalence of the Box-Jenkins-Åström control law to the Kalman linear regulator', *Electronics Letters* **9**, 220–221.

Ichikawa, K. (1985), *Control System Design Based on Exact Model Matching Techniques*, Springer-Verlag.

Ieko, T., Ochi, Y., Kanai, K., Hori, N. & Okamoto, K.-I. (1996), Digital redesign methods based on plant input mapping and a new discrete-time model, *in* 'Proc. of the 35th Conference on Decision and Control', Kobe, Japan, pp. 1569–1574.

Isermann, R. (1989), *Digital Control Systems*, Springer-Verlag. Vol. 1 and 2.

Ishitobi, M. (1992), 'Conditions for stable zeros of sampled systems', *IEEE Trans. Auto. Control* **vol. AC-37**, 1558–1561.

Ishitobi, M. (1993), 'Stable zeros of sampled low-pass systems', *Int. J. Control* **57**, 1485–1498.

Ismail, M. & Vakilazadian, H. (1989), 'The computer implementation of bilinear *s-z* transformation using new continued fraction algorithms', *IEEE Trans. Educ.* **E-32**, 270–279.

Jacquot, R. (1994), *Modern Digital Control Systems*, Marcel Dekker.

Janiszowski, K. (1993), 'A modification and the Tustin transformation', *IEEE Trans. Auto. Control* **AC-38**, 1313–1316.

Jury, E. (1958), *Sampled-Data Control Systems*, Wiley.

Jury, E. (1964), *Theory and Application of the Z-transform*, John Wiley.

Jury, E. & Chan, O. (1973), 'Combinatorial rules for some useful transformations', *IEEE Trans. Circuit Theory* **CT-20**, 476–480.

Kabamba, P. & Hara, S. (1993), 'Worst-case analysis and design of sampled-data control systems', *IEEE Trans. Auto. Control* **AC-38**, 1337–1357.

Kailath, T. (1968), 'An innovation approach to least squares estimation, Part I: Linear filtering in additive white noise,', *IEEE Trans. Automat. Contr.* **AC-13**, 646–654.

Kailath, T. (1980), *Linear Systems*, Prentice-Hall.

Kailath, T. & Frost, P. (1968), 'An innovations approach to least-squares estimation, Part II: Linear smoothing in additive white noise', *IEEE Trans. Automat. Contr.* **AC-13**, 655–660.

Khargonekar, P. & Sivashankar, N. (1991), '$\mathcal{H}_2$ optimal control for sampled-data systems', *System and Control Letters* **17**, 425–436.

Kohn, R. & Ansley, C. (1982), 'Computing the likelihood and their derivatives for a Gaussian ARMA model', *J. Statist. Computn Simuln* **15**, 229–263.

Kowalczuk, Z. (1983), 'A normal triangle and normal methods of transformations in the discrete approximation of continuous systems', *Arch. Automat. Telemech.* **28**, 15–32.

Kowalczuk, Z. (1993), 'Discrete approximation of continuous-time systems: a survey', *IEE Proc. G* **140**, 264–278.

Kowalczuk, Z. & Suchomski, P. (1997), Generalized predictive control of delay systems, *in* 'CD-ROM Proc. of the 4th European Control Conference, ECC'97', Eur. Union Control Assoc., Brussels, Belgium.

Krauss, P. (1996), Predictive control with linear process models in state-space, PhD thesis, Rheinisch-Westfälische Technische Hochschule, Aachen. (in German).

Krauss, P. & Rake, H. (1994), 'Multivariable predictive controller with Kalman filter', *Journal A* **35**(3), 38–43. Belgium.

Krauss, P. & Rake, H. (1995), Design for a multivariable predictive controller in state space, *in* 'Proc. of the 3rd European Control Conference, ECC'95', Eur. Union Control Assoc., Rome, Italy, pp. 3679–3584.

Krauss, P., Daß, K. & Bünte, T. (1994), 'Predictive controller with Kalman filter for state estimation', *Automatisierungstechnik* **42**, 533–539. (in German).

Kuo, B. (1970), *Discrete-Data Control Systems*, Prentice-Hall.

Kuo, B. & Peterson, D. (1973), 'Optimal discretization of continuous-data control system', *Automatica* **9**, 125–129.

Kushner, H. (1972), *Introduction to Stochastic Control*, Holt, Rinehart and Winston.

Kučera, V. (1991), *Analysis and Design of Discrete Linear Control Systems*, Prentice-Hall International.

Kučera, V. (1972), 'State space approach to discrete optimal control', *Kybernetika* **8**, 233–251.

Kučera, V. (1973), 'A review of the matrix Riccati equation', *Kybernetika* **9**, 42–61.

Kuzin, L. (1962), *Analysis and Design of Discrete Control Systems*, Mashgiz, Moscow. (in Russian).

Kwakernaak, H. & Sivan, R. (1972), *Linear Optimal Control Systems*, Wiley-Interscience.

Kwon, W. & Byun, D. (1989), 'Receding horizon tracking control as a predictive control and its stability properties', *Int. J. Control* **15**, 1807–1824.

Kwon, W., Choi, H., Byun, D. & Noh, S. (1992*a*), 'Recursive solution of generalized predictive control and its equivalence to receding horizon tracking control', *Automatica* **28**, 1235–1238.

Kwon, W., Lee, Y. & Noh, S. (1992*b*), Partition of GPC into a state observer and state feedback controller, *in* 'Proc. of the 1992 American Control Conference', American Automatic Control Council, pp. 2032–2036.

Kwon, W., Noh, S. & Lee, Y. (1992*c*), Stability guaranteed generalized predictive control and its equivalence to receding horizon tracking control, *in* 'Proc. of the 1992 American Control Conference', American Automatic Control Council, pp. 2037–2041.

Lam, K. (1982), 'Design of stochastic discrete-time linear optimal regulators', *Int. J. Syst. Sci.* **13**, 971–1000.

Lam, K. (1989*a*), 'State-space formulations for the filtered CARMA model', *Int. J. Control* **50**, 871–881.

Lam, K. (1989*b*), 'Using the filtered CARMA and CARIMA models for state-space self-tuning control', *Int. J. Systems Sci.* **20**, 2461–2470.

Lampe, B. & Rosenwasser, Y. (1993), Design of hybrid analog-discrete systems by parametric transfer functions, *in* 'Proc. of the 32nd Conference on Decision and Control', IEEE CSS, San Antonio, TX, pp. 3897–3898.

Lee, J., Morari, M. & Garcia, C. (1994), 'State-space interpretation of Model Predictive Control', *Automatica* **30**, 707–717.

Lennartson, B. (1990), 'On the choice of controller and sampling period for linear stochastic control', *Automatica , no.3, May 1990, pp.573-8* **26**, 573–578.

Lennartson, B. & Söderström, T. (1989), 'Investigation of the intersample variance in sampled-data control', *Int. J. Control* **50**, 1587–1602.

Lennartson, B., Söderström, T. & Sun, Z.-Q. (1989), 'Intersample behavior as measured by continuous-time quadratic criteria', *Int. J. Control* **49**, 2077–2083.

León de la Barra, B. (1997), Frequency domain properties of discrete time zeros, unpublished paper.

Lindorff, D. (1965), *Theory of Sampled-Data Control Systems*, Wiley.

Lindquist, A. & Picci, G. (1979), 'On the stochastic realization problem', *SIAM J. Contr. Optimiz.* **17**, 365–390.

Ljung, L. & Söderström, T. (1983), *Theory and Practice of Recursive Identification*, MIT Press.

Maine, R. & Iliff, K. (1981), 'Formulation and implementation of a practical algorithm for parameter estimation with process and measurement noise', *SIAM J. Appl. Math.* **41**, 558–579.

Malvar, H. (1985), 'Comments on 'Bilinear transformation by synthetic division'', *IEEE Trans. Auto. Control* **AC-30**, 414.

Matko, D. (1990), A new approach to discrete and continuous-time generalized predictive control, *in* 'Proc. of the 11th IFAC World Congress', Tallin, USSR, pp. 186–191.

McReynolds, S. & Bryson, A. (1967), A successive sweep method for solving optimum programming problems, *in* 'Proc. ACC', Troy, NY, pp. 22–25.

Meditch, J. (1969), *Stochastic Optimal Linear Estimation and Control*, McGraw-Hill.

Mélard, G. (1984), 'A fast algorithm for the exact likelihood of autoregresive-moving average models', *Applied Statistics* **33**, 104–113. Algorithm AS 197.

Middleton, R. & Goodwin, G. (1990), *Digital Control and Estimation: A Unified Approach*, Prentice-Hall International.

Morf, M., Sidhu, G. & Kailath, T. (1973), 'Some new algorithms for recursive estimation in constant, linear, discrete-time systems', *IEEE Trans. Auto. Control* **AC-19**, 315–323.

Ogata, K. (1987), *Discrete-time Control Systems*, Prentice-Hall International.

O'Malley, R. & Jameson, A. (1975), 'Singular perturbations and singular arcs, Part I', *IEEE Trans. Auto. Control* **AC-20**, 218–226.

O'Malley, R. & Jameson, A. (1977), 'Singular perturbations and singular arcs, Part II', *IEEE Trans. Auto. Control* **AC-22**, 326–335.

Ordys, A. (1989), Transient states in minimum-variance control, PhD thesis, Silesian Technical University, Faculty of Automatic Control, Electronics and Computer Science, Gliwice, Poland. (in Polish).

Ordys, A. (1993), 'Model-system parameter mismatch in GPC control', *Int. J. Adaptive Control and Signal Processing* **7**, 239–253.

Ordys, A. & Clarke, D. (1993), 'A state-space description for GPC controllers', *Int. J. Systems Sci.* **23**, 1722–1744.

Ordys, A. & Grimble, M. (1996), A multivariable dynamic performance predictive control with application to power generation plant, *in* 'Proc. of the 13th IFAC World Congress', San Francisco, CA, pp. 79–84.

Parthasarathy, R. & Jayasimha, K. (1984), 'Bilinear transformation by synthetic division', *IEEE Trans. Auto. Control* **AC-29**, 575–576.

Petkov, P., Christov, N. & Konstantinov, M. (1991), *Computational Methods for Linear Control Systems*, Prentice Hall.

Phillips, C. & Nagle, H. (1990), *Digital Control System Analysis and Design*, Prentice-Hall International.

Power, H. (1967), 'The mechanics of the bilinear transformation', *IEEE Trans. Educ.* **E-10**, 114–116.

Power, H. (1968), 'Comments on 'The mechanics of the bilinear transformation'', *IEEE Trans. Educ.* **E-11**, 159.

Richalet, J., Rault, A., Testud, J. & Papon, J. (1978), 'Model Predictive Heuristic Control: Applications to industrial processes', *Automatica* **14**, 413–428.

Rosenwasser, Y. & Lampe, B. (1997), *Digital Control in Continuous Time: Analysis and Design in Frequency Domain*, B.G. Teubner. (in German).

Rostgaard, M., Lauritsen, M. & Poulsen, N. (1996), 'A state space approach to the emulator based GPC design', *System & Control Letters* **28**, 291–301.

Rouhani, R. & Mehra, R. (1982), 'Model Algorithmic Control (MAC): Basic theoretical properties', *Automatica* **18**, 401–414.

Saberi, A., Sannuti, P. & Chen, B. (1995), *H_2 Optimal Control*, Prentice-Hall.

Santina, M., Stubberud, A. & Hostetter, G. (1994), *Digital Control Systems Design*, Saunders College Publishing.

Scokaert, P. (1997), 'Infinite horizon generalized predictive control', *Int. J. Control* **66**, 161–171.

Scokaert, P. & Rawlings, J. (1998), 'Constrained linear quadratic regulation', *IEEE Trans. on Auto. Control* **AC-43**, 1163–1169.

Scott, D. (1994), 'A simplified method for the bilinear *s-z* transformation', *IEEE Trans. Educ.* **E-37**, 289–292.

Shieh, L., Kasavaraju, R. & Tsai, J. (1995), 'Digital redesign of a continuous-time controller using the Padé and inverse-Padé approximation method', *J. of the Franklin Institute* **322b**, 433–442.

Sinha, N. (1972), 'Estimation of transfer function of continuous system from sampled data', *Proc. IEE* **119**, 612–614.

Sinha, N. & Rao, G. (1991), *Identification of Continuous-Time Systems: Methodology and Computer Implementation*, Kluwer.

Sirisena, H. (1968), 'Optimal control of saturating linear plants for quadratic performance indices', *Int. J. Control* **12**, 739–752.

Sirisena, H. (1985), 'Ripple-free deadbeat control of SISO discrete systems', *IEEE Trans. Auto. Control* **AC-30**, 168–170.

Sobolev, S. (1977), 'On the roots of Euler polynomials', *Dokl. Akad. Nauk SSSR* **235**, 297–. English trans. in *Soviet Math. Dokl.* **18**, 935-938, 1977.

Söderström, T. (1984), On computing continuous time counterparts to ARMA models, Technical report, Uppsala University. Report UPTEC 84 100 R.

Söderström, T. (1991), 'On computing stochastic continuous-time models from ARMA models', *Int. J. Control* **53**, 1311–1326.

Soeterboek, A., Verbruggen, H. & van den Bosch, P. (1991), On the design of the Unified Predictive Controller, *in* 'Proc. of the 11-th IFAC World Congress', Tallin, USSR.

Taube, B. & Lampe, B. (1992), LQGPC - a predictive design as tradeoff between LQG and GPC, *in* 'Proc. of the 33th Conference on Decision and Control', Tucson, AZ, pp. 3579–3581.

Tesfaye, A. & Tomizuka, M. (1995), 'Zeros of discretized continuous systems expressed in the Euler operator', *IEEE Trans. Auto. Control* **AC-40**, 743–747.

Titchmarsh, E. (1986), *The Theory of the Riemann Zeta Function*, Clarendon Press, Oxford.

Tsypkin, Y. (1958), *Theory of Pulse Systems*, Fizmatgiz, Moscow. (in Russian).

Tustin, A. (1947), 'A method of analyzing the behaviour of linear systems in terms of time series', *JIEE (London)* **94**, 130–142.

Uchida, K. & Shimemura, E. (1976), 'Optimal control of linear stochastic systems with quadratic criterion under classical information structure: On certainty equivalence', *Trans. SICE* **12**, 89–95.

Unbehauen, H. & Rao, G. (1987), *Identification of Continuous Systems*, North-Holland.

Urikura, S. & Nagata, A. (1987), 'Ripple-free deadbeat control for sampled-data systems', *IEEE Trans. Auto. Control* **AC-32**, 474–482.

van der Shaft, A. & Willems, J. (1984), 'A new procedure for stochastic realization of spectral density matrix', *SIAM J. Contr. Optimiz.* **22**, 845–856.

van Loan, C. (1977), 'Computing integrals involving the matrix exponential', *IEEE Trans. Auto. Control* **AC-23**, 395–404.

Warwick, K. (1987), 'Optimal observers for ARMA models', *Int. J. Control* **46**, 1493–1503.

Warwick, K. (1990), 'Relationship between Åström control and the Kalman linear regulator - Caines revisited', *J. Optimal Control: Applications and Methods* **11**, 223–323.

Warwick, K. (1992), State-space structures for ARMA models, *in* 'Proc. of the Workshop on System Structure and Control', Prague, Czechoslovakia.

Warwick, K. & Peterka, V. (1991), Optimal observer solution for predictive and LQG optimal control, *in* 'Proc. of the IEE Conference 'Control 91'', Edinburgh, UK, pp. 768–772.

Watson, W. (1976), 'Box, Jenkins-Åström and Kalman linear control laws and their equivalence', *Proc. IEE* **123**, 377–380.

Węgrzyn, S. (1960), 'Different z-transforms of the unit step function and their application to the analysis of sampled-data systems', *Bull. Pol. Acad. Sci.* **8**, 311–313. (in French).

Węgrzyn, S. (1963, 1970, 1980), *Fundamentals of Automatic Control*, PWN, Warsaw. (in Polish), French translation: *Les Bases de l'Automatique Industrielle*, Dunod, Paris, 1965.

Weller, S. (1998), 'Comments on 'Zeros of discretized continuous systems expressed in the Euler operator'', *IEEE Trans. Auto. Control* **AC-43**, 1308–1310.

Weller, S., Moran, W., Ninness, B. & Pollington, A. (1997*a*), On the limiting zeros of sampled-data systems with first-order holds, *in* 'Proc. of Control 97', Sydney, Australia, pp. 306–311.

Weller, S., Moran, W., Ninness, B. & Pollington, A. (1997*b*), Sampling zeros and the Euler-Frobenius polynomials, *in* 'Proc. of the 36th Conference on Decision and Control', IEEE CSS, San Diego, CA, pp. 1471–1476.

Willems, J. (1971), 'Least squares stationary optimal control and the algebraic Riccati equation', *IEEE Trans. Auto. Control* **AC-16**, 621–634.

Williamson, D. (1991), *Digital Control and Implementation*, Prentice-Hall International.

Wolovich, W. (1994), *Automatic Control Systems: Basic Analysis and Design*, Saunders College Publishing.

Wymer, C. (1972), 'Econometric estimation of stochastic differential equation systems', *Econometrica* **40**, 565–577.

Yackell, R., Kuo, B. & Singh, G. (1974), 'Digital redesign of continuous systems by matching of states at multiple sampling periods', *Automatica* **10**, 105–111.

Yamamoto, Y. (1994), 'A function space approach to sampled-data systems and tracking problems', *IEEE Trans. Auto. Control* **AC-39**, 703–713.

Ydstie, B. (1984), Extended Horizon Adaptive Control, *in* 'Proc. of the 9th IFAC World Congress', Budapest, Hungary, pp. 133–137.

Yoneya, A., Hashimoto, Y. & Togari, Y. (1992), 'Model following controller with specified sensitivity function', *IEEE Trans. Auto. Control* **AC-37**, 1582–1584.

Contributions to the Theory of Discrete-Time Control for Continuous-Time Systems

Abstract

Åström-Hagander-Sternby and Hagiwara-Yuasa-Araki theorems on limiting zeros of pulse transfer functions of sampled-data systems with respectively zero-order and first-order holds are extended by determining the accuracy of the asymptotic results for both the discretization and the intrinsic zeros when the sampling interval is small. Closed form formulae are derived that express the degree of the principal term of Taylor expansion of the difference between the true zeros and asymptotic ones as a function of the relative degree of the underlying continuous-time system, and the value of the corresponding coefficient itself.

A systematic approach to a class of approximations to the pulse transfer function of a system consisting of a zero-order hold and a linear continuous-time plant is presented. It is based on the asymptotic result of Åström, Hagander & Sternby (1984) on zeros of sampled systems at high sampling rates, and on the bilinear transformation. Model matching control, robust control and identification are suggested as possible areas of application. Superiority of the approximations considered over a δ-operator based truncated approximation of Goodwin et al. (1986) is shown.

Discrete-time models of sampled-data control systems are addressed when both a continuous-time plant and a discrete-time controller have a feedthrough. A new state-space model appropriate for the closed-loop modeling, and formulae for calculating the related discrete-time pulse transfer functions are derived. Intersample phenomena are studied and the feasibility of that model to describe systems with parasiting dynamics is emphasized.

Two approaches to the synthesis of a discrete-time model reference controller for a continuous-time system are presented and compared. The first one, purely discrete, bases on the discrete-time model of a dynamic system and on a discrete quadratic infinite horizon performance index while the second is based on the continuous-time integral performance index. When the sampling time tends to zero the control variable in the former problem does not converge to its continuous time prototype whereas in the latter does. The relative order of the continuous-time plant itself and the relationship between the model and plant relative orders are shown to be crucial for the design and control system behavior at high sampling rates.

A class of second-order continuous-time stochastic processes, which can be thought

as models of disturbances, is characterized and the issue of their sampling is discussed. As a result of sampling, discrete second-order random processes described by linear time-invariant state-space models are obtained. Equivalent representations with the number of noise inputs reduced to one are presented. In contrast to the innovations approach these representations have time-invariant parameters. The relationship with ARMA models is discussed and the Representations Theorem is generalized to a class of nonstationary processes. The identification issue of continuous-time processes is discussed.

A unified approach to the MV, LQG and GPC control problems based on the input-output and state-space representations of Box-Jenkins models will be presented. Its two main advantages are: an integral action of the controller attained with a realistic stationary model of the disturbance, and a reduction of the computational complexity. Moreover, it has been shown that employing Chandrasekhar equations can improve the computational efficiency for receding-horizon control problems as compared to the use of Riccati equations. The approach has also been shown to be an efficient design method for the optimal infinite horizon control systems.

Discrete-time control of continuous-time systems driven by ZOH with pulse amplitude modulation and disturbed by a stationary Gaussian process with a rational spectral density is dealt with. The algorithms considered have the form of a linear feedback from the Kalman filter. Certain time functions that characterize the performance of the continuous-time system with discrete feedback are considered. A methodology of their calculation is developed. Some results of the related works in the area are generalized and extended.

Przyczynki do teorii sterowania dyskretnego procesów ciągłych

Streszczenie

W pracy rozszerza się twierdzenia Åströma-Hagandera-Sternbyego oraz Hagiwary-Yuasy-Arakiego o zerach granicznych transmitancji impulsowych układów z ekstrapolatorami rzędu zerowego i pierwszego poprzez określenie dokładności wyników asymptotycznych dla zer wewnętrznych oraz zer dyskretyzacji dla małych okresów próbkowania. Wyprowadza się formuły wyrażające stopień członu głównego rozwinięcia Taylora różnicy pomiędzy zerami dokładnymi a asymptotycznymi jako funkcje względnego rzędu wyjściowego układu ciągłego oraz wartość współczynnika członu głównego.

Prezentuje się systematyczne podejście do klasy aproksymacji transmitancji impulsowej dla układu z ekstrapolatorem pierwszego rzędu bazujące na wyniku Åströma, Hagandera i Sternbyego (1984) dotyczącego zer asymptotycznych przy wysokich częstotliwościach próbkowania oraz na transformacji biliniowej. Pokazuje się wyższość rozważanych aproksymacji nad tak zwaną aproksymacją obciętą Goodwina i inn. (1986), bazującą na operatorze δ. Jako możliwe obszary zastosowań sugeruje się sterowanie według zadanego modelu, sterowanie odporne oraz identyfikację.

Wprowadza się nowy model w przestrzeni stanu przydatny do modelowania układów regulacji dyskretnej oraz odpowiadające mu transmitancje impulsowe w przypadku gdy zarówno obiekt ciągły jak i regulator dyskretny posiadają zerowy rząd względny. Bada się zjawiska pomiędzy chwilami próbkowania oraz podkreśla się przydatność tego modelu do opisu systemów z pasożytniczą dynamiką.

Przedstawia się i porównuje dwa podejścia do syntezy regulatorów dyskretnych z modelem odniesienia dla systemu ciągłego. Pierwsze, czysto dyskretne, bazuje na modelu dyskretnym systemu dynamicznego oraz na dyskretnym kwadratowym wskaźniku jakości podczas gdy drugie bazuje na wskaźniku ciągłym. Gdy okres próbkowania zmierza do zera zmienna sterująca w pierwszym problemie nie zmierza do swego prototypu ciągłego, podczas gdy w drugim zmierza. Pokazano, że kluczowe znaczenie dla projektowania oraz zachowania się układu regulacji ma względny rząd obiektu ciągłego oraz relacja pomiędzy względnymi rzędami modelu i obiektu.

Charakteryzuje się pewną klasę gaussowskich stochastycznych procesów ciągłych oraz dyskutuje się zagadnienie ich dyskretyzacji. W wyniku próbkowania otrzymuje się procesy dyskretne opisane przez niezależne od czasu reprezentacje w przestrzeni stanu. Przedstawia się reprezentacje równoważne z liczbą wejść losowych zredukowaną do 1. W przeci-

wieństwie do reprezentacji innowacyjnych reprezentacje te mają parametry niezależne od czasu. Dyskutuje się również zagadnienie identyfikacji procesów ciągłych.

Prezentuje się ujednolicone podejście do problemów MV, LQG oraz GPC bazujące na reprezentacjach modelu Boxa-Jenkinsa w przestrzeni stanu oraz wejściowo-wyjściowych. Jego zaletami są: osiągnięcie działania całkującego regulatora przy realistycznym stacjonarnym modelu zakłóceń oraz zmniejszenie złożoności obliczeniowej. Ponadto pokazuje się, że wykorzystanie równań Chandrasekhara poprawia skuteczność obliczeniową dla problemów ze skończonym horyzontem. Pokazuje się również, że podejście to jest efektywną metodą projektowania układów optymalnych o nieskończonych wskaźnikach jakości.

Wyznacza się charakterystyki układów ciągłych z zakłóceniem w postaci stacjonarnych procesów gaussowskich o wymiernej gęstości spektralnej regulowanych za pomocą dyskretnych w czasie algorytmów generujących sygnał stały pomiedzy chwilami próbkowania. Algorytmy te mają postać liniowych sprzężeń zwrotnych od wyjścia filtru Kalmana.