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ON THE CONNECTION BETWEEN FIELDS AND PARTICLES

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In the present paper the connection between field equations and equations of motion of particles is studied for the case of classical electrodynamics. The equations of motion and the energy and momentum of the particles are derived from the corresponding notions for the field. The mass of the particle is purely electromagnetic and is uniquely determined either by the self-force or by the self-energy and self-momentum. Most of the results are independent of the structure of the particles.

Introduction. The problem of deriving equations of motion of the particles from the field equations is a very old one. Its solution, however, was hindered by the appearance of infinite — and therefore meaningless — quantities on the one hand and by the difficulties in formulating the problem in a relativistically invariant way on the other hand.

Nowadays the formalism of the field theory seems to be developed far enough to enable us to surmount both hindrances.

In a theory where all properties of particles are deducible from the properties of the fields there is no place for the notion of a mechanical mass. This is a point which is used widely against the possibility of such a theory. For example, it is impossible to explain the mass of the neutron as its electromagnetic mass. However, it seems that these arguments do not exclude the notion of mass as originating solely in the fields if one takes into account all fields contributing to this mass as e. g. the meson fields. Besides we are not at all sure whether all fields and all kinds of particles are already known.

On the other hand an unambigous theory, free of infinities, demands an extended source model of some kind which, alternatively, may be viewed as a point source model with action at

a distance. In such a model certain functions must be introduced corresponding to the dimensions of the particle or to the distance at which action between a point particle and the field takes place. These functions are to a large extent arbitrary and they certainly may be assumed to be different not only for different particles but also for interactions of the same particle with different fields. The amount of arbitrariness introduced hereby seems to be sufficient to fit the field-masses with experimental data.

It is generally felt nowadays that the problem of elementary masses is that one which will forward the future development of the theory. From this point of view it seems interesting to study the connection of the structure of elementary particles with their masses.

There are other points which make the notion of a field mass very attractive. One of them is the fact pointed out by Feynman (1948) that only in the framwork of a pure field theory it is possible to explain classically pair creation and annihilation. Introduction of a mechanical mass would spoil this possibility. Another important point is that there are no free particles in a pure field theory. As the notion of a free particle seems to have no physical meaning (unless as a limiting case), this seems to speak in favour of a field-mass.

§ 1. General remarks. In this paper we consider only those types of field equations which are deducible from a variational principle. We restrict ourselves to Lagrangians containing first order derivatives of the fields since higher order derivatives would bring nothing physically new into the theory as pointed out elsewhere (Rzewuski 1951). There seem to be two general types of Lagrangians describing fields interacting with particles. One type gives account of particles by means of inhomogeneities, the other by means of nonlinearities. In this paper only the first type will be discussed. However, the considerations of this paragraph apply also to the second one.

An important demand on the theory is that its equations are free of divergent and therefore meaningless terms. We shall, therefore, consider a theory of the McManus (1948) type which may be viewed either as a theory of extended source or a theory of action at a distance.

The principle of least action has the general form

$$\delta \int L\left[x, A_\mu(x), \frac{\partial A_\mu(x)}{\partial x_\nu}\right] dx = 0 \qquad (1,1)$$

and the field equations are the correspondent Euler-Lagrange equations

$$\frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial \frac{\partial A_\mu}{\partial x_\nu}} - \frac{\partial L}{\partial A_\mu} = 0. \tag{1,2}$$

To make a complete transition from the field picture to the particle picture it is necessary to consider not only the field equations but also the conceptions of energy-momentum and angular momentum densities. They follow, as is well known, from the action principle (1,1) if we study its change due to translations and rotations of space-time. The energy-momentum tensor $T_{\mu\nu}$ and the angular momentum tensor

$$M_{\lambda\mu\nu} = x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda} \tag{1,3}$$

are not conserved if the Lagrangian depends explicitly on the coordinates of space-time. Indeed in this case

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = -\left(\frac{\partial L}{\partial x_\mu}\right)_{\text{expl.}}, \tag{1,4}$$

the subscript indicating differentiation with respect to the variable x contained explicitly in L,

$$\frac{\partial M_{\lambda\mu\nu}}{\partial x_\lambda} = T_{\nu\mu} - T_{\mu\nu} + x_\mu \frac{\partial T_{\nu\lambda}}{\partial x_\lambda} - x_\nu \frac{\partial T_{\mu\lambda}}{\partial x_\lambda}. \tag{1,5}$$

This accounts for the fact that there is exchange of energy and momentum with the sources.

To the densities $T_{\mu\nu}$ and $M_{\lambda\mu\nu}$ there correspond the usual integral quantities: the energy-momentum four-vector

$$T_\nu = \int d\sigma_\mu T_{\mu\nu} \quad (\sigma \text{ spacelike}) \tag{1,6}$$

and the angular momentum tensor

$$M_{\mu\nu} = \int d\sigma_\lambda M_{\lambda\mu\nu}, \quad M_{\mu\nu} = -M_{\nu\mu} \tag{1,7}$$

To make the transition to the particle picture it is necessary to have solutions of the field equations corresponding in some way to particles. Denoting the world lines of the particles by $\xi^n_\mu(\tau_n)$, $n = 1, 2, \ldots$, with the proper time τ_n as parameter, we shall demand, therefore, the solutions $A_\mu(x)$ of the field equations (1,2) to be functionals of the functions ξ^n_μ, $\dot{\xi}^n_\mu$ and possibly also higher order deri-

vatives. There is a large amount of arbitrariness in the choice of these functionals. However, if one form is chosen, the equations of motion for the particles follow from the action principle (1,1) expressed by means of the particle coordinates and its derivatives. The energy-momentum and angular momentum are similarly deducible from the corresponding quantities (1,6) and (1,7) for the field.

An example of a functional representation for $A_\mu(x)$ is

$$A_\mu(x)=\sum_n \int_{\tau_0}^{\tau_1} R_\mu^n(x, \xi^n, \dot{\xi}^n, \ldots)\, d\tau_\mu, \tag{1,8}$$

the only restriction on R_μ being that (1,8) satisfies as a function of x the field equations (1,2). In (1,8) the assumption has been made that A_μ is a sum of functionals each depending on the world line of one particle only.

In the next paragraphs we shall study in some detail the connection case between the field picture and the particle picture for the simple of electrodynamics. We hope to give a corresponding study for other fields, e. g. a non linear, in subsequent papers.

§ 2. Electrodynamics. The equations of motion. The field equations of electrodynamics are obtained from the action (1,1) with the specific Lagrangian

$$L=\tfrac{1}{4} f_{\mu\nu} f_{\mu\nu}+A_\mu \tilde{s}_\mu, \tag{2,1}$$

where

$$f_{\mu\nu}=\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu} \tag{2,2}$$

is the rotation of the field A_μ and $\tilde{s}_\mu(x)$ is supposed to represent its sources. The Lagrangian (2,1) depends explicitly on x and belongs therefore to the first type discussed in § 1. The field eqnations following from (1,1) and (2,1) are

$$\frac{\partial f_{\mu\nu}}{\partial x_\mu}=\tilde{s}_\nu. \tag{2,3}$$

To obtain herefrom the Maxwell equations it is necessary to add the subsidiary condition

$$\frac{\partial A_\mu}{\partial x_\mu}=0, \tag{2,4}$$

which changes (2,3) into

$$\frac{\partial^2 A_\nu}{\partial x_\mu^2}\equiv \Box A_\nu=\tilde{s}_\nu. \tag{2,5}$$

The solution of this equation has the general form

$$A_\nu(x) = \int G(x-x')\, dx'\, \tilde{s}_\nu(x'), \tag{2,6}$$

where $G(x-x')$ is a Green function of the equation (2,5) determined partially by the inhomogeneous equation

$$\Box\, G(x-x') = \delta(x-x'). \tag{2,7}$$

According to § 1 we have now to choose such a functional form for the field $\tilde{s}_\nu$ which would describe particles. Point particles are described by the well known formula

$$s_\nu(x) = \sum_n e_n \int_{-\infty}^{+\infty} \delta(x-\xi^n)\, \dot{\xi}^n_\nu\, d\tau_n, \tag{2,8}$$

extended particles by

$$\tilde{s}_\nu(x) = \sum_n e_n \int_{-\infty}^{+\infty} F(x-\xi^n)\, \dot{\xi}^n_\nu\, d\tau_n = \int F(x-x')\, dx'\, s_\nu(x'). \tag{2,9}$$

For relativistic reasons F must be a function of the invariant $(x_\mu - \xi^n_\mu)^2$. To account for the small dimensions of electrons F must be different from zero in the neighbourhood of ξ^n_μ only. This seems to contradict te fact that F depends on $(x_\mu - \xi^n_\mu)^2$ which is small in the neighbourhood of the whole light cone through ξ^n_μ. McManus (1948) has shown the very important fact that for a certain class of functions the contributions from points on the light cone distant from ξ^n_μ cancel and the electron is smeared over a finite space-time region only. However, for the considerations to follow the function F may be kept quite arbitrary. Only in the interpretation of the results we shall sometimes be forced to restrict it in a convenient manner.

Now, following the general method described in § 1, we may express the action principle in terms of the coordinates of the particles. For this end it is usefull to rewrite (2,1) using (2,5) and neglecting a four-divergence which would contribute an unimportant constant term to the action principle

$$\begin{aligned} L &= \frac{1}{4} f_{\mu\nu} f_{\mu\nu} + A_\mu \tilde{s}_\mu = \frac{1}{2} f_{\mu\nu} \frac{\partial A_\nu}{\partial x_\mu} + A_\mu \tilde{s}_\mu \\ &= -\frac{1}{2} A_\nu \frac{\partial f_{\mu\nu}}{\partial x_\mu} + A_\mu \tilde{s}_\mu = \frac{1}{2} A_\mu \tilde{s}_\mu. \end{aligned} \tag{2,10}$$

Introducing here (2,6) and (2,9), we get

$$\int L dx = \tfrac{1}{2} \sum_n \sum_m e_n e_m \int d\tau_n \int d\tau_m \dot{\xi}^n_\mu \tilde{G}(\xi^n - \xi^m) \dot{\xi}^m_\mu \qquad (2,11)$$

which is Fokkers action integral without the term proportional to the mechanical mass of the particle. $\tilde{G}$ stands here for

$$\tilde{G}(x-x') = \int\int F(x-x'')\,dx''\,G(x''-x''')\,dx'''\,F(x'''-x'). \qquad (2,12)$$

Denoting

$$\tilde{A}_\nu(x) = \int F(x-x')\,dx' A_\nu(x') = \sum_n e_n \int d\tau_n \tilde{G}(x-\xi^n)\,\dot{\xi}^n_\nu \qquad (2,13)$$

and using again (2,6) and (2,9) the action (2,11) may be written as follows

$$\int L\,dx = \tfrac{1}{2} \sum_n e_n \int d\tau_n L_n = 0 \qquad (2,14)$$

with

$$L_n = \sum_m e_m \int d\tau_m \dot{\xi}^n_\mu \tilde{G}(\xi^n - \xi^m)\,\dot{\xi}^m_\mu = \dot{\xi}^n_\mu \tilde{A}_\mu(\xi^n). \qquad (2,15)$$

The variational principle

$$\delta\, \tfrac{1}{2} \sum_n e_n \int d\tau_n L_n \qquad (2,16)$$

yields immediately the equations of motion of the particle

$$0 = e_n \sum_m \tilde{f}^m_{\mu\nu}(\xi^n)\,\dot{\xi}^n_\nu, \qquad n = 1, 2, \ldots. \qquad (2,17)$$

These are the Lorentz equations of motion with the inertia term hidden in the force exerted by the n-th particle on itself

$$f^{\text{self}}_\mu(\tau_n) = e_n \tilde{f}^n_{\mu\nu}(\xi^n)\,\dot{\xi}^n_\nu. \qquad (2,18)$$

To exhibit this inertia term it is necessary to make some assumptions concerning the F-function. If F belongs to that class of functions which extend the electron over a finite space-time domain only, then the world line of the particle has a certain width (instead of filling out the whole space-time). According to the properties of the G-function, interactions occur along the light cone. For the

action of a not to fast electron on itself this means that charge elements interact only when they are close to each other, the distance being of the order of magnitude of the electron radius. Now leaving out the irrelevant index n we may write (2,18) with help of (2,2), (2,6) and (2,9) in the form

$$\begin{aligned} & e^2 \int d\tau' \left\{ \frac{\partial \tilde{G}(\xi-\xi')}{\partial \xi_\mu} \dot{\xi}'_\nu - \frac{\partial \tilde{G}(\xi-\xi')}{\partial \xi_\nu} \dot{\xi}'_\mu \right\} \dot{\xi}_\nu \\ & = e^2 \int d\tau' \, \tilde{G}'[(\xi-\xi')^2] \cdot 2[(\xi_\mu - \xi'_\mu)\dot{\xi}'_\nu - (\xi_\nu - \xi'_\nu)\dot{\xi}'_\mu]\dot{\xi}_\nu , \end{aligned} \tag{2,19}$$

where $\xi' = \xi(\tau')$ and $\tilde{G}'$ means differentiation with respect to the argument. According to our assumptions about F, τ' is close to τ

$$\tau' - \tau = r \tag{2,20}$$

and we may expand (2,19) in powers of this difference, getting for the first two terms of the expansion

$$\begin{aligned} & -\frac{e^2}{2}\int \tilde{G}'(-r^2)\, 2r^2\, dr \cdot \ddot{\xi}_\mu - \frac{e^2}{3}\int \tilde{G}'(-r^2)\, 2r^3\, dr \cdot (\dot{\xi}_\mu \dot{\xi}_\nu \dddot{\xi}_\nu + \ddot{\xi}_\mu) + \dots \\ & = -\frac{e^2}{2}\int \tilde{G}(-r^2)\, dr \cdot \ddot{\xi}_\mu - \frac{2}{3} e^2 \int \tilde{G}(-r^2)\, r\, dr \cdot (\dot{\xi}_\mu \dot{\xi}_\nu \dddot{\xi}_\nu + \ddot{\xi}_\mu) + \dots . \end{aligned} \tag{2,21}$$

In the limit $F \to \delta$ of a point source, $\tilde{G}[(\xi_\mu - \xi_\mu)^2] \to G[(\xi'_\mu - \xi'_\mu)^2]$ which has the character of a $\delta[(\xi_\mu - \xi'_\mu)^2]$-function. Thus

$$\begin{aligned} & \int \tilde{G}(-r^2)\, dr \to \infty, \quad \int \tilde{G}(-r^2)\, r\, dr \to \text{const}, \\ & \int \tilde{G}(-r^2)\, r^n\, dr \to 0. \quad (n \geqslant 2) \end{aligned} \tag{2,22}$$

For processes in which the particles do not come to close to each other the first two terms in (2,21) are already a very good approximation.

$$\frac{e^2}{2}\int \tilde{G}(-r^2)\, dr = m \tag{2,23}$$

is the electromagnetic mass of the particle.

$$-e^2 \int \tilde{G}(-r^2)\, r\, dr \cdot (\dot{\xi}_\mu \dot{\xi}_\nu \dddot{\xi}_\nu + \ddot{\xi}_\mu) + \dots = f_\mu \tag{2,24}$$

is the damping force, so that equation (2,17) may now be written

$$m\ddot{\xi}^n_\mu = f^n_\mu + e_n \sum_{m \neq n} \tilde{f}^m_{\mu\nu}(\xi^n)\dot{\xi}^n_\nu, \quad n=1,2,\ldots \tag{2,25}$$

In the case where the rapidly varying motion of the particles or the structure of the function F do not allow the expansion (2,21), the correct equations (2,17) have to be used.

§ 3. Electrodynamics. The energy-momentum and angular momentum. It is necessary for consistency of the theory that the notion of energy-momentum of the particles follows from the corresponding quantity for fields and that the definition of mass obtained hereby is identical with that obtained from the equations of motion.

The energy-momentum tensor $T_{\mu\nu}$ corresponding to the Lagrangian (2,1)

$$T_{\mu\nu} = T'_{\mu\nu} + \frac{\partial}{\partial x_\alpha}(f_{\nu\alpha} A_\mu) \tag{3,1}$$

contains a divergence of a tensor antisymmetrical in the indices ν and α. Thus

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = \frac{\partial T'_{\mu\nu}}{\partial x_\nu}, \tag{3,2}$$

where $T'_{\mu\nu}$ is the conventional energy-momentum tensor

$$T'_{\mu\nu} = f_{\mu\alpha} f_{\nu\alpha} - \tfrac{1}{4}\delta_{\mu\nu} f^2_{\alpha\beta} - \delta_{\mu\nu} A_\alpha \tilde{s}_\alpha + A_\mu \tilde{s}_\nu. \tag{3,3}$$

For this tensor we have

$$\begin{aligned} \frac{\partial T'_{\mu\nu}}{\partial x_\nu} &= -A_\nu \frac{\partial \tilde{s}_\nu}{\partial x_\mu} = f_{\mu\nu}\tilde{s}_\nu + \frac{\partial}{\partial x_\nu}(A_\mu \tilde{s}_\nu) - \frac{\partial}{\partial x_\mu}(A_\nu \tilde{s}_\nu) \\ \frac{\partial T'_{\mu\nu}}{\partial x_\mu} &= f_{\mu\nu}\tilde{s}_\mu + \frac{\partial}{\partial x_\mu}(A_\mu \tilde{s}_\nu) - \frac{\partial}{\partial x_\nu}(A_\mu \tilde{s}_\mu) \end{aligned} \tag{3,4}$$

The particle momentum P_ν is not given directly by the field momentum T_ν as would be the case for point sources. As there is action at a distance the particle momentum is, so to speak, shifted in respect to the field momentum. It is easy, however, to find the expression for P_ν if one considers two conditions it has to fulfill. First it must go over into $-T_\nu$ in the limiting case of point particles ($F(x-x') \to \delta(x-x')$). Secondly the totality of sources defined by P_ν must cancel out the totality of sources defined by the first

equation (3,4). These two conditions fix the form of P_ν uniquely. Apart from irrelevant constant terms

$$P_\nu = -\int_{-\infty}^{\sigma} \tilde{f}_{\nu\mu}\, s_\mu\, dx. \tag{3,5}$$

The first condition is fulfilled since we may rewrite $-T_\nu$ in the following way

$$\begin{aligned} -T_\nu &= -\int_\sigma d\sigma_\mu\, T'_{\mu\nu} = -\int_{-\infty}^{\sigma} \frac{\partial T'_{\mu\nu}}{\partial x_\mu}\, dx \\ &= -\int_{-\infty}^{\sigma} \left[f_{\mu\nu}\tilde{s}_\mu + \frac{\partial}{\partial x_\mu}(A_\mu \tilde{s}_\nu) - \frac{\partial}{\partial x_\nu}(A_\mu \tilde{s}_\mu) \right] \end{aligned} \tag{3,6}$$

again leaving out irrelevant constant contributions from infinitely remote surfaces. Now for $F(x-x) \to \delta(x-x')$ the first term in (3,6) becomes identical with (3,5). The contribution from the two other terms is

$$\int_\sigma (d\sigma_\nu s_\mu - d\sigma_\mu s_\nu) A_\mu. \tag{3,7}$$

Here the factor $s_\mu(x)$ contains $\delta(x-\xi^n)$ (cf. (2,8)) so that only these points of σ contribute in which the world lines of the particles hit the surface. Since other points do not affect the value of (3,7) we may always choose σ to be orthogonal to the world lines. In this case (3,7) may be written

$$\int_\sigma d\sigma_\nu (s_\mu - s_\mu) A_\mu \equiv 0. \tag{3,8}$$

To prove the second condition we let σ move towards infinity

$$\begin{aligned} P_\nu(\infty) &= -\int_{-\infty}^{+\infty} \tilde{f}_{\nu\mu}\, s_\mu\, dx = -\int_{-\infty}^{+\infty} s_\mu \left(\frac{\partial \tilde{A}_\mu}{\partial x_\nu} - \frac{\partial \tilde{A}_\nu}{\partial x_\mu} \right) dx \\ &= -\int_{-\infty}^{+\infty}\int s_\mu \left(\frac{\partial F(x-x')}{\partial x_\nu} A'_\mu - \frac{\partial F(x-x')}{\partial x_\mu} A'_\nu \right) dx\, dx' \\ &= -\int_{-\infty}^{+\infty} \left(-\frac{\partial \tilde{s}_\mu}{\partial x_\nu} A_\mu + \frac{\partial \tilde{s}_\mu}{\partial x_\mu} A_\nu \right) dx \\ &= -\int_{-\infty}^{+\infty} \left[f_{\nu\mu}\tilde{s}_\mu + \frac{\partial}{\partial x_\mu}(A_\nu \tilde{s}_\mu) - \frac{\partial}{\partial x_\nu}(A_\mu \tilde{s}_\mu) \right] dx = -\int_{-\infty}^{+\infty} \frac{\partial T'_{\nu\mu}}{\partial x_\mu}\, dx. \end{aligned} \tag{3,9}$$

In order to express P_ν in terms of the coordinates and velocities of the particles we insert (2,8) into (3,5) and get

$$P_\nu = -\sum_n e_n \int_{-\infty}^{+\infty} d\tau_n \int_{-\infty}^{\sigma} dx\, f_{\nu\mu}\, \delta(x - \xi^n)\, \dot{\xi}^n_\mu = -\sum_n e_n \int_{-\infty}^{\tau_n^\sigma} d\tau_n \tilde{f}_{\nu\mu}(\xi^n)\, \dot{\xi}^n_\mu . \tag{3,10}$$

Here τ_n^σ is that value of τ_n for which the world line of the n-th particle intersects the surface σ.

First we notice that due to the equations of motion (2,17) P_ν is independent of the proper times τ_n^σ. In fact its value is zero for all τ_n^σ. Thus P_ν is conserved. In the field picture the energy-momentum density tensor had a non vanishing divergence according to local exchanges of energy-momentum between field and sources. In the particle picture the total momentum of the particles is conserved in time since all contributions from local exchanges of energy with the field vanish.

(3,10) consists of two parts. The self-momentum of the particles

$$P_\nu^{\text{self}} = \sum_n e_n \int_{-\infty}^{\tau_n^\sigma} d\tau_n \tilde{f}^n_{\nu\mu}(\xi^n)\, \dot{\xi}^n_\mu \tag{3,11}$$

and the momentum due to interactions with other particles

$$P_\nu^{\text{int}} = \sum_{n \neq m} \sum e_n \int_{-\infty}^{\tau_n^\sigma} d\tau_n \tilde{f}^m_{\nu\mu}(\xi^n)\, \dot{\xi}^n_\mu \tag{3,12}$$

(Because of $P_\nu \equiv 0$ we may arbitrarily choose the sign in (3,10)). As already mentioned, for consistency of the theory it is necessary that the definition of mass following from the equation of motion is the same as that which may be obtained from the expression for the momentum. That it is so follows immediately from the fact that for the self-force (2,18) and the self-momentum of the n-th particle (n-th term of (3,11)) the relation

$$\frac{dP_\nu^{\text{self}}}{d\tau_n^\sigma} = f_\nu^{\text{self}}(\tau_n^\sigma) \tag{3,13}$$

holds.

(3,10) may be rewritten so as to contain only particle coordinates and velocities. Using for this purpose equation (2,13) we get

$$P_\nu = \sum_n \sum_m e_n e_m \int_{-\infty}^{\tau_n^\sigma} d\tau_n \int_{-\infty}^{+\infty} d\tau_m \left\{ \frac{\partial \tilde{G}(\xi^n - \xi^m)}{\partial \xi_\nu^n} \dot{\xi}_\mu^m - \frac{\partial \tilde{G}(\xi^n - \xi^m)}{\partial \xi_\mu^n} \dot{\xi}_\nu^m \right\} \dot{\xi}_\mu^n$$

$$= \sum_n \sum_m e_n e_m \left\{ - \int_{-\infty}^{\infty} d\tau_m \tilde{G}[\xi^n(\tau_n^\sigma) - \xi^m] \dot{\xi}_\nu^m + \int_{-\infty}^{\tau_n^\sigma} d\tau_n \int_{-\infty}^{\infty} d\tau_m \frac{\partial \tilde{G}(\xi^n - \xi^m)}{\partial \xi_\nu^n} \dot{\xi}_\mu^n \dot{\xi}_\mu^m \right\}. \tag{3,14}$$

Assuming $\tilde{G}$ to be symmetric in ξ^n and ξ^m we get (cf. Feynman 1948)

$$\int_{-\infty}^{\tau_n^\sigma} d\tau_n \int_{-\infty}^{\tau_m^\sigma} d\tau_m \frac{\partial \tilde{G}(\xi^n - \xi^m)}{\partial \xi_\nu^n} \dot{\xi}_\mu^n \dot{\xi}_\mu^m$$

$$= - \int_{-\infty}^{\tau_n^\sigma} d\tau_n \int_{-\infty}^{\tau_m^\sigma} d\tau_m \frac{\partial \tilde{G}(\xi^n - \xi^m)}{\partial \xi_\nu^n} \dot{\xi}_\mu^n \dot{\xi}_\mu^m = 0 \tag{3,15}$$

and

$$P_\nu = \sum_n \sum_m e_n e_m \left\{ - \int_{-\infty}^{+\infty} d\tau_m \tilde{G}[\xi^n(\tau_n^\sigma) - \xi^m] \dot{\xi}_\nu^m + \int_{-\infty}^{\tau_n^\sigma} d\tau_n \int_{\tau_m^\sigma}^{\infty} d\tau_m \frac{\partial \tilde{G}(\xi^n - \xi^m)}{\partial \xi_\nu^n} \dot{\xi}_\mu^n \dot{\xi}_\mu^m \right\}. \tag{3,16}$$

Similar considerations yield for the total angular momentum of the particles the expression

$$N_{\nu\mu} = \int_{-\infty}^{\sigma} (x_\nu \tilde{f}_{\mu\alpha} s_\alpha - x_\mu \tilde{f}_{\nu\alpha} s_\alpha) \, dx \tag{3,17}$$

going over into

$$N_{\nu\mu} = \sum_n e_n \int_{-\infty}^{\tau_n^a} d\tau_n [\xi_\nu^n \tilde{f}_{\mu\alpha}(\xi^n) - \xi_\mu^n \tilde{f}_{\nu\alpha}(\xi^n)] \ddot{\xi}_\alpha^n . \tag{3,18}$$

We notice the important fact that most of the considerations of this paper are quite ingependent of the form of the function F. Special assumptions were needed to exhibit the inertia term. In this case $F(x)$ had to be large only for small values of the variables. This assumption is admissible for processes in which we may regard the particles as moving at large distances from each other. For close collisions we shall have F-functions which do not have this property. The possibility of a transition from the field picture to the particle picture, however, is not influenced by this fact.

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β-SPECTRUM OF RaE IN THE LOW ENERGY REGION*

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(received January 6, 1952)

Electron sensitive nuclear research emulsion was used as an electron detector in a magnetic lens spectrograph for the investigation of the β-spectrum of RaE in the low energy region. The intensities were found by counting of some 35 000 seperate electron tracks. Very thin sources were prepared by a volatilization method. The images obtained in the spectrograph were checked by a special method to prove the uniformity of the specific surface activity of the sources used. The spectrum was investigated within the energy band from 100 keV down to 4.6 keV. The results show that the energy distribution corresponding to this forbidden transition cannot be expressed simply in terms of $(E—E_0)^4$. It seems that the energy distribution curve cuts the $P(E)$ coordinate.

Introduction. The β-decay of Ra E is classed as a j-forbidden transition. This is confirmed by a half-life (5 days) long in comparison with the high upper energy limit (1.17 MeV) and by a particular shape of the energy distribution curve which obviously differs from the Fermi allowed form. On the other hand β-emission is unaccompanied by γ-rays and therefore the β-spectrum cannot be complex. Konopinski has pointed out that a combination of two types of interaction can explain the observed distribution at least in a certain energy region. The corresponding term in the Fermi allowed distribution formula has to be replaced by $(E—E_0)^4$.

Many papers have been published about the shape of the energy distribution curve of the β-spectrum of Ra E especially in the low energy region. Some authors (Waltner and Roggers 1948, 1949) suggested that this curve falls down to zero at zero energy which would be in strong contradiction to the Fermi theory applied to an element

* A part of this work was presented at the XIIIth Conference of Polish Physicists at Cracow, December 6, 1950.

of high Z-value, and would be difficult to understand even in the case of a forbidden transition.

There are many effects which can distort the spectrum in this region: the straggling effect in the source, the electron scattering in the spectrometer and the distortion introduced by the detector. Waltner and Roggers used a cloud chamber. They introduced the radioactive source in vapour form and so they could avoid the straggling effect. Their results were supported by rather poor statistics of the tracks. Langer and his co-workers (1950) used a window-less Geiger counter to avoid other disturbing effects.

The author tried to solve this problem by using electron sensitive nuclear research emulsions and very carefully prepared thin sources. In this way be hoped to check the results by Waltner and Roggers and on the other hand to test the usefullness of the emulsion as an electron detector in general.

This problem consists mainly of:

(a) the construction of a spectrograph for the recording of electrons in emulsions,

(b) working out of the best conditions for the emulsion processing to get very well defined low energy tracks at the emulsion surface,

(c) the choice of a proper method for the preparation of very thin Ra E sources,

(d) the checking of the efficiency of the emulsion as a detector for electrons of very low energies.

All these points but the last one were succesfully dealt with.

The β-spectrograph. One can easly obtain high luminosity and a well defined image using a magnetic lens spectrograph with uniform field. In designing the spectrograph advantage was taken of some calculations given by E. Persico and Du Mond (1949). The resolving power was about 1.8%. The emission angle was equal to $27°20' \pm 2°20'$. 2.2% of the total spherical emission of the source could pass through the solid angle defining diaphragm, but due to the geometry of the detector plate it had to be reduced to 1/4 of this value. The distance between the source and the plate was about 30 cm. The lead stop protecting the plate from γ-radiation of Th $B+C+C''$, which was used for preliminary experiments, was more than 10 cm thick.

The source diameter could not be larger than 1,6 mm and actually was ca 1,1 mm. It was prepared on a zapon film which was coated with an extremely thin layer of gold to increase its conductivity. The whole thickness of the source, zapon and gold taken together,

was not greater than $4-4\frac{1}{2}$ $\mu g/cm^2$. The film was suspended on a special Al-ring fastened to a support similar to a microscopic stage. An empty space deep enough (20 cm) to eliminate the back-scattering was left behind the source. It was not possible for a scattered electron to get through all the diaphragms unless it were scattered at least twice or three times. Moreover the scattered electrons which could eventually reach the plate produced tracks of different lengths,

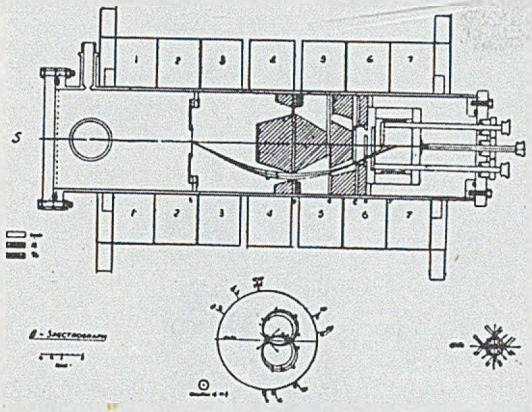

Fig. 1. (β) spectrograph; (S) source and its support; (B) solid angle defining diaphragm; (C) energy defining diaphragm, „ring-shaped focus"; (G), (H) baffles; (T) „line-shaped focus", plateholder and plate; (1—7) magnetic coils.

whereas the electrons running directly from the source all gave more or less identical tracks. During an exposure taken for 24 hours without any magnetic field no tracks were recorded.

The plate was placed along the axis of the spectrograph, the line-shaped focus lying in the plane of the emulsion (Fig. 1, 2). It was possible to take ten exposures on the same plate not getting in the air. To avoid electron dissipation all over the plate electrons which could reach its surface at very small angles had to be eliminated. Hence it proved necessary to reduce the utilized solid angle in the plane perpendicular to the axis to one fourth of its original value. All the diaphragms were partly closed and the opening of the energy defining diaphragm was just reduced to 1/4. All the openings were twisted to suit the helical form of the electron trajectories. Eventually the electrons reached the surface of the emulsion at angles between 18^0 and 30^0 inside the contour of a rectangular image 12 mm × 2.3 mm (1.5 mm only when the source diameter was reduced to 1.1 mm).

The main parts of the spectrograph can be seen on the photograph. (Fig. 2).

The magnetic field was produced by 7 coils. Its uniformity was controlled in numerous points by means of several rotating coils. After some corrections had been made the axial symmetry of the field was very good but the intensity of the field was dropping slightly along

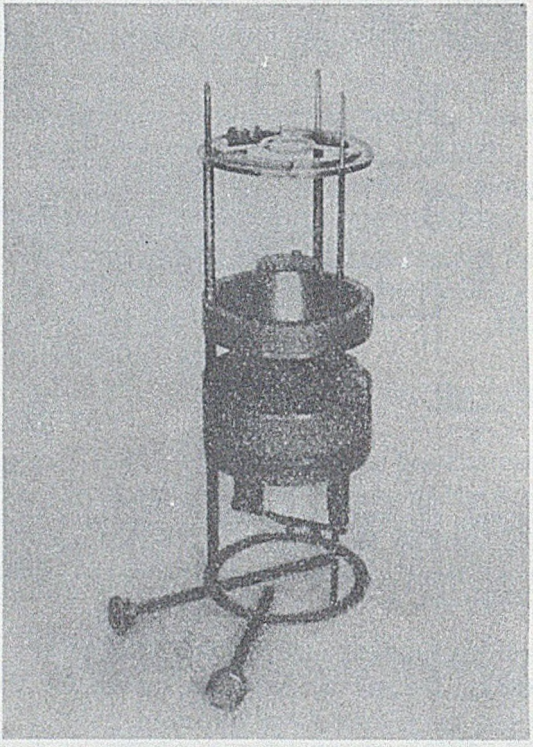

Fig. 2. The inner parts of the spectrograph.

the axis towards the ends; at the beginning of the electron trajectories it was ca 0.7% smaller than in the centre of the coil. Some small corrections in the position and size of the diaphragms were introduced in connection with this non-uniformity.

The proper position of the source which should be exactly on the axis of the spectrograph could be found by using a special rotating coil. In this way one could fix the best position of the microscopic stage supporting the Al-ring and the source. Therefore the source could be easily replaced by a new one.

The whole spectrograph was shielded by lead walls. Under these conditions exposures could be taken for many hours without increasing the number of background tracks.

The current flowing through the coils was supplied by a battery and stabilized to 0.1%. It was controlled continuously by the usual compensation method with a standard cell. The spectrograph was placed 2 meters away from any iron objects and 4 meters away from the rheostats. The spectrograph axis was directed parallel to the earth's magnetic field.

Intensity distribution within the image and source control. The diagram, Fig. 3, shows the calculated intensity distri-

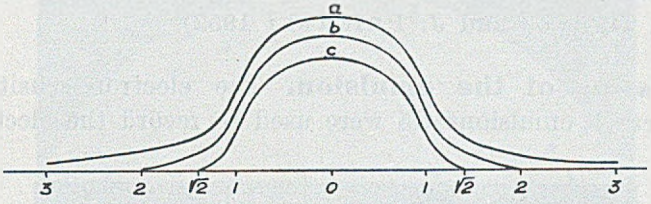

Fig. 3.

bution across the image when the openings of the diaphragms were reduced to (a) 180°, (b) 120°, (c) 90°, uniform activity of the source being assumed. In case (c) the limits of the image are well defined. Some intensity distributions calculated for different assumptions concerning the source uniformity are shown in other diagrams (Fig. 4).

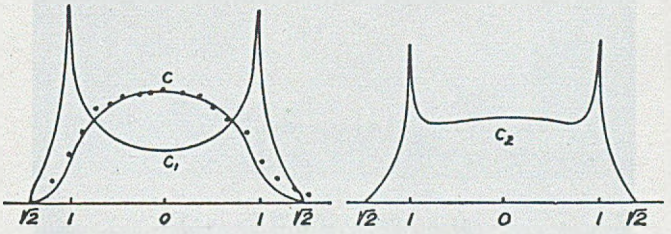

Fig. 4.

The curve C_1 corresponds to the case of a circular source active only on the periphery of the circle and C_2 is a superposition of C and C_1. It should be a rather usual distribution for sources prepared by ordinary methods when a drop of radioactive substance is dried by evaporation. The author's experimental points lie rather close to a curve calculated for an uniform source. The wings of the experimental distribution curve show that the contamination of the zapon film outside the radioactive spot was negligible.

These tests of the source are very useful although they cannot detect all irregularities of the source. They give a record of the source behaviour during the exposure without running any risk of destroying the zapon film, as often happens when autoradiographs are taken. It is an exclusive method for the control of the results obtained when the source is destroyed during the exposure by α-particles. Sometimes there is no need to count seperate tracks, one could easily obtain the whole intensity distribution curve from plate blackenings using a microphotometer.

Some other details about source preparation were published previously (M. Danysz and J. Pniewski 1952).

Processing of the emulsion. The electron sensitive Ilford nuclear research emulsions G5 were used to record the electrons. The

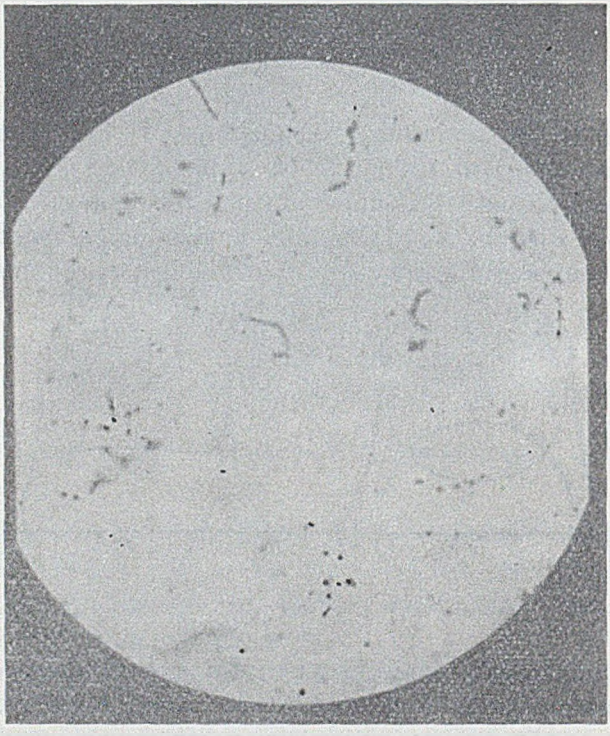

Fig. 5. 57 keV — electron tracks.

spectrum was investigated in the low energy region only, hence the emulsion thickness could be reduced to 25 μ. This facilitated the processing of the emulsions. It was important to have the surface of the

emulsion very clear, otherwise very short tracks could be confused or even lost among the background grains.

It proved to be advisable not to agitate the developer dish. Development at 18°C: azol — 18 ml, KBr — 930 mg, H_2O — 500 ml, for 20 min. no agitation;

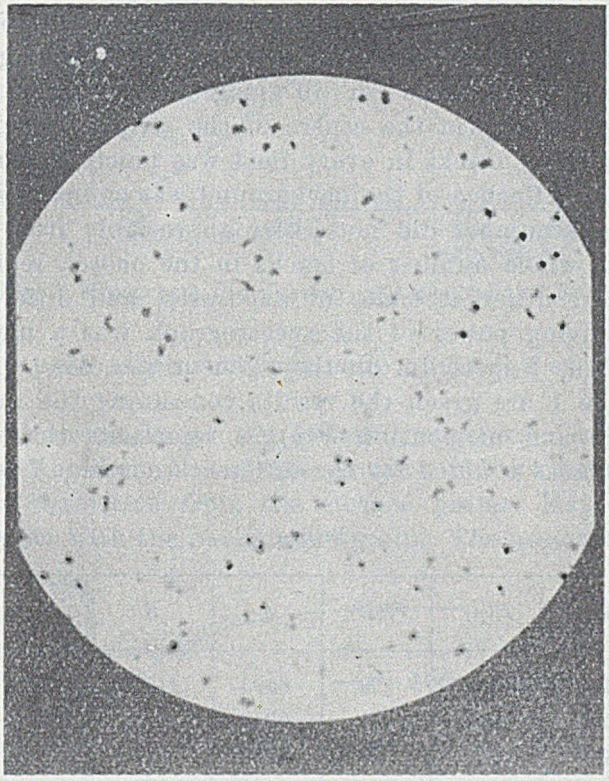

Fig. 6. 15 keV — electron tracks.

Stop bath: $NaHSO_3$ — 25 g, tap water 500 ml for 10 min; fixation: plain hypo solution for 15 min. (Dainton 1951).

The photographs (Figures 5 and 6) show some electron tracks for different energies.

Microscopy of individual tracks. Six well exposed plates were obtained, but after a very careful examination only four of them were scanded to get the β-spectrum of Ra E between 4,6 and 100 keV. All electron images obtained on different plates corresponded

to 11 energy values. The images had the shape of rectangular stripes 12 mm × 1,5 mm. The tracks in these images were examined through a Cooke microscope at magnification exceeding 1200 ×.

At first the counting was extended over all fields of view in a zone 50 μ wide perpendicular to the long side of the stripe. The numbers of tracks obtained for each field were used to get the intensity distribution across the stripe. From the points similar to those shown in Fig. 4 the symmetry line of the stripe could be found, and only those tracks were taken into account which belonged to a well defined region — the same for all images. This region was always chosen symmetrically in the centre of the stripe. In this region the average number of tracks in every field was much higher than in the wings and the influence of the background was comparatively smaller. Moreover their number did not differ appreciably from one field to another. The whole number of tracks in the chosen region corrected for the decay, different times of exposures, and different value of absolute resolving power of the spectrograph was a measure of the intensity in the β-spectrum on the momentum scale.

In Table 1 are given the results concerning the distribution of energy $P'(E)$ and momentum $P(p)$; n represents the total number of electron tracks counted for the different energies, N is the number

Table 1.

E keV	$H\varrho$	$P(p)$	$P'(E)$	n	N	N_{max}	X
4.6	251	28.5	59	6003	5	300	90
7.1	286	42.5	71	6217	9		92.8
12.9	386	63	80	3336	16		95.1
15.4	422	80	93	1443	23		98.6
16.8	441	80	90	2483	24	200	97.6
21.2	496	100	100	6720	34	150	100.0
28.4	578	121	105	1075	50		101
37.1	661	145	112	2090	69	100	102
56.9	826	180	115	737	106	50	101.5
72.1	936	211	122	1247	141		102.1
97.6	1102	240	123	718	188	20	96

of tracks per hour recorded on the plate in this spectrograph in a field of view 100 μ × 100 μ when the activity of the Ra E source was 0.1 mC, N_{max} — shows the maximal number of recorded tracks which

could be still easily identified on that area. To test the purity of the Ra E sources exposures were repeated at long time intervals while the β-decay was in progress. Their comparison was especially useful

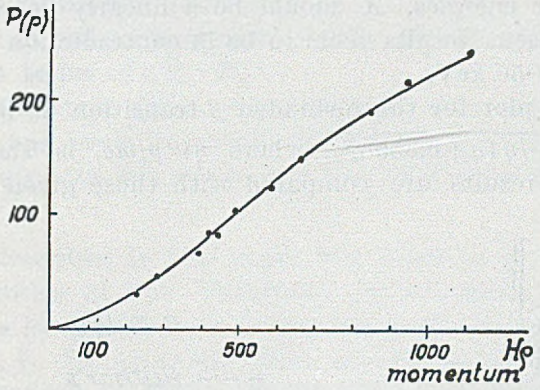

Fig. 7. Momentum distribution in β-spectrum of RaE.

at the energy point where the maximum of Ra D β-spectrum was to be expected.

The electron tracks at the lowest energy were composed of 1 or 2 grains and even the size of these grains could not serve as a proper criterion for distinction from the surface grains. Hence they were counted together with the background grains. The necessary correction

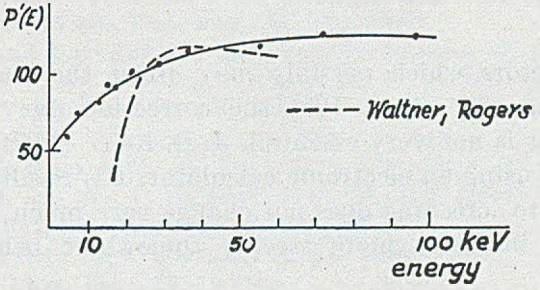

Fig. 8. Energy distribution in β-spectrum of RaE.

could be easily made by subtracting the background grain number as counted outside the image.

In Figs 7 and 8 are shown the graphs of the momentum and energy distributions and also a plot obtained by Waltner and Rogers (1949). The present results differ from those obtained by these authors in a cloud chamber. The intercept of energy plot at the zero energy point is different from 0. One should expect this result for an element with a high Z value.

The last column of Table 1 shows the values of $X=\sqrt[4]{P(p)/p^2F(Z,E)}$. If the suggestion of a combination of two interactions made by Konopiński is correct and the approximation given by him is valid down to lower energies, X should be a linearly decreasing function of E. The present results seem to be in contradiction with this view at least below 50 keV.

A Fermi plot for the forbidden β-transition in Ra E with a coordinate $\sqrt{P(p)/\eta\,(\eta+0.355\,\eta^2)}$, where $\eta=p/mc$, is shown in Fig. 9. The author's results are compared with those given by Wu (1949).

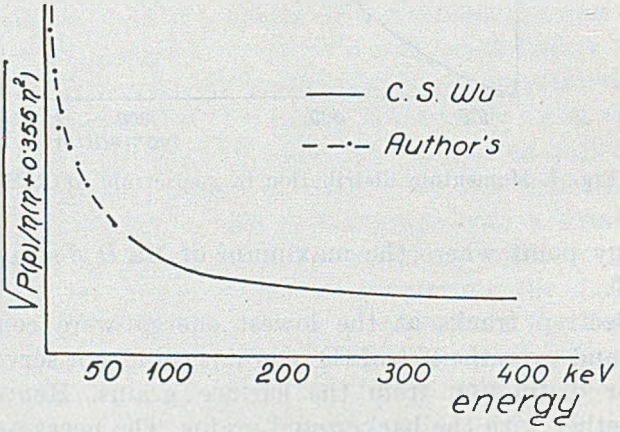

Fig. 9. The Fermi plot.

As to factors which possibly may affect the true shape of the β-spectrum one can expect that the correction due to the electron screening effect is not very essential. J. R. Reitz (1950) has calculated this correction using an electronic calculator. His results show that the reduction due to screening does not change very much with the energy and probably increases more steeply somewhere below 10 or even 5 keV.

Maybe some low energy electrons were not recorded due to backscattering at the surface of the emulsion or due to non-uniform sensitivity of the emulsion. This problem is discussed separately (J. Pniewski 1952) but one can expect that this error is not serious above 5 keV.

Summary

It was proved that the nuclear research emulsions can be used as a detector in β-spectrography and that this method has certain advantages at the low energy end of the β-spectrum.

The Ra E β-spectrum was obtained for energies from 100 keV down to 4,6 keV.

The intercept of the energy distribution curve at zero energy point seems to be different from zero.

The energy distribution function below 50 keV cannot be interpreted simply in terms of $(E-E_0)^4$.

The Fermi plot is given and compared with previous results.

A method has been described for analyzing of spectrographic images. In this way the behaviour and the uniformity of the sources were controlled.

The work described in this paper was made in the George Holt Physics Laboratories at the University of Liverpool. The Author wishes to express his gratitude to Professors S. Pieńkowski, J. Rotblat and W. H. B. Skinner for their encouragement and kind interest. He is indebted to Messrs M. Danysz and D. G. E. Martin for their valuable help and many discussions. He is also much obliged to the Komisja Popierania Twórczości Naukowej i Artystycznej przy Prezydium Rady Ministrów for granting of the scholarship which enabled him to carry out this research in Liverpool.

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A METHOD OF PREPARATION OF VERY THIN RaE SOURCES

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(received January 6, 1952)

A volatilization method of preparation of comparatively strong but extremely thin RaE sources for β-spectroscopy is described. A source 50 μC per 1 mm^2, supported by a conductive film, the whole less than 4 μg/cm^2 thick was prepared. The authors believe that the method described is new in many details.

Introduction. It has been recognized for a long time that the usual method of evaporation of a small drop containing a solution of radioactive substance has some disadvantages. The active deposit which is formed by the evaporation process does not exhibit the required uniformity. At the periphery of the spot some crystals are formed and the radioactivity at the centre is much weaker. Hence the effective thickness of the source at the periphery is much greater than the average value calculated over the whole spot. Such a source cannot be used for investigations of the β-spectrum in the low energy region. To avoid distortion of the β-spectrum caused by straggling in the supporting film and in the source itself at 5kev sources should be used not thicker than 4—5 μg/cm^2.

Langer and his co-workers (1948, 1950) suggest to wet the supporting film with a solution of insulin in water. It facilitates the uniform spreading of the radioactive substance over the entire region covered by the insulin solution. It seemed to the authors that this method is not applicable if the source is very small.

The supporting film must be made of a material which is not a very poor conductor, otherwise it would not be possible to keep the source at the constant zero potential.

The authors were able overcome these difficulties by applying the volatilization method. They were successfull in getting a uni-

* Presented at the XIIIth Conference of Polish Physicists of Cracow, December 6, 1950.

form Ra E spot source slightly larger than 1 mm in diameter as strong as 80 μC supported on a zapon film 2—3 $\mu g/cm^2$ thick. The film was covered with ca $1\frac{1}{2}$ $\mu g/cm^2$ gold coating. The increase of the conductivity of the surface was sufficient to remove the charging effects caused by the leaking out of negative charge in the β-emission of Ra E or α-emission of Polonium.

Volatilization of Ra E. The authors collected Ra E at the end of a rounded nickel rod by the usual electrochemical method. It was important to protect the whole rod except the end against any contamination. The whole activated area was probably not much

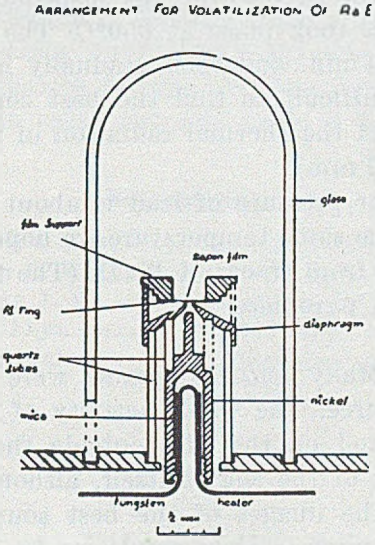

Fig. 1.

larger than 5 mm^2. The other end of the rod was thicker (10 mm) and drilled inside. The hollow was provided for the placing of a tungsten heater. All the other parts of the special arrangement for volatilization, the position of the film and the diaphragm can be clearly seen on the drawing, Fig. 1. The diaphragm was supported and spaced from the nickel rod by two quartz tubes.

The whole apparatus was placed in high vacuum (the external vacuum bell is not shown in the diagram). One can notice that Ra E atoms which leave the nickel rod and do not condense directly on the zapon film cannot contaminate it on any other place beyond the spot unless they are scattered many times on their way. All the parts of the apparatus could be quickly evacuated through the open-

ings marked on the diagram by dashed lines. It was proved to be important to keep the air pressure at a low level — less than 10^{-3} mm Hg. In this way it is possible to avoid the oxydation of the RaE layer (bizmuth) which makes the volatilization more difficult. It also diminishes the scattering of Ra E atoms. The scattered atoms stuck to the walls of the apparatus only when its surface was carefully washed.

The zapon film was composed of two layers each $1\frac{1}{2}$ $\mu g/cm^2$ thick, supported on a thin Al-ring which was stuck on a firm brass support.

At first the nickel rod with the Ra E deposit was heated up to 500—550°C and held at this temperature in high vacuum for ten minutes in order to remove more volatile impurities of Ra E. After this preliminary heating the zapon film was suspended above the rod and volatilization of Ra E took place at 650°C. The temperature was increased slowly for 10 min. and then gradually lowered.

It was rather difficult to find the best conditions under which the zapon could stand the thermal radiation of the nickel rod placed very near to it (ca 2 mm).

Since the vapour pressure of lead is about 10 times lower than that of bizmuth at the same temperature we hoped this method might serve to purify Ra E from traces of Ra D. (The temperature was controlled by an optical pyrometer).

The source. Many autoradiograms were taken to prove the uniformity of the source. The whole activity of the substance spread over the Al — ring and on the film outside the spot was less than 0.3% of the activity of the source itself, although their surface was 100 times greater. The images of the best sources were reproduced on electron sensitive nuclear plate emulsions by means of a magnetic lens spectrograph. In this way they could be investigated without running any risk of breaking the zapon film.

Each zapon film after being activated was thinly coated with gold by evaporation in vacuum of a small gold leaf placed 25 cm away from the film. The authors found that the resistance of the film under the conditions of their experiments was about $10^{12}\,\Omega$ and after drying the film for many hours in high vacuum it increased to about $10^{14}\,\Omega$. A coating of gold 10 $\mu g/cm^2$ thick would be sufficient for metallic conductivity but even $1\frac{1}{2}$ $\mu g/cm^2$ could reduce the resistance at least to the previous value ($10^{12}\,\Omega$) which was low enough to keep the potential difference between the source and the earth below 10 volts.

It proved to be more convenient not to coat the side of the zapon film which was stuck to the Al-ring but the opposite one. A very small potential difference was adequate to obtain the connection between

aluminium on one side of the film and gold on the other. It would be also better to collect the Ra E and gold deposits on opposite sides of the zapon film.

It was not possible to use any strong source prepared in this way for a time longer than 10—14 days. They always cracked under the influence of the α-particles of polonium produced by the β-decay of Ra E.

This work has been done in George Holt Physics Laboratories at the University of Liverpool. The authors take pleasure in tkanking Professor W. H. B. Skinner for his kind interest. They wish to thank Mr D. G. E. Martin for valuable help and many discussions and are also indebted to the Staff of the Radiochemical Centre for collaboration in collecting of the Ra E deposit. They are also greatly obliged to the Komisja Popierania Twórczości Naukowej i Artystycznej przy Prezydium Rady Ministrów for the granting of foreign scholarships which enabled them to carry out this work in Liverpool.

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ON THE RELATION BETWEEN THE ENERGY AND THE NUMBER OF GRAINS FOR LOW ENERGY TRACKS IN PHOTOGRAPHIC EMULSIONS

By J. PNIEWSKI, Institute of Experimental Physics, University of Warsaw

(received January 6, 1952)

The relation between the energy and the number of grains at low energies for Ilford electron sensitive emulsion G5 was investigated. The observations were extended over energies from 37,1 keV down to 4,2 keV. A good separation of joined or overlapping grains was achieved by resoaking the emulsion before microscopic examination.

In some cases it may be more advantageous to apply electron sensitive photographic emulsion for the study of β-spectra instead of using other methods, in particular for measurements in the low energy region. Hence it would be important to extend the determination of the range-energy relation down to very low energy. Unfortunately such measurements are very difficult due to very large straggling. Moreover the tracks are usually curled so that the measurements of ranges are unfeasible. Grain counting is more practicable, nevertheless fluctuations remain as a source of troubles. The author has found that the results are more consistent if one can separate the grains in these parts of the track which are going steeply down or up in the emulsion.

The thickness of the emulsion after processing and drying is usually shrinked by a factor of 2,7. One can plump the emulsion by resoaking it in water. A coverglass gives a good protection from drying. In this work the soaked emulsion was covered with a large coverglass 170 μ thick. To make it tight the edges and the uncovered parts of the emulsion outside the glass were coated with paraffin. Under these conditions the thickness of the emulsion increases enormously. An unprocessed emulsion 25 μ thick shrinked after processing to 9 μ and when resoaked increased its thickness to 50 μ. The virtual thickness

of the emulsion observed through the microscope with immersion oil was equal to 60 μ. Limited depth of focus at large magnification, e. g. 1500 ×, helps to separate the grains of curled tracks. Many tracks which seemed plane in a dry emulsion appeared to be skew after the emulsion had been soaked, i. e. their grains were found at different depths. The counting of single tracks is very laborious in a plumped emulsion but on the other hand their identification is rendered easier.

Ilford G 5 electron sensitive emulsions exposed in a magnetic lens β-spectrograph were used for the observations. Processing conditions: azol-developer (Dainton 1951, Pniewski 1952).

The diagrams shown in Fig. 1 give the distributions of the grain number in the tracks of β-particles for a given energy; the number

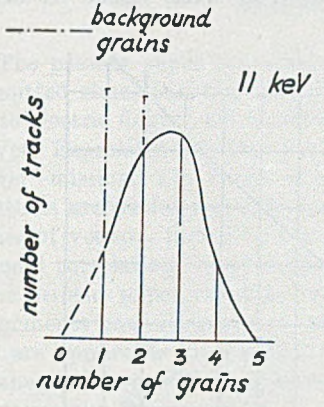

Fig. 1a.

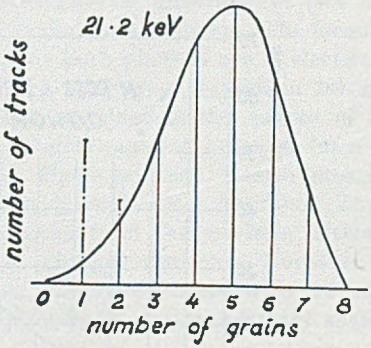

Fig. 1b.

of surface or background grains was previously subtracted. In this way the average number of grains was found for several energy values. By extrapolation of the distribution curves one obtains a basis for estimation of an important factor: the ratio of electrons which owing to backscattering at the surface of the emulsion are not recorded. The shape of the distribution curves for 21,2 kev and 11 kev shows that in these two cases the percentage of backscattered electrons is negligible. This is not so clear at lower energies but even at 7 kev the majority of the electrons produce two-grains tracks and most of the remaining — single-grain „tracks" Three-grain tracks are rare. The effect of background is more pronounced, but the numbe of unrecorded „zero-grain tracks" should not be large and probably can be disregarded as well.

The author noticed that even at 3—4 kev electrons are easily recorded by one-grain „tracks". It seems that above 5 kev the number

of unrecorded electrons could not appreciably change the results based on the counting of tracks.

Table of the average number of grains in the electron tracks for different energies.

keV	4,2	7,5	11	21,2	28,4	30	37,1	40	50	60
B. Zając M. Ross						11 ±0,4		13,8 ±0,5	20,4 ±0,8	22,4 ±0,6
Author's	1	1,5—2	3 ±0,5	5 ±0,5	8 ±0,5		12,5 ±1			

In this table the number of grains is given for different energy values and the results obtained are compared with those of Miss B. Zając and Miss M. Ross (1949).

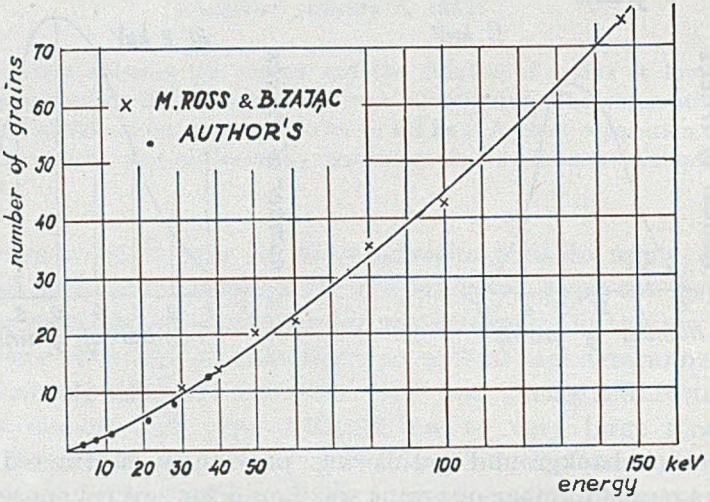

Fig. 2.

It should be emphasized that the number of grains in a twisted track may be reduced when the electron leaves the emulsion before being stopped.

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DIRECTIONAL CORRELATION FOR TWO-QUANTA EMISSION

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The present paper contains correlation formulae for directions of two photons emitted simultaneously in a single act of two-quanta emission. The formulae apply to electric double-dipole radiation, i. e. where both photons are of electric dipole type. Especially interesting is the transition $j'=0 \to j=0$ forbidden for one-quantum emission. The forms of matrix elements obtained in the course of the calculations are useful not only for the electric dipole moment but also for a wider class of vectors, including i. a. the magnetic dipole moment. Measurements of directional correlations of photons are admittedly possible for γ radiation. Therefore the present paper refers to two quanta emission from the nucleus, although the arguments equally apply to any system of charged particles. Moreover the results are applicable after a slight and obvious change of coefficients to other simultaneous processes of second order (in double-dipole approximation) such as for instance the Raman effect.

1. Introductory Note

Kramers and Heisenberg were the first to indicate the possibility of an induced two-quanta emission (1925). On the correspondence principle they evaluated the possibility of this process. On the basis of the Dirac theory of radiation M. Göppert-Mayer (1931) has given the general formula of probability of the two-quanta emission. The probability of the two-quanta emission from the metastable state $2s$ of hydrogen and from the metastable states $(1s\ 2s)^1S_0$, $(1s\ 2s)^3S_1$ of helium was obtained by G. Breit and E. Teller (1940). The probability of the two-quanta emission for the transition $j'=0 \to j=0$ was estimated by R. G. Sachs (1940). J. A. Wheeler (1947) investigated the process of the two-quanta emission by means of the correspondence principle.

The initial idea of the present paper is due to Professor W. Rubinowicz, to whom I am indebted for his kind interest in my work.

2. Starting Formulae

The subject of our consideration will be the interacting system composed of the nucleus and the electromagnetic radiation. The nucleus is taken as a system of particles, while the radiation is presented by means of a (vector) field quantized in accordance with the Bose-Einstein statistics (Dirac Radiation Theory).

The Schrödinger equation for our system in the interaction picture has the form

$$i\hbar\dot{\Psi}_I = H'_I\Psi_I, \tag{2.1}$$

where

$$\left.\begin{aligned} \Psi_I &= e^{\frac{i}{\hbar}H^0 t}\Psi \\ H'_I &= e^{\frac{i}{\hbar}H^0 t}H'e^{-\frac{i}{\hbar}H^0 t} \end{aligned}\right| \tag{2.2}$$

Operators and state-functions without an index sign are given in the Schrödinger picture. In these expressions H^0 denotes the Hamiltonian of the unperturbed system and H' is the Hamiltonian of the interaction.

We shall now consider the matrix representation given by the orthonormal set of eigenvectors of the operator H^0. Denoting these by $|n\rangle$ and the eigenvalues by E_n we have

$$H^0|n\rangle = E_n|n\rangle$$

and

$$\Psi_I = \sum_n |n\rangle b_n, \quad \text{where} \quad b_n = \langle n|\Psi_I\rangle.$$

In the above representation equation (2.1) takes the form

$$i\hbar\dot{b}_n = \sum_{n'}\langle n|H'|n'\rangle e^{\frac{i}{\hbar}(E_n - E_{n'})t} b_{n'}. \tag{2.3}$$

Below we introduce notations giving more specific characteristic of the system nucleus-radiation, without considering their interaction:

$$\left.\begin{aligned} |n\rangle &= |\alpha j m N_1 N_2 \ldots\rangle = |\alpha j m\rangle|N_1 N_2 \ldots\rangle \\ E_n &= E_{\alpha j m} + \sum_{\vec{k}l} N_{\vec{k}l}\hbar\omega, \quad \omega = ck, \quad \vec{k} = \vec{k}^0 k, \end{aligned}\right| \tag{2.4}$$

where j is the quantum number of the total angular momentum $\vec{J}$ of the nucleus: $J' = \sqrt{j(j+1)}\,\hbar$, m is the magnetic quantum number de-

fining J_z: $J'_z = m\hbar$, α represents all other labels of the states of the nucleus, and finally $N_1, N_2, \ldots, N_{\vec{k}l}, \ldots$ respectively are occupation numbers of oscillators representing the numbers of photons (contained in the field) with given wave vectors $\vec{k}$ and linear polarizations defined by unit vectors $\vec{e}_{\vec{k}l}$, so that

$$\vec{e}_{\vec{k}l} \cdot \vec{k} = 0 \quad (l = 1, 2). \tag{2.5}$$

Let us consider the transition of the nucleus from the state $\sum_{m'} |\alpha' j' m'\rangle a_{m'}$ at the time $t = 0$ to the state $|\alpha j m\rangle$ with the simultaneous emission of two photons having wave vectors $\vec{k}_1$, $\vec{k}_2$ and linear polarization given by $\vec{e}_{\vec{k}_1 l_1}$, $\vec{e}_{\vec{k}_2 l_2}$. Dropping from our notation all unchanging $N_{\vec{k}l}$, viz.

$$|n\rangle = |\alpha j m N_{\vec{k}_1 l_1} N_{\vec{k}_2 l_2}\rangle,$$
$$b_n = b_{\alpha j m}(N_{\vec{k}_1 l_1}, N_{\vec{k}_2 l_2}),$$

we have the following states

initial: $\sum_{m'} |\alpha' j' m' N_{\vec{k}_1 l_1} N_{\vec{k}_2 l_2}\rangle a_{m'}$

intermediate: $|\alpha'' j'' m'' N_{\vec{k}_1 l_1} + 1 \quad N_{\vec{k}_2 l_2}\rangle$ or $|\alpha'' j'' m'' N_{\vec{k}_1 l_1} N_{\vec{k}_2 l_2} + 1\rangle$

final: $|\alpha j m N_{\vec{k}_1 l_1} + 1 \quad N_{\vec{k}_2 l_2} + 1\rangle$.

We have therefore in our case the initial conditions

$$\text{for} \quad t = 0, \quad b_{\alpha' j' m'}(N_{\vec{k}_1 l_1}, N_{\vec{k}_2 l_2}) = a_{m'} \quad \text{other} \quad b's = 0.$$

The intermediate states appearing here are of a virtual character, i. e. the energy conservation law does not apply in the transitions to these states — the transitions to these states are virtual transitions:

$$E_{\alpha' j' m'} + E_{\alpha'' j'' m''} + \hbar\omega_s \quad (s = 1, 2).$$

In particular if all initial $N_{\vec{k}l} = 0$, we shall have a spontaneous two-quanta emission, in other cases we shall have a two-quanta emission induced by radiation.

In our calculation we shall use the non-relativistic form of the interaction Hamiltonian H'. This approximation is justified by small velocities ascribed to nucleons in the nucleus. We have therefore

$$H' = \sum_i \left[-\frac{e_i}{m_i c} \vec{p}_i \cdot \vec{A}(\vec{x}_i) + \frac{e_i^2}{2 m_i c^2} A^2(\vec{x}_i) \right], \tag{2,6}$$

where the operator of the vector potential has the following expansion in Fourier series:

$$\vec{A}(\vec{x}) = \sqrt{\frac{hc}{G}} \sum_{\vec{k}} \sum_{l=1}^{2} \frac{1}{\sqrt{k}} \vec{e}_{\vec{k}l} (a_{\vec{k}l} e^{i\vec{k}\cdot\vec{x}} + a^{*}_{\vec{k}l} e^{-i\vec{k}\cdot\vec{x}}). \tag{2.7}$$

G is here the volume of the period cube. Quantities $a^{*}_{\vec{k}l}$ and $a_{\vec{k}l}$ are the emission and absorption operators of the photon characterized by $\vec{k}$ and $\vec{e}_{\vec{k}l}$ with well known calculation properties.

Substituting (2.7) into (2.6), we see that the interaction Hamiltonian takes the form

$$\begin{aligned} H' = &\sum_{\vec{k}l} (a_{\vec{k}l} H_{\vec{k}l} + a^{*}_{\vec{k}l} H_{-\vec{k}l}) + \sum_{\vec{k}_1 l_1} \sum_{\vec{k}_2 l_2} (a_{\vec{k}_1 l_1} a_{\vec{k}_2 l_2} H_{\vec{k}_1 \vec{k}_2 l_1 l_2} \\ &+ a_{\vec{k}_1 l_1} a^{*}_{\vec{k}_2 l_2} H_{\vec{k}_1 - \vec{k}_2 l_1 l_2} + a^{*}_{\vec{k}_1 l_1} a_{\vec{k}_2 l_2} H_{-\vec{k}_1 \vec{k}_2 l_1 l_2} + a^{*}_{\vec{k}_1 l_2} a^{*}_{\vec{k}_2 l_2} H_{-\vec{k}_1 - \vec{k}_2 l_1 l_2}) \end{aligned}$$

where

$$\begin{aligned} H_{\vec{k}l} &= -\sqrt{\frac{h}{Gck}}\, \vec{e}_{\vec{k}l} \cdot \sum_i \frac{e_i}{m_i} \vec{p}_i\, e^{i\vec{k}\cdot\vec{r}_i} \\ H_{\vec{k}_1 \vec{k}_2 l_1 l_2} &= \frac{h}{2Gc\sqrt{k_1 k_2}}\, \vec{e}_{\vec{k}_1 l_1} \cdot \vec{e}_{\vec{k}_2 l_2} \sum \frac{e_i^2}{m_i}\, e^{i(\vec{k}_1 + \vec{k}_2)\cdot\vec{r}_i}. \end{aligned} \tag{2.8}$$

Solving the system of differential equations (2.3) in a formal way through the successive aproximation method, we get the following formulae for the amplitudes of the transitions between the previously mentioned initial and final states (cf. e. g. Heitler 1944, Chap. III, § 14)

$$\begin{aligned} &b_{ajm}(N_{\vec{k}_1 l_1} + 1, N_{\vec{k}_2 l_2} + 1) \\ &= \sqrt{N_{\vec{k}_1 l_1} + 1}\, \sqrt{N_{\vec{k}_2 l_2} + 1} \sum_{m'} \Bigg[-2 \langle ajm | H^{*}_{\vec{k}_1 \vec{k}_2 l_1 l_2} | a'j'm' \rangle \\ &+ \sum_{a''j''m''} \Bigg(\frac{\langle ajm | H^{*}_{\vec{k}_1 l_1} | a''j''m'' \rangle \langle a''j''m'' | H^{*}_{\vec{k}_2 l_2} | a'j'm' \rangle}{E_{a''j''m''} - E_{a'j'm'} + \hbar\omega_2} \\ &+ \frac{\langle ajm | H^{*}_{\vec{k}_2 l_2} | a''j''m'' \rangle \langle a''j''m'' | H^{*}_{\vec{k}_1 l_1} | a'j'm' \rangle}{E_{a''j''m''} - E_{a'j'm'} + \hbar\omega_1} \Bigg) \Bigg] \\ &\times a_{m'} \frac{e^{\frac{i}{\hbar}(E_{ajm} - E_{a'j'm'} + \hbar\omega_1 + \hbar\omega_2)t} - 1}{E_{ajm} - E_{a'j'm'} + \hbar\omega_1 + \hbar\omega_2}. \end{aligned} \tag{2.9}$$

We can now determine in precise terms the aim of the present paper, which is: to find the probability of the two-quanta emission of photons with directions given by unit vectors $\vec{k}_1^0$ and $\vec{k}_2^0$, in the transition of the nuclei (in the given ensemble) from all states with determined $\alpha' j'$ to all states with determined αj. These probabilities will be found from the formula (2.9) when the following operation will be carried out on the quantites $|b_{\alpha jm}(N_{\vec{k}_1 l_1}+1, N_{\vec{k}_2 l_2}+1)|^2$:

(1) Averaging over the initial states $\sum_{m'} |\alpha' j' m'\rangle a_{m'}$ of nuclei appearing in the investigated ensemble.

(2) Summing over the final magnetic states m of the nucleus.

(3) Summing over the frequencies (lengths of $\vec{k}$ vectors) and over the two directions of polarization for each photon.

In this way we get the following correlation formula

$$W_{\alpha' j';\alpha j}(\vec{k}_1^0, \vec{k}_2^0) = \sum_{k_1 l_1} \sum_{l_1 l_2=1}^{2} \sum_{m'} \langle |b_{\alpha jm}(N_{\vec{k}_1 l_1}+1, N_{\vec{k}_2 l_2}+1)|^2 \rangle_{av}. \quad (2.10)$$

In averaging over initial states of the nuclei, we shall assume that the initial distribution $a_{m'}$ of the nuclei among the magnetic states is random, viz.

$$|a_{m'}|^2 = \frac{1}{N},$$

where N is the normalizing factor. Therefore

$$a_{m'} = \frac{1}{\sqrt{N}} e^{i\delta_{m'}}$$

and consequently

$$\langle a_{m'} a_{m''} \rangle_{av} = \frac{1}{N} \langle e^{i(\delta_{m'} - \delta_{m''})} \rangle_{av} = \frac{1}{N} \delta_{m' m''}.$$

On this assumption we get from (2.9) the relation:

$$\begin{aligned}
&\langle |b_{\alpha jm}(N_{\vec{k}_1 l_1}+1, N_{\vec{k}_2 l_2}+1)|^2 \rangle_{av} \\
&= (N_{\vec{k}_1 l_1}+1)(N_{\vec{k}_2 l_2}+1) \sum_{m'} \Bigg| -2\langle \alpha jm | \overset{*}{H}_{\vec{k}_1 \vec{k}_2 l_1 l_2} | \alpha' j' m' \rangle \\
&+ \sum_{\alpha'' j'' m''} \Bigg(\frac{\langle \alpha jm | \overset{*}{H}_{\vec{k}_1 l_1} | \alpha'' j'' m'' \rangle \langle \alpha'' j'' m'' | \overset{*}{H}_{\vec{k}_3 l_3} | \alpha' j' m' \rangle}{E_{\alpha' j'' m''} - E_{\alpha' j' m'} + \hbar\omega_2} \\
&+ \frac{\langle \alpha jm | \overset{*}{H}_{\vec{k}_1 l_1} | \alpha'' j'' m'' \rangle \langle \alpha'' j'' m'' | \overset{*}{H}_{\vec{k}_2 l_2} | \alpha' j' m' \rangle}{E_{\alpha'' j'' m''} - E_{\alpha' j' m'} + \hbar\omega_1} \Bigg) \Bigg| \\
&\times 2 \frac{1 - \cos(E_{\alpha jm} - E_{\alpha' j' m'} + \hbar\omega_1 + \hbar\omega_2) \frac{t}{\hbar}}{(E_{\alpha jm} - E_{\alpha' j' m'} + \hbar\omega_1 + \hbar\omega_2)^2}.
\end{aligned} \quad (2.10a)$$

The strong maximum for $E_{\alpha jm} - E_{\alpha' j' m'} + \hbar\omega_1 + \hbar\omega_2 = 0$ expresses the energy conservation law. Since $E_{\alpha'' j'' m''} - E_{\alpha' j' m'} + \hbar\omega_s \neq 0$ $(s = 1,2)$, for the two-quanta emission we have a continuous spectrum of frequencies.

We shall now consider the matrix elements appearing in (2.10a). In calculating them we shall assume that the dimensions of fhe nucleus are small in comparison with the γ wave lengths. Using $\vec{X}$ as the symbol of the position of the mass center of the nucleus (for $t = 0$) and writing $\vec{x}_l = \vec{X} + \vec{x}'_l$, we have

$$e^{i\vec{k}\cdot\vec{x}_l} = (1 + i\vec{k}\cdot\vec{x}'_l + \ldots)\, e^{i\vec{k}\cdot\vec{X}} \approx e^{i\vec{k}\cdot\vec{X}}.$$

Breaking the series after the first term we get the electric dipole approximation for each of the two photons, and we shall speak hereafter of electric double-dipole radiation. In the above approximation the matrix elements, after making use of (2.8) will take the following form

$$\langle \alpha jm | \overset{*}{H}_{\vec{k}_1 \vec{k}_2 \lambda_1 \lambda_2} | \alpha' j' m' \rangle$$

$$= \frac{h}{2Gc\sqrt{k_1 k_2}} \vec{e}_{\vec{k}_1 \lambda_1} \cdot \vec{e}_{\vec{k}_2 \lambda_2} \sum_i \frac{e_i}{m_i} e^{-i(\vec{k}_1 + \vec{k}_2)\cdot\vec{X}} \delta_{\alpha jm,\, \alpha' j' m'} = 0,$$

as $|\alpha jm\rangle \neq |\alpha' j' m'\rangle$, and

$$\langle \alpha jm | \overset{*}{H}_{\vec{k} l} | \alpha' j' m' \rangle$$

$$= -\sqrt{\frac{h}{Gck}}\, \vec{e}_{\vec{k} l} \cdot \langle \alpha jm \Big| \sum_i \frac{e_i}{m_i} \vec{p}_i \Big| \alpha' j' m' \rangle\, e^{-i\vec{k}\cdot\vec{X}}$$

$$= -\sqrt{\frac{h}{Gck}} \frac{i}{\hbar} (E_{\alpha jm} - E_{\alpha' j' m'})\, \vec{e}_{\vec{k} l} \cdot \langle \alpha jm | \vec{P} | \alpha' j' m' \rangle\, e^{-i\vec{k}\cdot\vec{X}},$$

where

$$\vec{P} = \sum_i e_i \vec{x}_i \tag{2.11}$$

is the electric dipole moment of the nucleus.

The second expansion term of the matrix element

$$\langle \alpha jm | \overset{*}{H}_{\vec{k}_1 \vec{k}_2 \lambda_1 \lambda_2} | \alpha' j' m' \rangle$$

has the form

$$\frac{h}{Gc\sqrt{k_1 k_2}}\, \vec{e}_{\vec{k}_1 \lambda_1} \cdot \vec{e}_{\vec{k}_2 \lambda_2} \frac{e}{m} i(\vec{k}_1 + \vec{k}_2) \cdot \langle \alpha jm | \vec{P} | \alpha' j' m' \rangle.$$

Its order of magnitude for the nucleus (contrariwise to the case of the atom) might be compared with that of the first expansion term of

$$\frac{\langle ajm|\overset{*}{H}_{\vec{k}_1 l_1}|a''j''m''\rangle\langle a''j''m''|\overset{*}{H}_{\vec{k}_2 l_2}|a'j'm'\rangle}{E_{a''j''m''}-E_{a'j'm'}+\hbar\,_2}.$$

However for the electric double-dipole radiation the states $|ajm\rangle$ and $|a'j'm'\rangle$ have the same parity and $\langle ajm|\vec{P}|a'j'm'\rangle=0$ conformly to the generalized Laporte rule. We will deal with this case on the following pages.

In our approximation the probability function (2.10) takes the form (cf. the differences with Göppert-Mayer 1931)

$$W_{a'j';aj}(\vec{k}_1^0,\vec{k}_2^0)=\sum_{k_1k_2}\sum_{l_1l_2=1}^{2}\frac{h^2(N_{\vec{k}_1 l_1}+1)(N_{\vec{k}_2 l_2}+1)}{G\,c^2 k_1 k_2} \tag{2.12}$$

$$\sum_{m\,m'}\left|\vec{e}_{\vec{k}_1 l_1}\cdot\langle ajm|\vec{P}\vec{P}|a'j'm'\rangle_{\omega_2}\cdot\vec{e}_{\vec{k}_2 l_2}+\vec{e}_{\vec{k}_2 l_2}\cdot\langle ajm|\vec{P}\vec{P}|a'j'm'\rangle_{\omega_1}\cdot\vec{e}_{\vec{k}_1 l_1}\right|^2$$

$$\times\,2\,\frac{1-\cos(E_{ajm}-E_{a'j'm'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{ajm}-E_{a'j'm'}+\hbar\omega_1+\hbar\omega_2)^2},$$

in which we introduce the following notation:

$$\langle ajm|\vec{P}\vec{P}|a'j'm'\rangle_\omega=\sum_{a''j''m''}\frac{\langle ajm|\vec{P}|a''j''m''\rangle\langle a''j''m''|\vec{P}|a'j'm'\rangle}{\hbar^2\dfrac{E_{a''j''m''}-E_{a'j'm'}+\hbar\omega}{(E_{ajm}-E_{a''j''m''})(E_{a''j''m''}-E_{a'j'm'})}}. \tag{2.13}$$

The multiplication of the vectors here is the dyadic multiplication for which we shall not use the point sign, reserving it for the scalar multiplication.

From (2.12) and (2.13) we draw an interesting conclusion that there is a probability of the two-quanta double-dipole emission only when the nucleus has at least one state different from the initial and final states.

In further calculations we shall assume that the states of the nuclei $|ajm\rangle$ are degenerate with regard to m

$$E_{ajm}=E_{aj}. \tag{2.14}$$

3. The Matrix Elements

In the present chapter we shall calculate matrix elements defined by (2.13). We shall do this by calculating elements of a more general form, namely

$$\langle ajm|\vec{T}_1\vec{T}_2|a'j'm'\rangle_\omega = \sum_{\alpha''j''m''} \frac{\langle ajm|\vec{T}_1|\alpha''j''m''\rangle\langle\alpha''j''m''|\vec{T}_2|a'j'm'\rangle}{\frac{2}{\hbar}\frac{E_{\alpha''j''}-E_{a'j'}+\hbar\omega}{(E_{aj}-E_{\alpha''j''})(E_{\alpha''j''}-E_{a'j'})}}, \tag{3.1}$$

where $\vec{T}_1$ and $\vec{T}_2$ are arbitrary vectors satisfying the following commutation relations (Condon and Shortley 1935, 8[3])

$$[\vec{T},\vec{J}] = -i\hbar\,\vec{T}\times\check{I}. \tag{3.2}$$

$\vec{J}$ is here the vector of the total angular momentum, $\check{I}$ is the identity dyadic

$$\check{I} = \vec{i}\,\vec{i} + \vec{j}\,\vec{j} + \vec{k}\,\vec{k}. \tag{3.3}$$

The „˘" cap sign will be used to denote dyadics.

The class of vectors of $\vec{T}$ type contains among other vectors, also vectors $\vec{J}$, $\vec{x}_l$, $\vec{p}_l$, as well as any linear combination and the vector product of $\vec{T}$ type vectors (Condon and Shortley l. c.). This class includes therefore electric and magnetic dipole moments:

$$\vec{P} = \sum_l e_l\vec{x}_l, \quad \vec{M} = \frac{1}{2c}\sum_l \frac{e_l}{m_l}\,\vec{x}_l\times\vec{p}_l.$$

The matrix elements of a $\vec{T}$ vector have the following form (Condon and Shortley l. c)

$$\langle ajm|\vec{T}|a'j'm'\rangle = \vec{D}(\Delta m)\langle aj\,\vdots\,T\,\vdots\,a'j\rangle f_d(jm;j'm'), \tag{3.4}$$

where

$$\left.\begin{aligned} &\vec{D}(0) = \vec{k} && (\Delta j = j'-j, \quad \Delta m = m'-m)\\ &\vec{D}(\pm 1) = \tfrac{1}{2}(\vec{i} \pm i\,\vec{j}), \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} f_d(jm;jm) &= m\\ f_d(jm;j\,m\pm 1) &= \sqrt{(j\mp m)(j\pm m+1)}\\ f_d(jm;j+1\,m) &= \sqrt{(j+1)^2-m^2}\\ f_d(jm;j+1\,m\pm 1) &= \mp\sqrt{(j\pm m+1)(j\pm m+2)}\\ f_d(jm;j-1\,m) &= \sqrt{j^2-m^2}\\ f_d(jm;j-1\,m\pm 1) &= \pm\sqrt{(j\mp m)(j\mp m-1)}\\ \text{other } f_d\text{'s} &= 0. \end{aligned}\right\} \tag{3.6}$$

Expressions $\langle aj \vdots T \vdots a'j' \rangle$ in (3.4) are factors dependent on the $\vec{T}$ vector and the unknown set of eigenfunctions $|ajm\rangle$. They form a Hermitian matrix, which transforms with the change of the set of eigenvectors like the matrix of an observable,

Using (3.4) we may calculate the elements (3.1).

Defining

$$B(aj; j''; a'j') = \sum_{a''} \frac{\langle aj \vdots T_1 \vdots a''j'' \rangle \langle a''j'' \vdots T_2 \vdots a'j' \rangle}{\hbar^2 \frac{a_{a''j''} - E_{a'j'} + \hbar\omega}{(E_{aj} - E_{a''j''})(E_{a''j''} - E_{a'j'})}}, \tag{3.7}$$

we have by substituting (3.4) into (3.1):

$$\langle ajm | \vec{T}_1 \vec{T}_2 | a'j'm' \rangle_\omega \tag{3.8}$$
$$= \sum_{j''m''} \vec{D}(m''-m) \vec{D}(m'-m'') B(aj; j''; a'j') f_d(jm; j''m'') f_d(j''m'' j'm').$$

Irksome calculations carried out separately for distinct Δj and Δm give the following formula

$$\begin{aligned} \langle ajm | \vec{T} \vec{T} | a'j'm' \rangle_\omega = {} & \check{K}(\Delta m) \langle aj \vdots T_1 T_2 \vdots a'j' \rangle_\omega f_q(jm; j'm') \\ & + \tfrac{1}{3} \check{I} \langle ajm | \vec{T} \cdot \vec{T} | a'j'm' \rangle_\omega \\ & + \varepsilon(\Delta j) \check{D}(\Delta m) \langle aj \vdots T_1 \times T_2 \vdots a'j' \rangle_\omega f_d(jm; j'm'). \end{aligned} \tag{3.9}$$

The new notations used above are defined as follows

$$\left.\begin{aligned} \check{K}(0) &= \sqrt{\tfrac{2}{3}} (\vec{k}\vec{k} - \tfrac{1}{2}\vec{i}\vec{i} - \tfrac{1}{2}\vec{j}\vec{j}) \\ \check{K}(\pm 1) &= \tfrac{1}{2}[\vec{k}\vec{i} + \vec{i}\vec{k} \pm i(\vec{k}\vec{j} + \vec{j}\vec{k})] \\ \check{K}(\pm 2) &= \tfrac{1}{2}[\vec{i}\vec{i} - \vec{j}\vec{j} \pm i(\vec{i}\vec{j} + \vec{j}\vec{i})], \end{aligned}\right| \tag{3.10}$$

$$\left.\begin{aligned} \check{D}(0) &= \tfrac{1}{2}(\vec{i}\vec{j} - \vec{j}\vec{i}) = -\tfrac{1}{2}\vec{D}(0) \times \check{I} \\ \check{D}(\pm 1) &= \pm\tfrac{1}{2}[\vec{i}\vec{k} - \vec{k}\vec{i} \pm i(\vec{j}\vec{k} - \vec{k}\vec{j})] = i\vec{D}(\pm 1) \times \check{I}, \end{aligned}\right| \tag{3.11}$$

$$\left.\begin{aligned} \varepsilon(0) &= -1, \\ \varepsilon(\pm 1) &= +1 \end{aligned}\right| \tag{3.12}$$

$$\left.\begin{aligned} & \langle aj \vdots T_1 T_2 \vdots a'j \rangle_\omega \\ &= -B(aj; j-1; a'j) + B(aj; j; a'j) - B(aj; j+1; a'j) \\ & \langle aj \vdots T_1 T_2 \vdots a'j \pm 1 \rangle_\omega = B(aj; j; a'j \pm 1) + B(aj; j \pm 1; a'j \pm 1) \\ & \langle aj \vdots T_1 T_2 \vdots a'j \pm 2 \rangle_\omega = B(aj; j \pm 1; a'j \pm 2), \end{aligned}\right| \tag{3.13}$$

$$\begin{aligned}\langle ajm\,|\,\vec{T}_1\cdot\vec{T}_2\,|\,a'j'm'\rangle_\omega &= \langle aj\,\vdots\,T_1\cdot T_2\,\vdots\,a'j\rangle\,\delta_{jj'}\,\delta_{mm'}\\ &= [j(2j-1)\,B(aj;j-1;a'j)+j(j+1)\,B(aj;j;a'j)\\ &\quad +(j+1)(2j+3)\,B(aj;j+1;a'j)]\,\delta_{jj'}\,\delta_{mm'},\end{aligned}\tag{3.14}$$

$$\begin{aligned}&\langle aj\,\vdots\,T_1\times T_2\,\vdots\,a'j\rangle_\omega\\ &= (2j-1)\,B(aj;j-1;a'j)+B(aj;j;a'j)-(2j+3);B(aj;j+1;a'j\\ &\langle aj\,\vdots\,T_1\times T_2\,\vdots\,a'j-1\rangle_\omega\\ &= (j-1)\,B(aj;j-1;a'j-1)-(j+1)\,B(aj;j;a'j-1)\\ &\langle aj\,\vdots\,T_1\times T_2\,\vdots\,a'j+1\rangle_\omega\\ &= j\,B(aj;j;a'j+1)-(j+2)\,B(aj;j+1;a'j+1),\end{aligned}\tag{3.15}$$

$$\begin{aligned}
f_q(jm;jm) &= \tfrac{1}{\sqrt{6}}[3m^2-j(j+1)]\\
f_q(jm;j\,m\pm1) &= \tfrac{1}{2}(2m\pm1)\sqrt{(j\mp m)(j\pm m+1)}\\
f_q(jm;j\,m\pm2) &= \tfrac{1}{2}\sqrt{(j\mp m)(j\mp m-1)(j\pm m+1)(j\pm m+2)}\\
f_q(jm;j+1\,m) &= \sqrt{\tfrac{3}{2}}m\sqrt{(j-m+1)(j+m+1)}\\
f_q(jm;j+1\,m\pm1) &= \tfrac{1}{2}(j\mp2m)\sqrt{(j\pm m+1)(j\pm m+2)}\\
f_q(jm;j+1\,m\pm2) &= \mp\tfrac{1}{2}\sqrt{(j\mp m)(j\pm m+1)(j\pm m+2)(j\pm m+3)}\\
f_q(jm;j-1\,m) &= \sqrt{\tfrac{3}{2}}m\sqrt{(j-m)(j+m)}\\
f_q(jm;j-1\,m\pm1) &= \tfrac{1}{2}(j\pm2m+1)\sqrt{(j\mp m)(j\mp m-1)}\\
f_q(jm;j-1\,m\pm2) &= \pm\tfrac{1}{2}\sqrt{(j\mp m)(j\mp m-2)(j\pm m+1)}\\
f_q(jm;j+2\,m) &= \sqrt{\tfrac{3}{2}}\sqrt{(j-m+1)(j+m+1)(j-m+2)(j+m+2)}\\
f_q(jm;j+2\,m\pm1) &= \mp\sqrt{(j-m+1)(j+m+1)(j\pm m+2)(j\pm m+3)}\\
f_q(jm;j+2\,m\pm2) &= \tfrac{1}{2}\sqrt{(j\pm m+1)(j\pm m+2)(j\pm m+3)(j\pm m+4)}\\
f_q(jm;j-2\,m) &= \sqrt{\tfrac{3}{2}}\sqrt{(j-m)(j+m)(j-m-1)(j+m-1)}\\
f_q(jm;j-2\,m\pm1) &= \pm\sqrt{(j-m)(j+m)(j\mp m-1)(j\mp m-2)}\\
f_q(jm;j-2\,m\pm2) &= \tfrac{1}{2}\sqrt{(j\mp m)(j\mp m-1)(j\mp m-2)(j\mp m-3)}
\end{aligned}$$

other $f_q's = 0$. (3.16)

Proceeding for the vector product in the way above indicated for the dyadic product, we have:

$$\begin{aligned}\langle ajm\,|\,\vec{T}_1\times\vec{T}_2\,|\,a'j'm'\rangle &= \sum_{a''j''m''}\frac{\langle ajm|\vec{T}_1|a''j''m''\rangle\times\langle a''j''m''|\vec{T}_2|a'j'm'\rangle}{\hbar^2\dfrac{E_{a''j''}-E_{a'j'}+\hbar\omega}{(E_{aj}-E_{a''j''})(E_{a''j''}-E_{a'j'})}}\\ &= \vec{D}(\Delta m)\langle aj\,\vdots\,T_1\times T_2\,\vdots\,a'j'\rangle f_d(jm;j'm').\end{aligned}\tag{3.17}$$

It is interesting to compare the forms (3.9) with the matrix elements of the dyadic $\vec{T}_1 \vec{T}_2$. The electric quadripole moment

$$\check{Q} = \tfrac{1}{2} \sum_l e_l \vec{x}_l \vec{x}_l$$

is a dyadic of the $\vec{T}_1 \vec{T}_2$ type, where $\vec{T}_1 \times \vec{T}_2 = 0$, for which we get the formulae obtained (for quadripole moments) by W. Rubinowicz (1930):

$$\begin{aligned} \langle \omega j m \,|\, \vec{T}_1 \vec{T}_2 \,|\, \alpha' j' m' \rangle = \check{K}(\Delta m) \langle \alpha j \vdots T_1 T_2 \vdots \alpha' j' \rangle f_q(jm; j'm') \\ + \tfrac{1}{3} \check{I} \langle \alpha j m \,|\, \vec{T}_1 \cdot \vec{T}_2 \,|\, \alpha' j' m' \rangle . \end{aligned} \tag{3.18}$$

Here the factors $\langle \alpha j \vdots \vec{T}_1 \vec{T}_2 \vdots \alpha' j' \rangle$, $\langle \alpha j m | \vec{T}_1 \cdot \vec{T}_2 | \alpha' j' m' \rangle$, $\langle \alpha j \vdots T_1 \times T_2 \vdots \alpha' j' \rangle$ are defined by the expressions

$$A(\alpha j; j''; \alpha' j') = \sum_{\alpha''} \langle \alpha j \vdots T_1 \vdots \alpha'' j'' \rangle \langle \alpha'' j'' \vdots T_2 \vdots \alpha' j' \rangle ,$$

just like the factors $\langle \alpha j \vdots \widehat{T_1 T_2} \vdots \alpha' j' \rangle$, $\langle \alpha j m | \widehat{\vec{T}_1 \cdot \vec{T}_2} | \alpha' j' m' \rangle$, $\langle \alpha j \vdots \widehat{T_1 \times T_2} \vdots \alpha' j' \rangle$, by $B(\alpha j; j''; \alpha' j')$. However, the expressions $\langle \alpha j \vdots T_1 \times T_2 \vdots \alpha j' \rangle$ disappear:

$$\langle \alpha j \vdots T_1 \times T_2 \vdots \alpha' j' \rangle = 0 \tag{3.18a}$$

when $\vec{T}_1 \times \vec{T}_2 = 0$, as in this case

$$0 = \langle \alpha j m | \vec{T}_1 \times \vec{T}_2 | \alpha' j' m' \rangle = \vec{D}(\Delta m) \langle \alpha j \vdots T_1 \times T_2 \vdots \alpha' j' \rangle f_d(jm; j'm'). \tag{3.19}$$

Obviously formula (3.18a) applies in the case when $\vec{T}_1 = \vec{T}_2 = \vec{P}$, because $\vec{P} \times \vec{P} = 0$.

Similarity of the quadripole elements (3.18) and our expressions (3.9) is evident. In our case, however, since expressions (3.15) generally speaking do not disappear, the transition elements (3.9) contain a „quadripole part" with the coefficient (3.13) as well as a „dipole part" with the coefficient (3.15).

It should se emphasized that the assumption of degeneracy ef the energy levels $E_{\alpha j m} = E_{\alpha j}$ played a useful part in our procedure enabling us to assemble unknown quantities into factors independent of m. B. Milianczuk (1931) used analogous procedure for calculating the intensity of forced dipole lines in the ordinary emission.

4. Correlation Formulae

Correlation formulae are obtained by insertion of the matrix elements (3.9) into formula (2.12). We shall keep in mind that the dyadics $\check{K}(\Delta m)$ and $\check{I}$ are symmetric and $\check{D}(\Delta m)$ antisymmetric:

$$\begin{aligned}
\vec{a}\cdot\check{K}(\Delta m)\cdot\vec{b} &= \vec{k}\cdot\check{K}(\Delta m)\cdot\vec{a},\\
\vec{a}\cdot\check{I}\cdot\vec{b} &= \vec{b}\cdot\check{I}\cdot\vec{a},\\
\vec{a}\cdot\check{D}(\Delta m)\cdot\vec{b} &= -\vec{b}\cdot\check{D}(\Delta m)\cdot\vec{a}.
\end{aligned}$$

This fact will prove useful. We have

$$W_{\alpha' j';\alpha j}(\vec{k}_1^0,\vec{k}_2^0) \tag{4.1}$$

$$= \sum_{k_1 k_2}\sum_{l_1 l_2=1}^{2} \frac{h^2(N_{\vec{k}_1 l_1}+1)(N_{\vec{k}_2 l_2}+1)}{G^2 c^2 k_1 k_2}\, 2\,\frac{1-\cos(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)^2}$$

$$\times \sum_{mm'} |\vec{e}_{\vec{k}_1 l_1}\cdot[\check{K}(\Delta m)(\langle \alpha j|P\widehat{P}|\alpha' j'\rangle_{\omega_2}+\langle \alpha j|P\widehat{P}|\alpha' j'\rangle_{\omega_1}) f_q(jm;j'm')$$

$$+\tfrac{1}{3}\check{I}(\langle \alpha j|P\widehat{\cdot}P|\alpha' j'\rangle_{\omega_2}+\langle \alpha j|P\widehat{\cdot}P|\alpha' j'\rangle_{\omega_1})\,\delta_{jj'}\delta_{mm'}+$$

$$+\varepsilon(\Delta j)\check{D}(\Delta m)(\langle \alpha j|P\widehat{\times}P|\alpha' j'\rangle_{\omega_2}-\langle \alpha j|P\widehat{\times}P|\alpha' j'\rangle_{\omega_1}) f_d(jm;j'm')]\cdot\vec{e}_{\vec{k}_2 l_2}|^2$$

$$= J_1(\alpha j;\alpha' j')\sum_{mm'}\Big(\sum_{l_1 l_2}|\vec{e}_{\vec{k}_1 l_1}\cdot\check{K}(\Delta m)\cdot\vec{e}_{\vec{k}_2 l_2}|^2\Big) f_q(jm;j'm')^2 +$$

$$+J_2(\alpha j;\alpha' j')\sum_{mm'}\Big(\sum_{l_1 l_2}|\vec{e}_{\vec{k}_1 l_1}\cdot\check{I}\cdot\vec{e}_{\vec{k}_2 l_2}|^2\Big)\tfrac{1}{3}\delta_{jj'}\delta_{mm'}+$$

$$+J_3(\alpha j;\alpha' j')\sum_{mm'}\Big(\sum_{l_1 l_2}|\vec{e}_{\vec{k}_1 l_1}\cdot\check{D}(\Delta m)\cdot\vec{e}_{\vec{k}_2 l_2}|^2\Big) f_d(jm;j'm')^2 +$$

$$+J_{12}(\alpha j;\alpha' j')\sum_{mm'}\Big(\sum_{l_1 l_2}\vec{e}_{\vec{k}_1 l_1}\cdot\check{K}(\Delta m)\cdot\vec{e}_{\vec{k}_2 l_2}\vec{e}_{\vec{k}_1 l_1}\cdot\check{I}^*\cdot\vec{e}_{\vec{k}_1 l_1}\Big) f_q(jm;j'm')\tfrac{1}{3}\delta_{jj'}\delta_{mm'}+$$

$$+\text{conj.}+$$

$$+J_{13}(\alpha j;\alpha' j')\sum_{mm'}\Big(\sum_{l_1 l_2}\vec{e}_{\vec{k}_1 l_1}\cdot\check{K}(\Delta m)\cdot\vec{e}_{\vec{k}_2 l_2}\vec{e}_{\vec{k}_1 l_1}\cdot\check{D}(\Delta m)^*\cdot$$

$$\cdot\vec{e}_{\vec{k}_2 l_2}\Big) f_q(jm;j'm') f_d(jm;j'm')+\text{conj.}+$$

$$+J_{23}(\alpha j;\alpha' j')\sum_{mm'}\Big(\sum_{l_1 l_1}\vec{e}_{\vec{k}_1 l_1}\cdot\check{I}\cdot\vec{e}_{\vec{k}_2 l_2}\vec{e}_{\vec{k}_1 l_1}\cdot\check{D}(\Delta m)^*\cdot$$

$$\cdot\vec{e}_{\vec{k}_2 l_2}\Big)\tfrac{1}{3}\delta_{jj'}\delta_{mm'} f_d(jm;j'm')+\text{conj.},$$

where

$$
\left.
\begin{aligned}
J_1(\alpha j;\,\alpha' j') &= \sum_{k_1 k_2} \frac{h^2(N_{k_1}+1)(N_{k_2}+1)}{G^2 c^2 k_1 k_2} 2\, \frac{1-\cos(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)^2} \,|\langle \alpha j|\!:\widehat{PP}\!:|\alpha' j'\rangle_{\omega_1}+\langle \alpha j|\!:\widehat{PP}\!:|\alpha' j'\rangle_{\omega_2}|^2 \\
J_2(\alpha j;\,\alpha' j') &= \sum_{k_1 k_2} \frac{h^2(N_{k_1}+1)(N_{k_2}+1)}{G^2 c^2 k_1 k_2} 2\, \frac{1-\cos(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)^2} \,|\langle \alpha j|\!:\widehat{P\cdot P}\!:|\alpha' j'\rangle_{\omega_1}+\langle \alpha j|\!:\widehat{P\cdot P}\!:|\alpha' j'\rangle_{\omega_2}|^2 \\
J_3(\alpha j;\,\alpha' j') &= \sum_{k_1 k_2} \frac{h^2(N_{k_1}+1)(N_{k_2}+1)}{G^2 c^2 k_1 k_2} 2\, \frac{1-\cos(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)^2} \,|\langle \alpha j|\!:\widehat{P\times P}\!:|\alpha' j'\rangle_{\omega_1}-\langle \alpha j|\!:\widehat{P\times P}\!:|\alpha' j'\rangle_{\omega_2}|^2 \\
J_{12}(\alpha j;\,\alpha' j') &= \sum_{k_1 k_2} \frac{h^2(N_{k_1}+1)(N_{k_2}+1)}{G^2 c^2 k_1 k_2} 2\, \frac{1-\cos(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)^2} (\langle \alpha j|\!:\widehat{PP}\!:|\alpha' j'\rangle_{\omega_1}+\langle \alpha j|\!:\widehat{PP}\!:|\alpha' j'\rangle_{\omega_2}) \\
&\qquad (\langle \alpha j|\!:\widehat{P\cdot P}\!:|\alpha' j'\rangle_{\omega_1}+\langle \alpha j|\!:\widehat{P\cdot P}\!:|\alpha' j'\rangle_{\omega_2})^* \\
J_{13}(\alpha j;\,\alpha' j') &= -\sum_{k_1 k_2} \frac{h^2(N_{k_1}+1)(N_{k_2}+1)}{G^2 c^2 k_1 k_2} 2\, \frac{1-\cos(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)^2} (\langle \alpha j|\!:\widehat{PP}\!:|\alpha' j'\rangle_{\omega_1}+\langle \alpha j|\!:\widehat{PP}\!:|\alpha' j'\rangle_{\omega_2}) \\
&\qquad (\langle \alpha j|\!:\widehat{P\times P}\!:|\alpha' j'\rangle_{\omega_1}-\langle \alpha j|\!:\widehat{P\times P}\!:|\alpha' j'\rangle_{\omega_2})^* \\
J_{23}(\alpha j;\,\alpha' j') &= -\sum_{k_1 k_2} \frac{h^2(N_{k_1}+1)(N_{k_2}+1)}{G^2 c^2 k_1 k_2} 2\, \frac{1-\cos(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)^2} (\langle \alpha j|\!:\widehat{P\cdot P}\!:|\alpha' j'\rangle_{\omega_1}+\langle \alpha j|\!:\widehat{P\cdot P}\!:|\alpha' j'\rangle_{\omega_2}) \\
&\qquad (\langle \alpha j|\!:\widehat{P\times P}\!:|\alpha' j'\rangle_{\omega_1}-\langle \alpha j|\!:\widehat{P\times P}\!:|\alpha' j'\rangle_{\omega_2})^*.
\end{aligned}
\right\} \qquad (4.2)
$$

We assumed above that the initial distribution of radiation among the polarizations will be random, i. e.

$$N_{\vec{k}l} = N_{\vec{k}}, \tag{4.3}$$

where $N_{\vec{k}}$ does not depend on the direction of polarization (e. g. for the spontaneous two-quanta emission we have $N_{\vec{k}l} = 0$).

Now we come to the calculation of the parts of expressions (4.1) which depend on the direction of the photons. We shall base our cal-

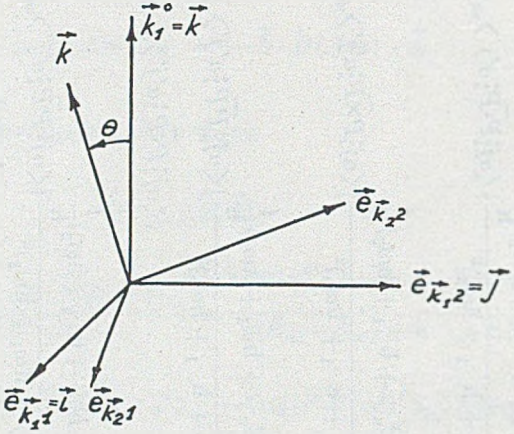

Fig. 1

culation on the following formula for an arbitrary dyadic $\check{A}$ and a symmetric or antisymmetric dyadic $\check{B}$ ($\vec{a}\check{B}\vec{b} = \pm \vec{b}\check{B}\vec{a}$)

$$\sum_{l_1 l_2 = 1}^{2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{A} \cdot \vec{e}_{\vec{k}_2 l_2}\, \vec{e}_{\vec{k}_1 l_1} \cdot \check{B}^* \cdot \vec{e}_{\vec{k}_2 l_2} = \sum_{l_1 = 1}^{2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{A} \cdot \vec{e}_{\vec{k}_1 l_1} \cdot \check{B}^* - \sum_{l_1 = 1}^{2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{A} \cdot \vec{k}_2^0\, \vec{e}_{\vec{k}_1 l_1} \cdot \check{B}^* \cdot \vec{k}_2^0 .$$

Here we made use of the orthogonality of the system of unit vectors $\vec{e}_{\vec{k}_1}$, $\vec{e}_{\vec{k}_2}$, $\vec{k}_2^0$:

$$\check{I} = \vec{i}\,\vec{i} + \vec{j}\,\vec{j} + \vec{k}\,\vec{k} = \vec{e}_{\vec{k}_1 1}\, \vec{e}_{\vec{k}_1 1} + \vec{e}_{\vec{k}_2 2}\, \vec{e}_{\vec{k}_2 2} + \vec{k}_2^0\, \vec{k}_2^0 .$$

Chosing the axis versors of the coordinate system in the directions of the propagation and of the two polarizations of the first quanta (see Fig. 1)

$$\vec{i} = \vec{e}_{\vec{k}_1 1}, \quad \vec{j} = \vec{e}_{\vec{k}_1 2}, \quad \vec{k} = \vec{k}_1^0$$

(here $\vec{k}$ does not designate the wave vector!) our formula is written:

$$\sum_{l_1 l_2=1}^{2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{A} \cdot \vec{e}_{\vec{k}_2 l_2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{B}^* \cdot \vec{e}_{\vec{k} l_2}$$
$$= (\vec{i} \cdot \check{A}) \cdot (\vec{i} \cdot \check{B}^*) + (\vec{j} \cdot \check{A}) \cdot (\vec{j} \cdot \check{B}^*) - \vec{i} \cdot \check{A} \cdot \vec{k}_2^0 \vec{i} \cdot \check{B}^* \cdot \vec{k}_2^0 - \vec{j} \cdot \check{A} \cdot \vec{k}_2^0 \vec{j} \cdot \check{B}^* \cdot \vec{k}_2^0. \tag{4.4}$$

When $\check{A} = \check{B}$ (4.4), is modified, as follows

$$\sum_{l_1 l_2=1}^{2} |\vec{e}_{\vec{k}_1 l_1} \cdot \check{A} \cdot \vec{e}_{\vec{k}_2 l_2}|^2 = |\vec{i} \cdot \check{A}|^2 + |\vec{j} \cdot \check{A}|^2 - |\vec{i} \cdot \check{A} \cdot \vec{k}_2^0|^2 - |\vec{j} \cdot \check{A} \cdot \vec{k}_2^0|^2. \tag{4.4a}$$

By the application of formulae (4.4) or (4.4a) to dyadice appearing in (4.1) we obtain the following expressions:

$$\left.\begin{aligned}
&\sum_{l_1 l_2=1}^{2} |\vec{e}_{\vec{k}_1 l_1} \cdot \check{K}(\Delta m) \cdot \vec{e}_{\vec{k}_2 l_2}|^2 = \begin{cases} \tfrac{1}{6}(1+\cos^2\theta) & \text{for } \Delta m = 0 \\ \tfrac{1}{2}(1-\cos^2\theta) & \text{for } \Delta m = \pm 1 \\ \tfrac{1}{2}(1+\cos^2\theta) & \text{for } \Delta m = \pm 2 \end{cases} \\
&\sum_{l_1 l_2=1}^{2} |\vec{e}_{\vec{k}_1 l_1} \cdot \check{D}(\Delta m) \vec{e}_{\vec{k}_2 l_2}|^2 = \begin{cases} \tfrac{1}{4}(1+\cos^2\theta) & \text{for } \Delta m = 0 \\ \tfrac{1}{2}(1-\cos^2\theta) & \text{for } \Delta m = \pm 1 \end{cases} \\
&\sum_{l_1 l_2=1}^{2} |\vec{e}_{\vec{k}_1 l_1} \cdot \quad \check{I} \quad \cdot \vec{e}_{\vec{k}_2 l_2}|^2 = 1+\cos^2\theta \\
&\sum_{l_1 l_2=1}^{2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{K}(0) \cdot \vec{e}_{\vec{k}_2 l_2} \vec{e}_{\vec{k}_1 l_1} \check{I}^* \cdot \vec{e}_{\vec{k}_2 l_2} = -\tfrac{1}{\sqrt{6}}(1+\cos^2\theta) \\
&\sum_{l_1 l_2=1}^{2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{K}(\Delta m) \cdot \vec{e}_{\vec{k}_2 l_2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{D}(\Delta m)^* \cdot \vec{e}_{\vec{k}_2 l_2} = \begin{cases} 0 & \text{for } \Delta m = 0 \\ \pm\tfrac{1}{2}(1-\cos^2\theta) & \text{for } \Delta m = \pm 1 \end{cases} \\
&\sum_{l_1 l_2=1}^{2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{I} \cdot \vec{e}_{\vec{k}_2 l_2} \vec{e}_{\vec{k}_1 l_1} \cdot \check{D}(0) \cdot \vec{e}_{\vec{k}_2 l_2} = 0.
\end{aligned}\right\} \tag{4.5}$$

Inserting formulae (4.5) into (4.1), making use of the definition of the quantities $f_q(jm; j'm')$ and $f_d\, jm; j'm')$, and carrying out tedious summing over m and m', we arrive at the correlation formulae (4.7). In the second summation of the two we use the following identities

$$\begin{aligned}
&\sum_m 1 = 2j+1 \quad \sum_m m^2 = \tfrac{1}{3} j(j+1)(2j+1) \\
&\sum_m m^4 = \tfrac{1}{15} j(j+1)(2j+1)(3j^2+3j+1).
\end{aligned} \tag{4.6}$$

During the calculations it will be observed that all mixed expressions in (4.1) disappear. These expressions would have given dipole-quadripole interference in double-dipole two-quanta emission.

Finally we obtain the following correlation formulae, dropping time factors which are independent of the directions of the photons,

$$
\begin{aligned}
&W^{\mathrm{rel}\,(\theta)}_{a'j\pm 2;\,aj} = 1 + \tfrac{1}{13}\cos^2\theta \\
&W^{\mathrm{rel}\,(\theta)}_{aj-1;\,aj} = 1 + \tfrac{1}{13}\cos^2\theta + \frac{J_3(aj;a'j-1)}{J_1(aj;a'j-1)}\,\frac{15}{13}\,\frac{1}{(j-1)(j+1)}\,(1-\tfrac{1}{3}\cos^2\theta) \\
&\qquad \text{for } j\neq 1 \\
&W^{\mathrm{rel}\,(\theta)}_{a'0;\,a1} = 1 - \tfrac{1}{3}\cos^2\theta \\
&W^{\mathrm{rel}\,(\theta)}_{a'j+1;\,aj} = 1 + \tfrac{1}{13}\cos^2\theta + \frac{J_3(aj;a'j+1)}{J_1(aj;a'j+1)}\,\frac{15}{13}\,\frac{1}{j(j+2)}\,(1-\tfrac{1}{3}\cos^2\theta) \\
&\qquad \text{for } j\neq 0 \\
&W^{\mathrm{rel}\,(\theta)}_{a'1;\,a0} = 1 - \tfrac{1}{3}\cos^2\theta \\
&W^{\mathrm{rel}\,(\theta)}_{a'j;\,aj} = 1 + \tfrac{1}{13}\cos^2\theta + \frac{J_3(ja;a'j)}{J_1(aj;a'j)}\,\frac{15}{13}\,\frac{3}{4}\,\frac{1}{(j-\frac{1}{2})\,(j+\frac{3}{2})}\,(1-\tfrac{1}{3}\cos^2\theta) \\
&\qquad \text{for } j\neq 0,\tfrac{1}{2} \\
&\qquad\qquad + \frac{J_2(aj;a'j)}{J_1(aj;a'j)}\,\frac{15}{13}\cdot\frac{1}{3}\,\frac{1}{(j-\frac{1}{2})\,j(j+\frac{1}{2})(j+\frac{3}{2})}\,(1+\cos^2\theta) \\
&W^{\mathrm{rel}\,(\theta)}_{a'\frac{1}{2};\,a\frac{1}{2}} = 1 - \tfrac{1}{3}\cos^2\theta + \frac{J_2(a\frac{1}{2};a'\frac{1}{2})}{J_3(a\frac{1}{2};a'\frac{1}{2})}\,\frac{4}{9}\,\frac{1}{j(j+1)}\,(1+\cos^2\theta) \\
&W^{\mathrm{rel}\,(\theta)}_{a'0;\,a0} = 1 + \cos^2\theta.
\end{aligned}
\tag{4.7}
$$

Here $W^{\mathrm{rel}}_{a'j';\,aj}(\vec{k}_1,\vec{k}_2) = W^{\mathrm{rel}\,(\theta)}_{a'j';\,aj}$ signify the relative probability for two-quanta emission accompanying the transition $a'j' \to aj$. The quantities $J_s(aj;a'j')$ $(s=1,2,3)$ defined in (4.2) are generally unknown. For their evaluation it is necessary to know the eigenfunctions of the given nucleus.

Especially interesting is the transition $a'0 \to a0$ forbidden in the one-quantum emission. Investigation of the directional correlations of the γ rays in this case may lead to the experimental discovery of the (double-dipole) two-quanta emission. Our formulae give in this instance the angular relation

$$W^{\mathrm{rel}\,(\theta)}_{a'0;\,a0} = 1 + \cos^2\theta.$$

As the formulae of D. R. Hamilton (1940) for (dipole-dipole) successive emission of two quanta give for the transition $a'0 \to a''1 \to a0$ the same angular relation: $1+\cos^2\theta$, the possible difficulty of deciding whether in such an instance simultaneous (double-dipole) transition or a successive (dipole-dipole) transition is taking place — cannot be solved merely by measuring correlations of γ rays.

In all other instances where formulae (4.7) do not contain unknown quantities $I_s(aj;a'j')$ $(s=1,2,3)$ — directional correlations for (double-dipole) simultaneous two-quanta emisson and (dipole-dipole)

successive emissions of two quanta (10) are identical. These are the instances

of successive emissions:			of simultaneous emissions:	
$a'j\pm 2$	$\to a''j\pm 1$	$\to aj$	$a'j\pm 2$	$\to aj$
$a'0$	$\to a''1$	$\to a1$	$a'0$	$\to a1$
$a'1$	$\to a''1$	$\to a0$	$a'1$	$\to a0.$

The above identity is the consequence of the fact that in these instances, as the result of the selection rules for dipole radiation (see (3.4) and (3.6)), there exist virtual intermediate states of the only one value j'' (with generally different a'').

However, it seems possible to find out experimentally whether in such transitions (as e. g. in the transition $a'0 \to a0$) in a particular case a two-quanta emission or a successive emission of two quanta is taking place. The following characteristic differences between the two types of emission are to be considered:

(1) for successive emission — an actual intermediate state of the nucleus, whose energy satisfies

$$E_{a'j'} > E_{a''j''} > E_{aj};$$

for simultaneous emission — virtual intermediate states of the nucleus, whose energies satisfy

$$E_{a''j''} > E_{a'j'} \quad \text{or} \quad E_{aj} > E_{a''j''};$$

(2) for successive emission — a discrete spectrum with two distinct frequencies; for simultaneous emission — a continuous spectrum.

Now in cases covered by formulae (4.7) containing unknown factors $I_s(aj, a'j')$ $(s=1,2,3)$ there exists, as a rule, virtual intermediate states with various j'' values. In these cases it may be expected that our correlation formulae should differ from the Hamilton formulae. To inquire into this possibility we shall try in the next section to evaluate the order of magnitude of the unknown factors in the formulae (4.7).

When the initial and final states have different parity, the electric double-dipole transitions for which our calculations apply, are forbidden by the generalized Laporte rule. In this case we must consider further expansion terms of the matrices $\langle ajm | H^*_{\vec{k}_1\vec{k}_2 l_1 l_2} | a'j'm' \rangle$ and $\langle ajm | H^*_{\vec{k}l} | a'j'm' \rangle$ appearing in (2.10). We get in this way the

transition probabilities for electric and magnetic double-multipole radiations of various orders.

It is not difficult to prove by means of (3.9), that the same correlation formula:

$$W^{\mathrm{rc}}_{\alpha' 0;\alpha 0}(\theta) = 1 + \cos^2\theta \tag{4.8}$$

as for electric double-dipole $\alpha' 0 \to \alpha 0$ transitions applies also for magnetic double-dipole $\alpha' 0 \to \alpha 0$ transition (both photons are of magnetic dipole type) and electric-magnetic double-dipole $\alpha' 0 \to \alpha 0$ transition (one photon is of electric dipole-, the other of magnetic dipole type). To prove it we make use of the following formulae

$$\begin{aligned} &\sum_{\lambda_1\lambda_2=1}^{2} |\vec{e}_{\vec{k}_1\lambda_1} \cdot (\check{I} \times \vec{k}^0_s) \cdot \vec{e}_{\vec{k}_2\lambda_2}|^2 = 1 + \cos^2\theta \quad (s = 1, 2), \\ &\sum_{\lambda_1\lambda_2=1}^{2} |\vec{e}_{\vec{k}_1\lambda_1} \cdot (\vec{k}^0_1 \times \check{I} \times \vec{k}^0_2)\, \vec{e}_{\vec{k}_2\lambda_2}|^2 = 1 + \cos^2\theta. \end{aligned} \tag{4.9}$$

The ratios of the orders of magnitude of the electric, electric-magnetic and magnetic double-dipole two-quanta emission probabilities are

$$1 : \left(\frac{2\pi R}{\lambda}\right)^2 : \left(\frac{2\pi R}{\lambda}\right)^4,$$

where $R \sim 10^{-13}$ is the radius of the nucleus. E. g., for γ rays of 1 Mev energy we have

$$\frac{2\pi R}{\lambda} \sim 5 \cdot 10^{-3}.$$

5. Estimation of Coefficients

We shall now evaluate the order of magnitude of the unknown factors appearing in the correlation formulae (4.7).

First we shall write formulae (3.13), (3.14), (3.15) in a slightly modified form:

$$\left.\begin{aligned} \langle aj \vdots T_1 \widehat{T}_2 \vdots a' j' \rangle_\omega &= \sum_{j''} {}_1(j, j'', j')\, B(aj; j''; a' j') \\ \langle aj \vdots T_1 \widehat{\cdot} T_2 \vdots a' j' \rangle_\omega &= \sum_{j''} a_2(j, j'', j')\, B(aj; j'' a' j') \\ \langle aj \vdots T_1 \widehat{\times} T_2 \vdots a' j' \rangle_\omega &= \sum_{j''} a_3(j, j'' j')\, B(aj; j''; a' j'), \end{aligned}\right\} \tag{5.1}$$

where

$$\begin{aligned}
&a_1(j,j\pm1,j)=-1 \qquad a_2(j,j-1,j)=j(2j-1) \qquad a_3(j,j-1,j-1)=j-1\\
&a_1(j,j,j)=1 \qquad a_2(j,j,j)=j(j+1) \qquad a_3(j,j,j-1)=-(j+1)\\
&a_1(j,j\pm1,j\pm1)=1 \qquad a_2(j,j+1,j)=(j+1)(2j+3) \qquad a_3(j,j-1,j)=2j-1\\
&a_1(j,j;j\pm1)=1 \qquad \text{other } a_2's=0 \qquad a_3(j,j,j)=1\\
&a_1(j;j\pm1;j\pm2)=1 \qquad\qquad c_3(j,j+1,j)=-(2j+3\\
&\text{other } a_1's=0 \qquad\qquad a_3(j,j,j+1)=j\\
&\qquad\qquad a_3(j,j+1,j+1)=-(j+2)\\
&\qquad\qquad \text{other } a_3's=0
\end{aligned} \tag{5.2}$$

Next we shall transform the expressions (4.2) which define the factors $I_s(\alpha j;\alpha' j')$ $(s=1,2,3)$ by making use of the fact that the function:

$$g_{\alpha j;\alpha' j'}(\omega_1+\omega_2)=2\frac{1-\cos(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)\frac{t}{\hbar}}{(E_{\alpha j}-E_{\alpha' j'}+\hbar\omega_1+\hbar\omega_2)^2} \tag{5.3}$$

has a strong maximum for:

$$\omega_1+\omega_2=\frac{1}{\hbar}(E_{\alpha' j'}-E_{\alpha j})=\omega_{\alpha j;\alpha' j'}, \tag{5.4}$$

while the other parts of the integrands in (4.2) are slowly varying with ω_1 and ω_2 — as a consequence of the fact that in the simultaneous processes

$$E_{\alpha'' j''}-E_{\alpha' j}+\hbar\omega_s\neq 0 \quad (s=1,2)$$

Instead of the sums over $\vec{k}_1$ and $\vec{k}_2$ we write integrals with respect to $\omega_1=ck_1$ $\omega_2=ck_2$ keeping in mind that the density (on the interval unit of ω) of oscillators of radiation field having the frequency ω and the direction of propagation contained in the solid angle $d\Omega$ is

$$\varrho(\omega)=\frac{G}{(2\pi c)^3}\omega^2 d\Omega, \tag{5.5}$$

where G is the volume of the cube of periodicity.

Assuming then that the initial radiation distribution is random:

$$N_{\vec{k}}=N \tag{5.6}$$

(e. g. $N_{\mathbf{k}}=0$ as for the spontaneous emission), we have

$$\left.\begin{aligned}
J_1(aj;a'j') &= tC\int_0^{\omega_{aj;a'j'}}(\omega_{aj;a'j'}-\omega_2)\,\omega_2\,|\langle aj:\widehat{PP}:a'j'\rangle_{\omega_2} \\
&\qquad +\langle aj:\widehat{PP}:a'j'\rangle_{\omega_{aj;a'j'}-\omega_2}|^2\,d\omega_2 \\
J_2(aj;a'j') &= tC\int_0^{\omega_{aj;a'j'}}(\omega_{aj;a'j'}-\omega_2)\,\omega_2\,|\langle aj:\widehat{P\cdot P}:a'j'\rangle_{\omega_2} \\
&\qquad +\langle aj:\widehat{P\cdot P}:a'j'\rangle_{\omega_{aj;a'j'}-\omega_2}|^2\,d\omega_2 \\
J_3(aj;a'j') &= tC\int_0^{\omega_{aj;a'j'}}(\omega_{aj;a'j'}-\omega_2)\,\omega_2\,|\langle aj:\widehat{P\times P}:a'j'\rangle_{\omega_2} \\
&\qquad -\langle aj:\widehat{P\times P}:a'j'\rangle_{\omega_{aj;a'j'}-\omega_2}|\,d\omega_2,
\end{aligned}\right\} \tag{5.7}$$

where we have assembled all constants into one quantity

$$C=\frac{(2\pi)^3(N+1)^2\,d\Omega_1\,d\Omega_2}{(2\pi c)^6} \tag{5.7a}$$

Now we come to the estimation of $I_s(aj,a'j')$ $(s=1,2,3)$. For this purpose we assume that the nuclei under consideration apart

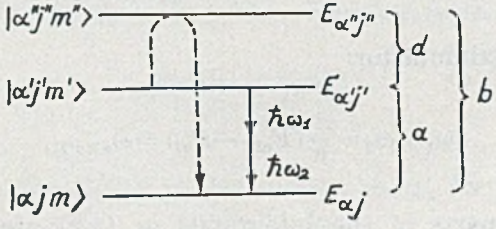

Fig. 2

from the initial energy level $E_{\alpha'j'}$ and the final energy level $E_{\alpha j}$ have only one intermediate energy level $E_{\alpha''j''}$, and that all energy levels are degenerate with respect to m (cf. R. G. Sachs (1940)). The situation is represented in Fig. 2.

In this case the formulae (5.1) are reduced to the following:

$$\left.\begin{aligned}
\langle aj:\widehat{T_1T_2}:a'j'\rangle_\omega &= a_1(j,j'',j')\,B(aj;a''j'';a'j') \\
\langle aj:\widehat{T_1\cdot T_2}:a'j'\rangle_\omega &= (a_2j,j''j')\,B(aj;a''j'';a'j') \\
\langle aj:\widehat{T_1\times T_2}:a'j'\rangle_\omega &= a_3(j,j'',j')\,B(aj;a''j'';a',j')
\end{aligned}\right\} \tag{5.8}$$

where

$$B(aj;a''j'';a'j')=\frac{\langle aj:T_1:a''j''\rangle\langle a''j'':T_2:a'j'\rangle}{\hbar^2\dfrac{E_{a''j''}-E_{a'j'}+\hbar\omega}{(E_{aj}-E_{a''j''})(E_{a''j''}-E_{a'j'})}} \tag{5.9}$$

Integrals (5.7), after the introduction of the following notations

$$\begin{aligned} &E_{a'j'}-E_{aj}=\hbar\omega_{aj;a'j'}=a, \quad E_{a''j''}-E_{aj}=b,\\ &E_{a''j''}-E_{a'j}=d, \quad \hbar\omega_2=x \end{aligned} \tag{5.10}$$

assume the form given below

$$J_1=t\frac{C}{\hbar^7}a_1^2|\langle aj\vdots P\vdots a''j''\rangle\langle a''j''\vdots P\vdots a'j'\rangle|^2 b^2d^2(b+d)^2\int_0^a\frac{(a-x)x}{(b-x)^2(d+x)^2}dx$$

$$J_2=t\frac{C}{\hbar^7}a_2^2|\langle aj\vdots P\vdots a''j''\rangle\langle a''j''\vdots P\vdots a'j'\rangle|^2 b^2d^2(b+d)^2\int_0^a\frac{a-x)x}{(b-x)^2(d+x)^2}dx \tag{5.11}$$

$$J_3=t\frac{C}{\hbar^7}a_3^2|\langle aj\vdots P\vdots a''j''\rangle\langle a''j''\vdots P\vdots a'j'\rangle|^2 b^2d^2\int_0^a\frac{(a-2x)^2(a-x)x}{(b-x)^2(d+x)^2}dx.$$

Through integration we get

$$\begin{aligned} &\int_0^a\frac{(a-x)x}{(b-x)^2(d+x)^2}dx=\frac{2}{(b+d)^2}\left[\frac{b^2+d^2}{b+d}\ln\frac{b}{d}-(b-d)\right]\\ &\int_0^a\frac{(a-2x)^2(a-x)x}{(b-x)^2(d+x)^2}dx=2\left[\frac{b^2+4bd+d^2}{b+d}\ln\frac{b}{d}-3(b-d)\right]. \end{aligned} \tag{5.12}$$

Finally, in our approximation, we have the following expressions for the factors in (4.7)

$$\begin{aligned} &\frac{J_3(aj;a'j')}{J_1(aj;a'j')}=a_3^2(j,j'',j')\left[1+2\frac{2\ln u-\left(u-\frac{1}{u}\right)}{\left(u+\frac{1}{u}\right)\ln u-\left(u-\frac{1}{u}\right)}\right]\\ &\frac{J_2(aj;a'j')}{J_1(aj;a'j')}=a_2^2(j;j'';j'), \end{aligned} \tag{5.13}$$

where

$$u=\frac{b}{d}=\frac{E_{a''j''}-E_a}{E_{a''j''}-E_{a'j'}} \quad (>0). \tag{5.14}$$

In particular for $u=2$ and $u=4$ have respectively

$$\frac{J_3}{J_1}\approx 2\,a_3^2 \quad \text{and} \quad \frac{J_3}{J_1}\approx a_3^2.$$

It can be seen that the coefficients in (4.7) are of the order of magnitude of one. It follows indeed that in certain cases the Hamilton

formulae differ from ours, e. g. for the transition $a'2 \to a''1 \to a2$ the former give (10)

$$1 + \frac{1}{133}\cos^2\theta,$$

while ours, according to the above estimation, give

$$1 + \tfrac{1}{13}\cos^2\theta + 1 - \tfrac{1}{3}\cos^2\theta + 1 + \cos^2\theta \sim 1 + \tfrac{1}{4}\cos^2\theta.$$

We will now estimate the order of magnitude of the absolute probability for the electric double-dipole two-quanta emission, e. g. for the transition $a'0 \to a0$. Formula (4.1) takes now the form (we write $W_{a'j';aj}(\vec{k}_1^0, \vec{k}_2^0) = W_{a'j';aj}{}^{(\theta)}$)

$$W_{a'0;a0}{}^{(\theta)} = \tfrac{1}{9} J_2(a0; a'0)(1 + \cos^2\theta)]. \tag{5.15}$$

Making use of the approximation formulae (5.11), we have

$$W_{a'0;a0}{}^{(\theta)} = t\frac{(2\pi)^4(N+1)^2}{3\hbar^7 c^6} d\Omega_1 d\Omega_2 |\langle aj \vdots P \vdots a''j''\rangle|^2 |\langle a''j'' \vdots P \vdots a'j'\rangle|^2 \cdot 2b^2d^2\left[\frac{b^2+d^2}{b+d}\ln\frac{b}{d} - (b-d)\right](1+\cos^2\theta). \tag{5.16}$$

The probability per unit of time for two-quanta emission in all directions is given by

$$w_{a'j';aj} = \frac{1}{t}\int\int \frac{W_{a'j';aj}{}^{(\theta)}}{d\Omega_1 d\Omega_2} d\Omega_1 d\Omega_2 = \frac{16\cdot 2}{t}\int_0^\pi \frac{W_{a'j';aj}(\theta)}{d\Omega_1 d\Omega_2}\sin\theta\, d\theta, \tag{5.17}$$

where $d\Omega_2 = \sin\theta\, d\theta\, d\varphi$ (if $\vec{k}_1^0$ points in the direction of the z-axis). So we have

$$w_{a'0;a0} = \frac{2048\pi^6(N+1)^2}{9\hbar^7 c^6} |\langle aj \vdots P \vdots a''j''\rangle|^2 |\langle a''j'' \vdots P \vdots a'j'\rangle|^2 b^2d^2\left[\frac{b^2+d^2}{b+d}\ln\frac{b}{d} - (b-d)\right]. \tag{5.18}$$

We estimate the expressions $\langle aj \vdots P \vdots a'j'\rangle$ as follows

$$|\langle aj \vdots P \vdots a'j'\rangle| \sim |\langle ajm | P | a'j'm'\rangle| \sim eR,$$

where $R \sim 10^{-13}$ is the radius of the nucleus. According to this estimation formula (5.18) gives e. g. for $b = 1\,\mathrm{Mev}$, $d = 0{,}5\,\mathrm{Mev}$ and $N = 0$ (spontaneous emission) the following order of magnitude for the electric double-dipole emission probability

$$w_{a'0;a0} \sim 10^4\ \mathrm{sec}^{-1}.$$

In our case, for electric-magnetic double-dipole emission we have $w_{a'0;a0} \sim 10^{-1}\ \mathrm{sec}^{-1}$ and for magnetic double-dipole emission,

$$w_{a'0;a0} \sim 10^{-6}\ \mathrm{sec}^{-1}.$$

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THERMAL PSEUDOHYSTERESIS OF THE DIELECTRIC CONSTANT OF FERROELECTRIC TITANATES*

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Measurements of the dielectric constant of barium titanate and barium-strontium titanate were made in various temperatures with an alternating current of 2 and 5 kHz frequency and a distinct anomaly in the dependence of the dielectric constant on temperature was discovered: the curve $\varepsilon = f(t)$ is different during the heating and the cooling of a sample. Values of the dielectric constant during cooling (above as well as below the Curie point) are higher than those measured at the same temperatures during heating. In the direct vicinity of the Curie point the value of the dielectric constant is higher by over 10 per cent during cooling than the maximum value observed during heating. We have called this phenomenon the thermal pseudohysteresis of the dielectric constant because of the characteristic appearance of the residue after the former thermal state.

Ferroelectrics of the titanate group possess an unusually large dielectric constant which is to a great extent dependent on temperature. The maximum value occurs in the Curie temperature. Other physical properties of the ferroelectrics also present at this point a number of anomalies. Examining the dielectric constant of titanates in various temperatures we observed in the vicinity of the Curie point an anomaly of the dielectric constant, consisting in the phenomenon that the dielectric constant depended not only on temperature but also on the former thermal state of the examined sample.

Experimental procedure

Measurements of the dielectric constant in various temperatures were made by means of a Sullivan bridge fed by an oscillator of acoustic frequencies. The examined samples consisted of polycrystalline

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barium titanate or barium-strontium titanate composed in 21,4 per cent of $SrTiO_3$ and in 78,6 per cent of $BaTiO_3$, produced by the Institute of Inorganic Chemistry of Wrocław Institute of Technology [1].

The samples were sintered in 1300°C during 2 hours. They had the shape of disks of the following dimensions: $BaTiO_3$ — diameter 11,50 mm, thickness 3,72 mm; $BaTiO_3$—$SrTiO_3$ — diameter 11,54 mm thickness 2,80 mm. The measuring condenser consisted of the sample with silvered plates. The silver plating was done by the thermogenetic method (Partington et al., 1949) by heating the sample painted with Ag_2O — paste in an electric furnace at 700°C. The condenser with soldered copper wires was placed in an empty metal container with a precise thermometer and three thermocouples. This container, was placed in an ultrathermostat for investigations up to 100°C, and in a special electric furnace for higher temperatures. The temperature was held constant within the limits of $\pm$ 0,1°C. The temperature was controlled by means of the thermo-couples. The thermo-couples of copper (0,18 mm diameter) and Eureka resistance wire (0,12 mm diameter) had a thermoelectric efficiency of 40 μV/1°C; a difference of temperature of 0,05°C could be detected by employing a galvanometer with the sensivity of $2 \cdot 10^{-9}$A per division of the scale. The junctions were placed so that the following differences of temperature could be established: between the wall of the container and the plate of the condenser, the plate of the condenser and the bulb of the thermometer, and between the bulb of the thermometer and the interior of the condenser. In the last case a small hole was drilled with a thin needle in the sample in order to introduce into it the junction of the thermo-couple.

Experimental results

Measurements of the dielectric constant were made by employing an alternating current, the frequency being 2 kHz and 5 kHz. The measuring potential difference amounts to 7,8 V maximum. The temperature range was 20—150°C for barium titanate and 20—75°C for barium-strontium titanate. Fig. 1 represents the curve of the dielectric constant plotted against temperature in the vicinity of the Curie point. Curves I, II, III represent consecutive cycles of heating and cooling for the same sample of barium-strontium titanate. Each cycle lasted 1 to 5 hours. It is obvious from the curves that changes of the dielectric constant are different during the rise and during the

[1] The Authors wish to express their thanks to Professor W. Trzebiatowski for providing the samples for the investigation.

fall of temperature. Corresponding curves were obtained for the barium titanate. It was observed that changes in the speed of heating did not essentially change the curve of heating (IIa, IIIa, apart from the curve Ia, which was the curve of a first heating), but

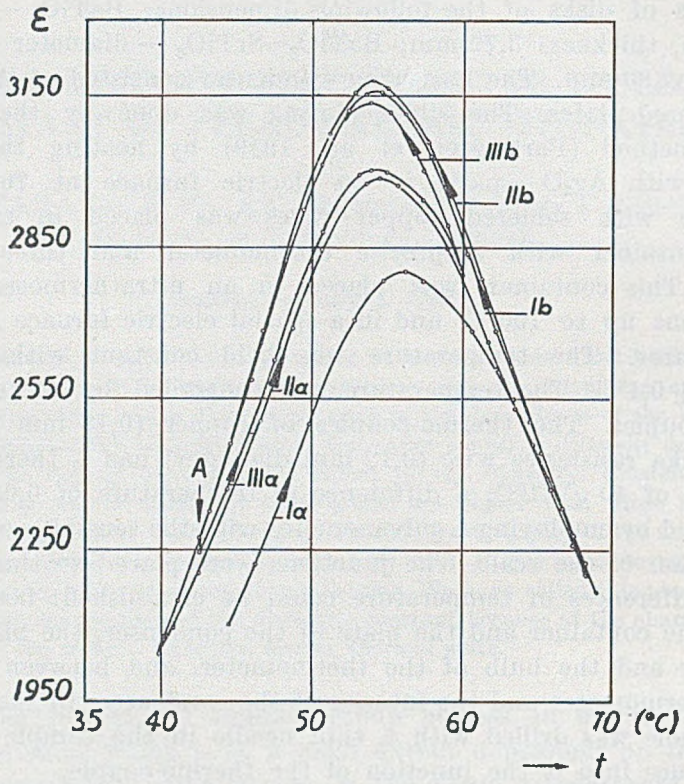

Fig. 1. The dependence of the dielectric constant of barium-strontium titanate on temperature for three consecutive cycles of heating and cooling.

changes in the speed of cooling markedly affected the curve of cooling (Ib, IIb, IIIb). It was observed that the shape of the curve depends not only on the speed of cooling but also on the previous temperature of the sample. The dielectric constant of the sample has a higher value during cooling than during heating; sometimes the differences exceed 10 per cent.

The temperature in which the maximum of the dielectric constant occurs depends on the direction of changes of temperature: during cooling the maximum occurs at a somewhat lower temperature than during heating. For barium titanate the maximum is observed at 127°C during the heating and at 125°C during the cooling: for barium-strontium titanate at 56°C during the heating and at 54°C dur-

ing the cooling. The different course of curves during the heating and the cooling processes is also distinct in closed thermal cycles not enclosing the Curie point, but lying above or below it. Fig. 2 represents typical curves $\varepsilon = f(t)$ for barium-strontium titanate below the Curie point (curve I) and for barium titanate above the Curie point (curve II). The figure shows that the closed cycle of heating and cool-

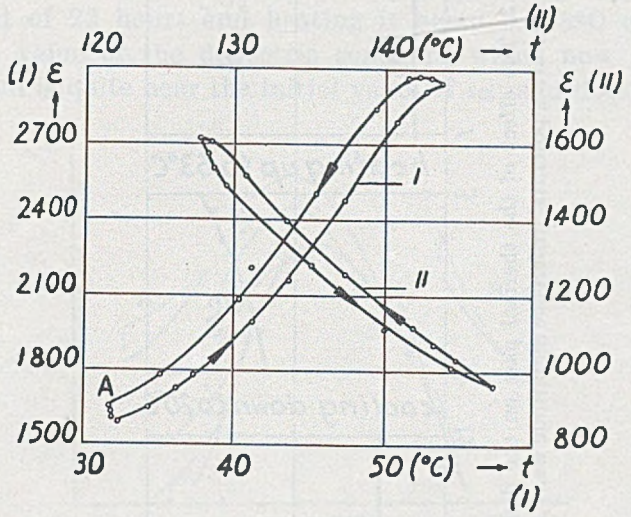

Fig. 2. Pseudohysteresis curves: I — $BaTiO_3$ — $SrTiO_3$ below the Curie point; II — $BaTiO_3$ above the Curie point.

ing below the Curie point resembles the hysteresis loop; it is composed of two parts: the ascending one and the descending one, and the descending one lies above the ascending one. Now above the Curie point the situation is different: the descending curve lies below the ascending one. The „residue" after the former thermal state appears here in a singular way: a high temperature in the past causes an increase of the dielectric constant, even if the dielectric constant in higher temperature is lower as is the case above the Curie point. For this reason we have called this phenomenon the thermal pseudohysteresis of the dielectric constant. This peculiar occurence takes place in a still more striking way in the direct vicinity of the Curie point, where the dielectric constant measured during cooling has a higher value than in the whole process of heating. In points *A* of Figs. 1 and 2 it is obvious that stopping the process of cooling causes a spontaneous falling of the value of the dielectric constant. These facts were corroborated by experiments, the results of which are represented by Fig. 3.

A sample of barium-strontium titanate, which was kept at room temperature, was heated in an ultrathermostat to 45°C and its dielectric constant was measured [2]: $\varepsilon_{45}=2133$. This value did not change

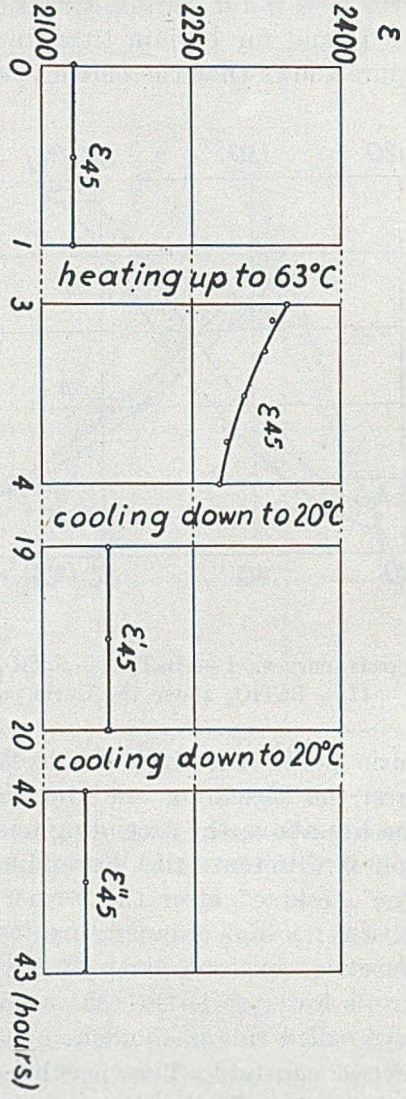

Fig. 3. The effect of the thermal past on the dielectric constant of $BaTiO_3$—$SrTiO_3$. The measuring potential difference amplitude was 0,3 V, thickness of the sample 2,80 mm.

in time. The sample was next heated to 63°C and cooled back to 45°C. The measured value of the dielectric constant after the temperature had become stable exceeded considerably the initial value ε_{45} and

[2] ε with an index below denotes the value of the dielectric constant measured when the temperature had been recently lower than the temperature at which the measurement was taken; with an index above — when the temperature had been higher.

amounted to $\varepsilon^{45}=2346$. This value diminished in the course of time as shown by the curve on Fig. 3. After a series of measurements which lasted one hour the sample was cooled to room temperature and after 15 hours was again heated to 45°C. The value on the constant $\varepsilon'_{45}=2169$ is now much lower than the former ε^{45} and only slightly higher than the initial value ε_{45}. It is noteworthy that this value does not diminish visibly in the course of time. Cooling of the sample again to 20°C for a period of 22 hours and heating it again to 45°C changes but slightly the value of the dielectric constant, which now amounts to $\varepsilon''_{45}=2145$ and is quite near the initial value of ε_{45}, and does not change

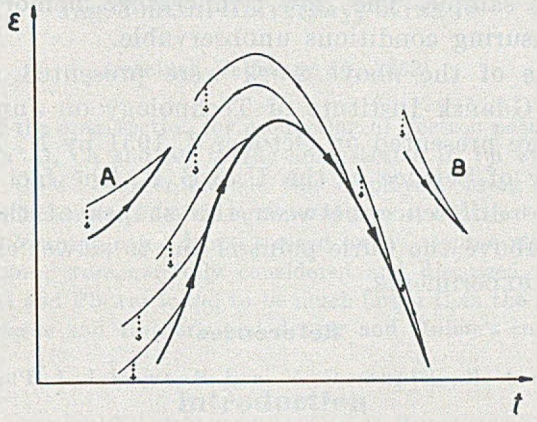

Fig. 4. Diagram of the dependence of ε of titanates on temperature *after several thermic cycles*. The thick curve — heating; an almost uniform dependence. Thin curve — cooling; the shape of these curves depends on the initial temperatures and on the speed of cooling.

in the course of time. Thus a change of the dielectric constant in the course of time occurs only if the sample before measuring was in a higher temperature, but it does not occur if the sample before the measuring was in a lower temperature. In this latter case an influence of a still more remote thermal past of the sample may appear only in a small degree, if then it had been in a higher temperature. Probably for this reason the value of ε'_{45} is somewhat higher than those of ε''_{45} and ε_{45}.

The results obtained may be represented on a general diagram, on Fig. 4, where one can see the shapes of curves $\varepsilon=f(t)$ during heating of the condenser with the dielectric, and during cooling from various initial temperatures. The dotted lines show the direction of the spontaneous change of the dielectric constant when the process of cooling is stopped at a certain temperature. In that case the value of the dielectric constant always diminishes. The curves *A* and *B* on Fig. 4

represent individual thermic cycles for temperatures above and below the Curie point.

The above mentioned observations may be interpreted as follows. The dielectric constant changes its value without any delay in heating process, nevertheless in cooling process the delay-effect takes placed. This effect appears as an excess of dielectric constant as regards to the value observed in heating process at the same temperature. This excess vanishes with time and proves about previous termal state of the sample, if there was a state of higher temperature than measuring one. This phenomenon may be called a „high temperature memory" of the sample. The „low temperature memory", if existing, was in our measuring conditious unobservable.

The results of the above work were presented at a scientifi, meeting of the Gdańsk Institute of Technology on June 8, 1951. Similar results were presented on October 3, 1951 by N. A. Roy (1951) to the Academy of Science of the U. S. S. R. The fact that Roy did not observe the difference between the shapes of the cooling and heating curves above the Curie point is due to slower changes of temperature in his experiments.

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ON THE DEPENDENCE OF THE CROSS-SECTION FOR PAIR PRODUCTION ON THE ATOMIC NUMBER Z

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The ratio of the cross-section for production of electron pairs by gamma ThC'' rays (2,62 MeV) in Al, Cu and Ag to the cross-section in Pb was measured. For this purpose the G.-M. counter method was employed, using counters for beta rays working in coincidence. Possible sources of errors, the most important among which are due to Compton coincidences caused by a Compton electron and a scattered gamma photon, were carefully considered and discussed. The ratio of the cross-sections in Al and Pb was found to be much larger than the ratio theoretically calculated from Bethe and Heitler's and Jaeger and Hulme's theories.

Introduction

The aim of the measurements was to obtain the ratio of the cross-section for the production of electron pairs in Al, Cu, and Ag by gamma rays of ThC'', to the cross-section in Pb. The results of these measurements were compared with values calculated on the basis of Bethe and Heitler's and Jaeger and Hulme's theories.

According to Bethe and Heitler (1934) who employed in their calculations Born's approximation, the cross-section for pair production is proportional to Z^2. According to Jaeger and Hulme (1936) whose calculations were more exact (they did not employ the above mentioned approximation), the dependence on Z of the cross-section for pair production is expressed by the formula

$$\sigma = a(Z/137)^2 + b(Z/137)^4$$

in which the first term gives the value of the cross-section for Born's approximation; the constants a and b for $h\nu = 5{,}2\, mc^2$ (2,62 MeV) calculated from experimental data for lead amount to $a = 6{,}95$ barns, $b = 4{,}62$ barns.

The dependence of the cross-section for pair production on the atomic number was measured by the cloud chamber method by

Chadwick, Blackett and Occhialini (1933, 1934) and by Bencke (1935). The results obtained point approximately to a quadratic dependence of the cross-section on Z, but the small amount of statistical material and errors due to the thickness of employed plates do not permit to choose between the two theories. Benedetti's (1935, 1936) and Bocciarelli's (1937) measurements, in which the number of positons produced by gamma rays in plates of various materials was measured with G.-M. counters, indicated certain deviations from the quadratic dependence on Z for light elements. These deviations were explained by Franchetti (1938) to be due to the background of Compton electrons. Groshev (1945) by means of the cloud chamber method obtained a quadratic dependence on Z of the cross-section for pair production by gamma rays of ThC'' in the chamber gas (nitrogen, krypton and xenon), but the precision of his measurements does not yet permit to decide between the two theories. Measurements of absorption in various materials, of gamma rays of large energies: 11,04 MeV, 13,73 MeV, 19,10 MeV (Adams 1948) and 88 MeV (Lawson 1949), for which the production of electron pairs is the main cause of absorption, proved inconsistent with theoretical calculations: for heavy elements (Pb and U) the cross section was too small, for light ones (Be and Al) too large. These deviations had a different sign than Jaeger and Hulme's correction, and were larger for gamma rays of greater energy, which is also inconsistent with Jaeger and Hulme's theoretical calculations. Recently there appeared a paper by Hahn, Baldinger and Huber (1951) who, by way of registering annihilation radiation by means of scintillation counters, counted the positons produced by gamma rays in various materials. Some deviations from the quadratic dependence on Z obtained by them for elements heavier than Fe were consistent with Jaeger and Hulme's theory. They did not investigate elements lighter than Fe, because Hulme and Jaeger's theory does not expect in those materials measurable deviations from the quadratic dependence.

Apparatus

The apparatus used in the present work was similar, except for slight modifications, to this of a former work (Hrynkiewicz 1950).

A sample of MsTh equivalent to about 40 mg of Ra was placed in a lead block. The gamma rays filtered through 25 mm of lead were practically monoenergetic with energy of 2,62 MeV. A beam of these rays, was collimated by means of a channel 30 cm long and directed towards a metal foil in which the pairs were produced. The foils were

set in a light aluminium frame in order to diminish the background due to the external layers of the lead block in which the collimating channel was bored. The G.-M. counters for beta rays with mica

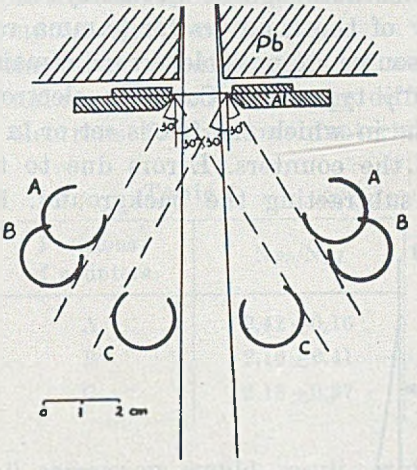

Fig. 1. Position of the counters.

windows (5 mg/cm^2), working in coincidence, were placed in a distance of about 5 cm from the foil. The mouth of the collimating channel, the foil frame and the setting of the counters are represented on Fig. 1.

Measurements and results

Proper coincidences, i. e. coincidences caused by electron pairs produced in the foil may belong to two types:

1. $\beta^-\beta^+$-caused by a negaton entering one counter and a positon entering the other.

2. $\beta^+\gamma$-caused by a positon registered in one counter and the annihilation photon of this positon producing an impulse in the second counter.

Of course the first type happens much more frequently.

Besides these the apparatus could register the following undesirable coincidences:

1. $\beta^-\gamma$-„Compton coincidences" caused by a Compton electron entering one counter, and a scattered gamma photon producing an impulse in the second counter;

2. $\gamma\gamma$-caused by registering in one counter a scattered photon registered previously in the other one;

3. Accidental coincidences.

The number of accidental coincidences did not exceed 0,05 per hour thanks to the good resolving time of the coincidence circuit; and they could be neglected as well as those of the $\gamma\gamma$ type which could be neglected because of the geometry of the apparatus and the small efficiency of the counters for gamma rays.

As the main source of possible errors remain the Compton coincidences of the $\beta^-\gamma$ type. The Compton electron may originate in the foil, in the frame in which the foil is set or in the layer of air between the foil and the counters. Errors due to the last two sources are eliminated by subtracting the background. For eliminating the

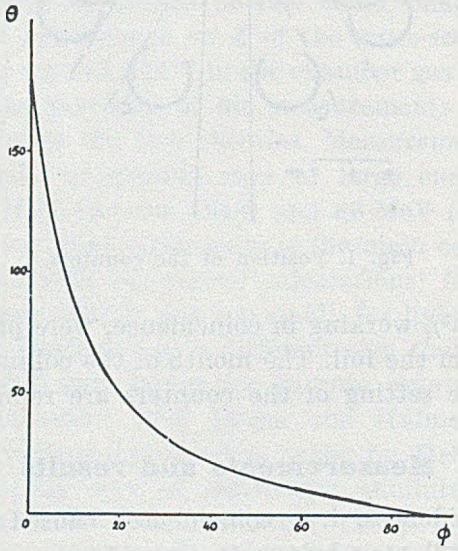

Fig. 2. The dependence of the angle between the directions of the incident and the scattered photon on the angle between the directions of the incident photon and the Compton electron for gamma rays of 2,62 MeV energy.

foil as a source of numerous Compton coincidences it is necessary to set the counters at a proper angle. It is obvious from Fig. 2 that for gamma rays of the energy of 2.62 MeV the angle between the directions of the negaton and of the scattered photon $(\Phi+\Theta)$ is never smaller than 60^0 and that for angles Φ smaller than 30^0 the corresponding angles Θ are larger than 30^0 and vice versa. If we set both counters symmetrically in such a way, that their effective volumes lie totally in a cone whose angle at the vertex is equal to 60^0, or outside such a cone, coincidences of the $\beta^-\gamma$ type will not be registered. Setting of the counters inside this cone is not convenient due to a large background of gamma rays. In our measurements the external setting

was employed, as represented by Fig. 1A. Conformity of obtained ratios of coincidences N_{Pb}/N_{Al} for Al (43,6 mg/cm²) and Pb (42,4 mg/cm²) foils for three settings of counters A, B and C (see Table 1 and Fig. 1) proves that the Compton coincidences $\beta^-\gamma$ (except those caused by Compton electrons scattered in the foil) are not registered and that the effect of differences of angular distribution of pair electrons on final results is small. If the differences of angular distribution of pair electrons for various materials are taken into

Table 1

Positions of counters	N_{Pb}/N_{Al}
A	2,42 ± 0,16
B	2,16 ± 0,41
C	2,18 ± 0,37

consideration, a small correction would result, which would enlarge the observed deviation for Al from theoretical calculations.

The ratios of the numbers of coincidences for Al, Cu and Ag foils to the number of coincidences for the lead foil were extrapolated to zero thickness in order to eliminate possible errors arising from the differences of absorption of produced electrons for the same surface density of various materials, and the scattering in the foil of Compton electrons from a 60° cone. The extrapolation to zero thickness eliminates these errors, as both effects mentioned above are proportional to the square of the thickness of the foil, while the number of pairs produced in the foil is linearly proportional to the thickness. This can be proved by the following simplified reasoning. The number of Compton electrons produced in a layer of the foil of thickness dh, lying in the distance h from the surface, is proportional to the thickness of this layer:

$$dn_0 = a \cdot dh.$$

The number of electrons produced in this layer, which will be scattered while passing through the remaining part of the foil is proportional to dn_0 and to the distance which they must still pass in the foil, i. e. to $H-h$, where H is the total thickness of the foil,

$$dn = b \cdot dn_0 (H-h),$$

whence

$$n = \int_0^H a \cdot b \cdot (H-h)\, dh = (abH^2)/2.$$

Therefore the number of scattered Compton electrons is proportional to H^2.

In the case of absorption in the foil of produced pair electrons a similar reasoning may be completed.

By passing to the limit ($H \to 0$) with the ratio of coincidences for the given element and for lead, the effects proportional to H^2 are eliminated.

The measurements consisted in registering the number of coincidences N for Al, Cu, Ag and Pb foils in dependence on surface density s. The results after substracting the background, are presented in Table 2 and on diagrams in Fig. 3. The ratio of coincidences N_{Pb}/N,

Table 2

Frame without foils	t (h) 107,3		N_0 (imp/hour) 8,8 ± 0,19
Element	s (mg/cm²)	t (h)	N (imp/hour)
Al	11,01	74,8	3,3 ± 0,33
	22,02	53,1	6,7 ± 0,42
	32,74	50,2	8,5 ± 0,44
	43,56	47,0	12,2 ± 0,49
Cu	11,68	56,2	5,0 ± 0,38
	23,01	85,8	8,9 ± 0,36
	29,92	72,2	11,4 ± 0,40
	41,60	55,5	15,0 ± 0,48
Ag	12,63	67,3	8,0 ± 0,39
	20,14	69,7	11,2 ± 0,41
	25,51	63,3	14,3 ± 0,45
	32,77	47,5	17,9 ± 0,54
	37,66	51,9	19,4 ± 0,54
Pb	9,91	76,2	9,9 ± 0,38
	20,94	63,4	17,5 ± 0,47
	30,85	57,8	23,3 ± 0,54
	42,36	36,7	29,5 ± 0,71

as a function of s, was then evaluated. The obtained values were graphically extrapolated to zero thickness (Fig. 4). In Fig. 4 there are also represented the values of the ratio N_{Pb}/N calculated from Bethe and Heitler's (o) and Jaeger and Hulme's ($\times$) theories. These values are compared with experimental values in Table 3. The ratio of the

Table 3

Element	Z	A	N_{Pb}/N		
			theoretical		experimental
			H.--B.	J.--H.	
Al	13	27,0	5,2	6,5	3,7 ±0,4
Cu	29	63,6	2,4	2,9	2,55±0,2
Ag	47	107,9	1,6	1,8	1,7 ±0,1
Pb	82	207,2	1,0	1,0	1,0

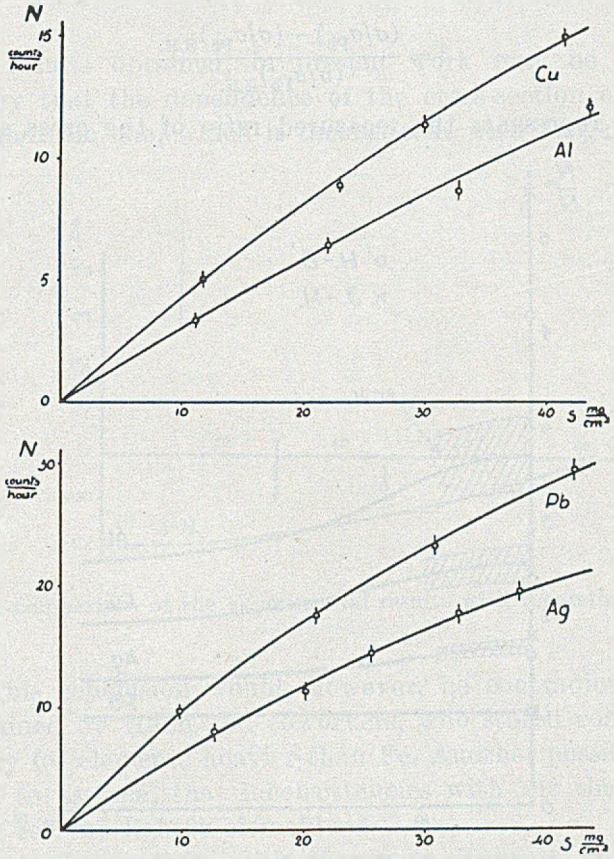

Fig. 3. The dependence of the number of coincidences on the surface density of the Al, Cu, Ag and Pb foils.

cross sections σ/σ_{Pb} calculated from experimental values of N/N_{Pb} are represented in Table 4; for comparison the theoretically calculated

Table 4

Element	σ/σ_{Pb}		
	theoretical		experimental
	H.—B.	J.—H.	
Al	0,025	0,020	0,035 ± 0,003
Cu	0,125	0,104	0,121 ± 0,009
Ag	0,328	0,286	0,306 ± 0,018
Pb	1,0	1,0	1,0

ratios are also given in this table. Values of the expression

$$\delta = \frac{(\sigma/\sigma_{Pb}) - (\sigma/\sigma_{Pb})_{B.H.}}{(\sigma/\sigma_{Pb})_{B.H.}}$$

where σ/σ_{Pb} represents the measured ratio of the cross sections, and

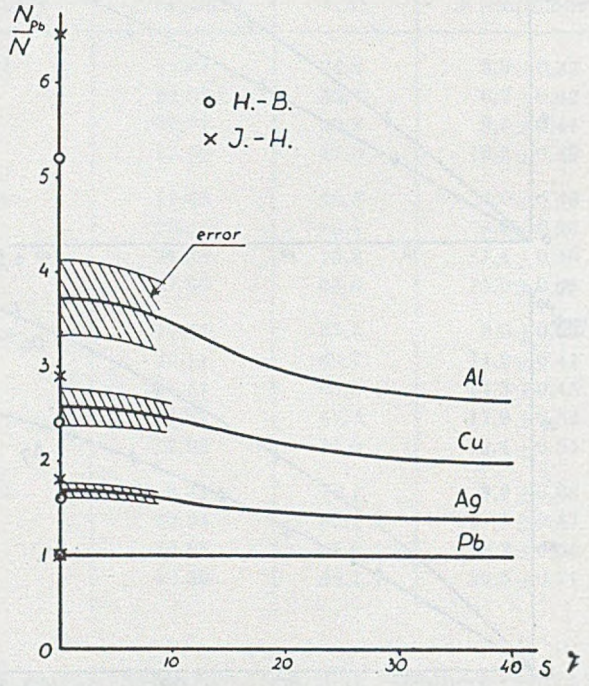

Fig. 4. The ratio of the number of coincidences N/N_{Pb} as a function of the surface density s.

$(\sigma/\sigma_{Pb})_{B.H.}$ represents the ratio calculated from Bethe and Heitler's theory, were calculated in order to make visible on a diagram the deviations from Bethe and Heitler's theory. The values δ are repre-

sented in Table 5 and compared with deviations expected by Jaeger and Hulme's theory, and presented on a diagram (Fig. 5) as a function of Z.

Table 5

Element	J.—H.	experimental
Al	—0,20	$+(0{,}40 \pm 0{,}12)$
Cu	—0,17	$-(0{,}03 \pm 0{,}07)$
Ag	—0,13	$-(0{,}07 \pm 0{,}05)$
Pb	0	0

The results obtained, in present work may be interpreted in such a way that the dependence of the cross-section on Z is weaker than a quadratic one, which is inconsistent with Jaeger and Hulme's

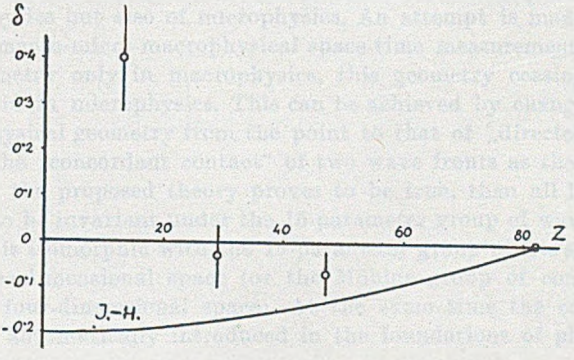

Fig. 5. A comparison of the experimental results with theoretical calculations.

theory. This conclusion would, however, be contradictory to the results obtained by Hahn and coworkers, who stated conformance with this theory for elements heavier than Fe. Another possible explanation would be to assume, that inconsistencies with the theory take place especially for the lighter elements.

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ON THE MICROSTRUCTURE OF THE WORLD. I. THE ELEMENTARY LENGTH*

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(received June 10, 1952)**

The main causes of the difficulties encountered by present day physics in the theory of elementary particles is attributed to (i) the indiscriminate use of Euclidean (or pseudo-Euclidean) metric both in microphysics and macrophysics and (ii) the recognition of the point (or the point-event) not only as the primitive notion of macrophysics but also of microphysics. An attempt is made to construct such a scheme of macro-micro-macrophysical space-time measurements which leads to Euclidean geometry only in macrophysics, this geometry ceasing to be even approximately valid in microphysics. This can be achieved by changing the primitive element of physical geometry from the point to that of „directed wave front" and by adopting the „concordant contact" of two wave fronts as their unique primitive relation. If the proposed theory proves to be true, than all laws of microphysics will have to be invariant under the 15-parameter group of wave-front transformations, which is isomorphic with the 15-parameter group of Lie's sphere transformations in three-dimensional space (or the Möbius group of conformal point-transformation in four-dimensional space). At the same time the concept of elementary length is automatically introduced in the foundations of physics.

§ 1. Introduction

1.0. The difficulties encountered in present day microphysics led many eminent physicists to express the view that the changes that have to be done in the foundations of quantum physics must necessarily be of a very drastic nature. It seems, however, that all the innovations introduced so far in this direction were not deep enough. The present paper is the first part of an attempt to formulate the foundations of physics in such a way as to take into consideration from the very beginning the essential difference between microphysics

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and macrophysics. The main idea is to choose as the primitive notion of the geometry on which the whole structure of physics has to rely the concept of wave rather than that of point (or point-event)[1]. It turns out that at the same time the concept of elementary length is built in in the foundations of physics, so to say, automatically.

1.1. We begin with a general remark of a rather vague nature. The fact that the dimensions of the atoms and the periods of light which they emit cannot be determined without knowledge of Planck's constant h suggest that the metric of physical space might be intimately connected with the process of quantization. The suspicion arises that after the introduction of Euclidean metric as background both for macroscopic and microscopic phenomena it might be already too late for a consistent quantization; in other words, it might be too late to build in the third, dynamical, universal constant h (considering the universal length as the first, static or geometrical, and the velocity of light c as the second, kinematical, universal constant.

§ 2. The macroscopic standpoint

2.0. On the other hand, it is well known that purely microphysical measurements do not exist but only macro-micro-macrophysical ones, and that — on the microphysical scale — all these measurements are performed from a very large distance (practically from infinity). The nearest distances from which we can observe an electron is of the order of magnitude of the grains of photographic emulsions (c. 1 μ); therefore, if we imagine the electron to be a sphere of 1 mm diameter, the nearest distance from which we could observe it would be some 1000 km from the 1 mm sphere. Many experiments with Geiger-Müller counters are comparable with the observation from the earth of a sphere of 1 mm diameter located at the distance of the sun (corresponding to a distance of the counters of c. 15 cm).

2.1. Let us then adapt the starting point of the new theory to this situation. We imagine a microscopically very large sphere S_R of radius R. Our measuring instruments are situated on this sphere and outside of it. It is well known that their functioning can be described by macroscopic concepts. This implies that on the S_R-sphere and outside of it ordinary Euclidean geometry prevails with an accuracy surpassing by far the possibilities of our detecting any discrepancies from its laws (at least as far as points and figures consisting of points are

[1] This view was already expressed by the author in his conference on Wave Mechanics and Relativity Theory at the 7th Meeting of Polish Physicists at Cracow, September 1934 (Weyssenhoff 1935).

concerned). But from the interior of S_R different „signals" arrive to us and we must find means to interpret them. Putting it more precisely, we may describe the situation, as follows. Near A on or outside of S_R we perform some experiments and then a short while afterwards we observe in other points $B, C, \ldots$ on or outside of S_R phenomena which did not occur before. Moreover, we can ascertain that the phenomena in $B, C, \ldots$ were not caused by any action propagated from A around or even through the interior of the sphere S_R by any known macroscopic process. We speak then of *microscopic* phenomena and we are just looking for a theory to describe them all in a consistent way. What right do we have to assume that Euclidean geometry prevailing on our „macroscopic standpoint" outside of S_R will also be adequate to describe all microscopic phenomena inside of S_R?

2.2. The general situation sketched above was often considered[2] or at least conceived by many physicists and it was also the starting point of Heisenberg's considerations leading to his S-matrix formalism. However, provided the theory advocated here should prove to be true, Heisenberg did not make the decisive step of discarding Euclidean geometry as background of quantum theory.

§ 3. The wave fronts and their geometry.

3.0. The task before us is to find such a geometry which will in a certain sense, yet to be specified, cease to be Euclidean *towards the inside*, and not *towards the outside* as Riemannian geometry and all sorts of kindred geometries do. At first this could seem impossible as none of the very different kinds of non-Riemannian geometries created by mathematicians and physicists of the 20th century show this peculiarity. On the contrary, all these geometries have the common property of becoming „linearized by passage to the infinitely small", they all become simpler in smaller and smaller regions, with decreasing dimensions most of them tend to become Euclidean. However, our doubts would be well-founded only if we were compelled to stick to the concept of point (in the sense of „ordinary" geometry) as basic element. But the whole situation turns to our favour if we are free to choose — following the great geometers of the 19-th century, beginning with Plücker — another primitive element.

3.01. There is still another reason for the abandoning of the primitive notion of four-dimensional point. It cannot be denied that this notion, the demonstrative act „here-now", is an essentially macro-

[2] See, e. g., Weyssenhoff (1928).

scopic concept and its transplantation to microphysics is a hypothesis which can be justified only by success. The leading idea of the present investigation is just the conviction that this success has proved to be lacking.

3.1. The question arises now what sort of geometric, or rather physico-geometric, entities have to take over the rôle of the discarded points in the foundations of physics. Undoubtedly, we cannot throw away the whole of contemporary physics and begin the whole structure anew, but we must take over from modern physics as much as we can, leaving away only those notions whose transfer from macrophysics to microphysics seems impermissible. In particular, we must thoroughly consider the leading ideas of modern natural philosophy and find out their most characteristic feature. As such we regard here the fundamental part played by the notion of wave and adopt therefore the wave (in a possibly simple form) as the primitive element out of which the whole edifice of physics has to be constructed. It will appear presently that in its most simple form this wave has to be understood as wave front (wave surface, surface of equal phase of a monochromatic wave, ...), which becomes a point in the particular case of a spherical wave front of vanishing radius.

3.2. The characteristic property of waves is that they comply with Huygens' principle. Now, this principle may be expressed without making any use of metric whatever and most probably also without the concept of point, but certainly not without that of contact between wave fronts. We take, therefore, this contact as characteristic relation between our wave fronts and we shall consider as having an objective meaning in microphysics only what is invariant under the group of wave-front transformations, i. e. transformations carrying over spherical wave fronts into spherical wave fronts without destroying their mutual contacts. This is in very few words our programme and we must now work it out in greater detail.

3.3. After having decided to choose the wave as the primitive notion of physical geometry, we must try to take it (at first) in its most simple form, at any rate not as a progressive wave, consisting of an infinity of wave fronts, but just as one wave front. As these wave fronts, however, have to play the rôle of elements out of which afterwards progressive waves will have to be formed, we will take them at once as *directed wave fronts* (whose one side is considered as positive, the other as negative, the direction of the front being from the negative side to the positive one). On a drawing we can indicate the direction of a wave front by an arrow of any direction as long as it is not tangent to the surface which represents the wave front

on that particular drawing. (Henceforth we shall assign positive values of the radii to divergent spherical wave fronts and negative to convergent ones).

3.4. On the other hand, the simplest waves with which we have to do in physics are plane waves and spherical waves, and it is clear that plane waves can never be wholly realized in nature as their fronts had then to extend to infinity. They may be considered, however, as limiting cases of spherical waves (emanating from infinitely remote sources) and as such they must also be included in our wave fronts. Moreover, our wave fronts must also include points as special cases of spherical wave fronts with vanishing radii. Thus finally our wave front manifold will consist of directed spheres, directed planes, and points, the difference between these three geometrical entities existing only with respect to the particular Euclidean mapping used. Objectively all wave fronts must be considered as equivalent as any of them can be carried over into any other by a suitably chosen wave-front transformation.

3.5. It is obvious that the wave fronts being directed we must attribute an objective meaning only to their *concordant contact* [3]. For instance, two spherical divergent wave fronts which are tangent in the ordinary sense (in a given Euclidean mapping) will be considered as being in concordant contact only when one of them lies within the other. This formulation of the question agrees evidently with the physical significance of the wave fronts. The contact of two wave fronts progressing in opposite directions is something accidental, existing only in the given Euclidean mapping, it does not constitute any objective relation between these wave fronts, and may be removed by a wave-front transformation. As an extreme case two overlapping spheres may be mentioned, one directed outwards, the other inwards; these spheres are not „properly tangent" (not in concordant contact, not tangent in the sense of Lie's geometry of spheres), by a suitably chosen wave-front transformation they may be carried over into two arbitrarily chosen spheres, planes, or points.

3.6. Of course, all the above considerations have only a preliminary character, they may help to form a mental picture of the situation but strictly speaking the concept of a wave front — similarly to all primitive notions in natural science — acquires only gradually its full meaning as the theory of which it is a part progresses. In our case in particular one of the essential points which did not appear till now

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[3] On continuity grounds a point must be considered as tangent to a directed sphere or plane if it lies on it.

clearly enough is that the spheres, planes, and points we are speaking of may be only considered as illustrations of the wave fronts (in a particular mapping of the wave-front manifold upon an auxiliary Euclidean space), but we must always keep in mind that they differ substantially from the wave fronts, as they consist of points, whereas the wave fronts, as primitive notions, have only a meaning as a whole, they are indivisible entities and do not consist of anything else. This property they share with the de Broglie waves. Our task is not to construct the wave fronts out of points, but to define points — or rather surface elements — in terms of the wave fronts.

§ 4. Lie's geometry of spheres

4.0. It happened, as it often does in history, that the mathematical formalism needed for the handling of a new physical theory has been created by mathematicians long before it could find immediate application. Indeed, mathematically our wave fronts are nothing else as the so called Lie spheres which play the rôle of primitive elements in an interesting kind of geometry created by Sophus Lie (1872) in the 19th century. As Lie's geometry of spheres — or, in Felix Klein's terminology, the higher geometry of spheres — is little known among physicists, we shall give here some information about it; this we shall do in a somewhat modified form adapted to the needs of physicists. More details can be found in Felix Klein's *Vorlesungen über höhere Geometrie* (1926) or in Wilhelm Blaschke's *Vorlesungen über Differentialgeometrie* Vol. III. *Differentialgeometrie der Kreise und Kugeln*, edited by Gerhard Thomsen (1929)[4]

4.1. The reader will remark that what follows now is to a great extent only a recapitulation in a little more precise form of what has been already said about wave fronts and their interrelations. Let us consider a three-dimensional Euclidean space, which we shall call henceforth the *auxiliary Euclidean space*, and in it the ensemble of all the „Lie spheres", that is, all the points, directed spheres, and directed planes. As every directed sphere may be characterized by the three coordinates of its center and the value of its radius (taken positive or negative according to whether the sphere is directed outwards or inwards), this ensemble forms a four-dimensional manifold. One calls region of this manifold every ensemble of Lie spheres which can be brought into one-to-one continuous correspondence with a region of points in a four-dimensional space.

[4] The author is not aware of any other text-book treating this subject a little more in detail.

4.2. The Lie spheres play the rôle of the elements of Lie's geometry and the concordant contact of these elements — a point being in contact with a sphere or plane when it lies respectively on this sphere or plane — the rôle of the unique, fundamental relation between them. On these two concepts one could surely base an axiomatic formulation of the whole higher geometry of spheres in a similar manner as, for instance, projective geometry has been evolved from the primitive notions of points, planes, and incidence; this — so far as the author is aware — has not yet been done, which must be considered rather as a pity from the physicists point of view.

4.3. Alternatively one can, however, define the same geometry in the sense of Klein's *Erlanger Programm* by its fundamental group, the rôle of this group being played by all the transformations which carry the Lie spheres one into another leaving undisturbed all their concordant contacts. The „Lie spheres" themselves can at the same time be also defined analytically as ordered sets of six hexaspherical coordinates (defined in § 6 below). There it will also be shown that the fundamental group of Lie's geometry of spheres depends on 15 independent parameters.

4.4. It is interesting that already Lie, the founder of the higher geometry of spheres, found it indicated to replace, on continuity grounds, ordinary spheres by directed spheres. For instance[5] if there are given two regions of the sphere manifold standing in a one-to-one continuous correspondence and we try to extend this correspondence to larger regions including the given ones, then it appears that this can be done only if we are concerned with directed spheres. With ordinary spheres the extended correspondence would in general be bi-univoque. Moreover, to secure continuity in the whole manifold of Lie spheres one must add a „point at infinity", corresponding in a certain sense to the plane at infinity of projective geometry (see § 6.7).

4.5. We have seen already that physical considerations also lead to the adoption of directed wave fronts. This may be vividly illustrated on the following example. Let us imagine a contracting spherical wave front whose radius becomes smaller and smaller till it vanishes altogether. If r could only assume non-negative values this would be a limiting process. But as it is, with directed spheres and negative as well as positive values of r, the process may proceed further. The values of r which initially have been negative pass through zero and become positive. The propagation of the convergent wave continues as a divergent wave. It is to be noticed that the particular Lie sphere

[5] Cf. Blaschke (1929), § 2.

which appeared thereby as a point does it only in the chosen Euclidean mapping of the wave-front manifold. In another mapping another sphere may play the same rôle; *every* sphere may be made to appear as a point in a suitably chosen mapping.

4.51. At the same time the difference between divergent and convergent waves -- and thus also between retarded and advanced potentials — loses its objective meaning in microphysics. It becomes only a matter of choice of the Euclidean mapping whether a given spherical wave front plays the part of a divergent or convergent wave.

4.52. The situation may be compared to a well-known example in special relativity theory: the relativistically uniformly accelerated motion of a particle. Departing from the common practice of physicists Born (1909), who first described it, called it hyperbolic motion as its world line is a hyperbola [6]. But in ordinary three-dimensional space it is a rectilinear motion with constant proper acceleration. Viewed from a fixed inertial frame of reference, the particle approches from infinity with the velocity of light (for $t=-\infty$), gradually it slows down till it comes to rest and then the motion proceeds further in a reversed direction and growing speed, till for $t=+\infty$ the velocity becomes again equal to c. At first glance it could seem that one point-event is singled out by the described motion, namely that one in which the particle reverses its direction of motion. But obviously every point of the trajectory has the same property in a suitably chosen inertial frame of reference. The analogy with the contracting and then expanding sphere is obvious.

4.6. Let us draw from the above comparison one more important conclusion. We notice that the equivalent of the Lorentz transformation of coordinates in relativity theory consists here in the change of the Euclidean mapping, rather than in the change of coordinates, and — as will become clearer in the sequel — this mapping itself depends on the geometrical configuration of the measuring apparatus.

§ 5. The pentaspherical coordinates

5.0. At first, one could fear that all calculations in the new theory based on the transformations of spheres into spheres might prove to be far more difficult than those we meet in present-day physics. It appears, however, that this is not quite the case, as the geometers of the 19th century have already found suitable coordinates, so called hexaspherical coordinates, in which Lie's transformations of spheres

[6] In the same terminology the uniformly accelerated motion of ordinary, non-relativistic, dynamics would have to be called parabolic motion.

into spheres (our wave-front transformations) become linear, and the determination of the invariants of the new group of transformations may proceed on similar lines as in relativity theory. There remain, it is true, some additive inconveniences caused by the new coordinates being homogeneous and not independent but fulfilling a quadratic condition, they may, however, be overcome without serious difficulties.

5.01. Before defining the hexaspherical coordinates, we must first of all introduce the so called pentaspherical coordinates [7]. The idea is to find such coordinates in which the equations of spheres become linear. In other words, we are going to „linearize the equation ofa sphere" though in a quite different manner as Dirac has done it with the wave equation [8]. At the same time we shall also make our coordinates homogeneous. It is well known that ordinary, unhomogeneous coordinates are the adequate tool for handling „local problems", but when transformations of whole spaces into themselves play a predominant rôle than the homogeneous coordinates become more expedient. This situation presents itself often in geometry and practically always in quantum physics.

5.1. Let us now denote ordinary rectangular coordinates in an auxiliary Euclidean space by $\dot{x}^m$ $(m=1,2,3)$ and the coordinates of the center of a sphere of radius r by x^m, then the equation of the sphere will read

$$(\dot{x}^1-x^1)^2+(\dot{x}^2-x^2)^2+(\dot{x}^3-x^3)=r^2 \tag{5,1}$$

or

$$2x^m\dot{x}^m-\dot{x}^m\dot{x}^m-(xx)=0 \tag{5,2}$$

where the sum over m is understood and we have put

$$(xx)\equiv x^m x^m-r^2. \tag{5,3}$$

In (5,2) only the second term destroys the linearity in the $\dot{x}$'s; we re-establish it simply by introducing $\dot{x}^m\dot{x}^m$ as a fourth coordinate, at the

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[7] The denominations pentaspherical and hexaspherical coordinates are not general enough, as they apply only to spheres in three dimensions. It would be better to call them respectively *lower* and *higher spherical coordinates*, as they are chiefly used (in Felix Klein's terminology) in the lower (Möbius) and higher (Lie) geometry of spheres. Then every sphere in an n-dimensional Euclidean (or pseudo-Euclidean) space would be characterized by $n+2$ lower and $n+3$ higher spherical coordinates.

[8] The procedure of Dirac could also be used in our case and the comparison of the two methods might provide another proof of the close connection between spinors and geometry of spheres.

same time making the coordinates homogeneous, which raises their number up to five. Thus, we put [9]

$$\overline{\eta}^1 : \overline{\eta}^2 \ \overline{\eta}^3 \ \overline{\eta}^5 : \overline{\eta}^6 = \dot{x}^1 : \dot{x}^2 : \dot{x}^3 : \frac{\dot{x}^m \dot{x}^m}{l} : l \tag{5,4}$$

Mathematicians have done it till now in a very similar manner writing only 1 instead of our l, a *constant with the dimensions of length* [10]. This is obviously permitted, but by so doing one fixes once for all the unit of length and hampers the insight into the physical meaning of the formulae. To retain the possibility of arbitrary changes of the units of measurements, l has been inserted in such a manner as to preserve dimensional homogeneity of the five terms on the right-hand side of (5,4).

An equivalent mode of writing the definition formulae (5,4) of the pentaspherical point-coordinates $\overline{\eta}$ is

$$\boxed{\overline{\eta}^m = \varrho \dot{x}^m, \quad \overline{\eta}^5 = \varrho \frac{\dot{x}^m \dot{x}^m}{l}, \quad \overline{\eta}^6 = \varrho l,} \tag{5,5}$$

where ϱ is an arbitrary factor subject only to the restriction of being different from zero (and being the same in all five equations (5,5)).

The five coordinates $\overline{\eta}$ are not independent as they fulfil identically the quadratic relation

$$\overline{\eta}^m \overline{\eta}^m - \overline{\eta}^5 \overline{\eta}^6 \equiv (\overline{\eta}^1)^2 + (\overline{\eta}^2)^2 + (\overline{\eta}^3)^2 - \overline{\eta}^5 \overline{\eta}^6 = 0. \tag{5,6}$$

The bars above the η's serve to remind us that we have to do with pentaspherical *point*-coordinates. These coordinates will turn out to be a special case of pentaspherical *sphere*-coordinates which we are going now to introduce.

5.2. With pentaspherical point-coordinates the equation of the sphere (5,2) reads

$$2 x^m \overline{\eta}^m - l \overline{\eta}^5 - \frac{(xx)}{l} \overline{\eta}^6 = 0, \tag{5,7}$$

an equation which may be given the form [11]

$$2((\eta \overline{\eta})) \equiv 2 \eta^m \overline{\eta}^m - (\eta^6 \overline{\eta}^5 + \eta^5 \overline{\eta}^6) = 0 \tag{5,8}$$

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[9] We omit intentionally the index 4 in order not to be compelled afterwards to change the meaning of some η's while introducing hexaspherical coordinates.

[10] We do not settle in advance the question whether l is a universal constant or whether it has different values for different particles, this being yet irrelevant in the present stage of our investigation.

[11] $((\eta \overline{\eta}))$ in (5,8) is the polar form of (5,6).

by putting

$$\eta^m = \varrho x^m, \quad \eta^5 = \varrho \frac{(xx)}{l}, \quad \eta^6 = \varrho^6 l. \tag{5,9}$$

The ratios of the five η in (5,9) define uniquely the given sphere, and the five independent variables η — as well as each of their five independent combinations — are called therefore the *pentaspherical sphere-coordinates*. In contradistinction to pentaspherical point-coordinates they are independent. Instead of (5,6) we have now

$$((\eta\eta)) = \varrho^2 r^2 \tag{5,10}$$

or

$$r = \frac{\sqrt{((\eta\eta))}}{\eta^6} l. \tag{5,10'}$$

5.3. In the above considerations η^6 has been different from zero. For $\eta^6 = 0$ we get planes instead of spheres. This may be directly seen by inserting (5,9) into our initial equation (5.2):

$$2\eta^m \dot{x}^m - \eta^6 \frac{\dot{x}^m \dot{x}^m}{l} - \eta^5 l = 0. \tag{5.11}$$

For $\eta^6 = 0$ this is an equation of a plane, η^1, η^2, η^3, and $-\frac{\eta^5 l}{2}$ playing the rôle of homogeneous plane coordinates.

5.4. It is easy to introduce a new sort of pentaspherical coordinates in such a way as to make ((...)) in (5.8) an algebraic sum of squares. It suffices to put, for instance,

$$\xi^m = \eta^m, \quad \xi^5 = \tfrac{1}{2}(\eta^6 - \eta^5), \quad \xi^6 = \tfrac{1}{2}(\eta^6 + \eta^5) \tag{5,12}$$

or equivalently

$$\eta^m = \xi^m, \quad \eta^5 = \xi^6 - \xi^5, \quad \eta^6 = \xi^6 + \xi^5. \tag{5,12'}$$

The same equations may be used for the point-coordinates $\bar{\xi}$ and $\bar{\eta}$. Then (5,10) becomes

$$((\xi\xi)) \equiv \xi^m \xi^m + (\xi^5)^2 - (\xi^6)^2 \equiv (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^5)^2 - (\xi^6)^2 = \varrho^2 r^2 \tag{5,13}$$

and instead of (5,10') we get

$$r = \frac{\sqrt{((\xi\xi))}}{\xi^5 + \xi^6} l. \tag{5,13'}$$

The pentaspherical point-coordinates $\bar{\xi}$ are subject to the equation

$$((\bar{\xi}\,\bar{\xi})) = 0 \tag{5,14}$$

which takes over the rôle of equation (5,6).

Finally, equations (5,9) defining the relations between pentaspherical sphere-coordinates and rectangular coordinates of the centres and the radii of the spheres become

$$\xi^m = \varrho x^m, \quad \xi^5 = \varrho \frac{l^2 - (xx)}{2l}, \quad \xi^6 = \varrho \frac{l^2 + (xx)}{2l}. \tag{5,15}$$

Both sorts of pentaspherical coordinates defined above, the η- and the ξ-coordinates, are called by different authors simply pentaspherical coordinates. We shall use them alternatively as sometimes one sort sometimes the other proves to be more convenient.

5.5. Non-singular linear point-transformations carrying over spheres into spheres are called Möbius transformations. Liouville has shown more than one hundred years ago that all conformal point-transformations of a Euclidean space of more than two dimensions on to itself carry over spheres into spheres, and are thus Möbius, or conformal, transformations. In three dimensions they can be represented by non-singular linear transformations of the pentaspherical point-coordinates leaving invariant the quadratic condition (5,6) for $\overline{\eta}$-coordinates or (5,14) for $\overline{\xi}$-coordinates. They form therefore a 10-parameter group. More generally in a Euclidean (or pseudo-Euclidean) space of n dimensions there are $n+2$ „lower spherical point-coordinates" and it may be easily seen that the number of essential parameters of the corresponding Möbius group is [12]

$$[(n+2)^2 - 1] - [\tfrac{1}{2}(n+2)(n+3) - 1] = \tfrac{1}{2}(n+1)(n+2) \tag{5,16}$$

In particular the group of conformal point-transformations in four-dimensional Minkowski space has 15 essential parameters (see § 8.0 below).

5.6. The Möbius transformations as point-to-point transformations cannot yet play the part of the wave-front transformations we are after. They are, it is true, linear in pentaspherical point-coordinates and transform spheres into spheres and tangent spheres into tangent spheres, but in addition they leave all angles invariant and, what is worse, they make no distinction between divergent and convergent spheres and never change points into spheres or *vice versa*. All that we know already about wave-front transformations suffices to foresee

[12] The transformation matrix has $(n+2)^2$ elements, the number of their independent ratios is one less. The number of coefficients of a quadratic form in $n+2$ variables is $(n+2)(n+3)/2$, but as not this form itself but only its vanishing is invariant this number must be diminished by one.

that to define these transformation in a simple manner we must use sphere coordinates rather than point coordinates. But even then the pentaspherical sphere-coordinates will not do and we are obliged to go over to hexaspherical coordinates.

§ 6. The hexaspherical coordinates

6.0. Without doubt the pentaspherical sphere-coordinates are not yet the coordinates best adapted to our purposes; they may be characterized as coordinates of undirected spheres as they do not discriminate between two spheres differing only in the sign of r and we are in search of coordinates of directed spheres.

6.1. We introduce therefore (on the fourth, yet unoccupied place) a sixth coordinate proportional to r itself, namely

$$\eta^4 = \varrho r. \tag{6,1}$$

Thus, from (5,9), we get

$$\boxed{\eta^m = \varrho x^m \quad \eta^4 = \varrho r, \quad \eta^5 = \varrho \frac{(xx)}{l}, \quad \eta^6 = \varrho l,} \tag{6,2}$$

where (xx) is given by (5,3). The six hexaspherical coordinates η^A $(A = 1, \ldots, 6)$ are not independent as they fulfil identically the quadratic relation

$$\langle \eta\eta \rangle \equiv (\eta\eta) - \eta^5\eta^6 \equiv (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 - (\eta^4)^2 - \eta^5\eta^6 = 0. \tag{6,3}$$

They are thus twice superabundant coordinates of directed spheres in a three-dimensional Euclidean space; there are ∞^4 spheres in this space but each sphere is characterized by six coordinates as these coordinates (i) are related by the quadratic relation (6,3) and (ii) are homogeneous, so that only their ratios matter.

Equations (6,2) solved for x^1, x^2, x^3, and r are

$$x^m = \frac{\eta^m}{\eta^6} l, \quad r = \frac{\eta^4}{\eta^6} l. \tag{6,4}$$

The third equation (6,2) gives nothing new, due to (6,3).

6.2. Hexaspherical ξ-coordinates may be defined similarly to the pentaspherical ξ-coordinates by their connection with the η-coordinates. An obvious generalization of (5,12) yields

$$\xi^m = \eta^m, \quad \xi^4 = \eta^4, \quad \xi^5 = \tfrac{1}{2}(\eta^6 - \eta^5), \quad \xi^6 = \tfrac{1}{2}(\eta^6 + \eta^5) \tag{6,5}$$

or equivalently

$$\eta^m = \xi^m \quad \eta^4 = \xi^4, \quad \eta^5 = \xi^6 - \xi^5, \quad \eta^6 = \xi^6 + \xi^5. \tag{6,5'}$$

Instead of (6,2) we get

$$\xi^m = \varrho x^m, \quad \xi^4 = \varrho r, \quad \xi^5 = \varrho \frac{l^2 - (xx)}{2l}, \quad \xi^6 = \varrho \frac{l^2 + (xx)}{2l}. \tag{6,6}$$

The hexaspherical ξ-coordinates fulfil identically the quadratic relation

$$\langle \xi\xi \rangle \equiv (\xi\xi) + (\xi^5)^2 - (\xi^6)^2 \equiv (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2 + (\xi^5)^2 - (\xi^6)^2 = 0. \tag{6,7}$$

Instead of (6,4) we have now

$$x^m = \frac{\xi^m}{\xi^5 + \xi^6} l, \quad r = \frac{\xi^4}{\xi^5 + \xi^6} l. \tag{6,8}$$

There are plainly no barred hexaspherical coordinates as (in three-dimensional space) there are no hexaspherical *point*-coordinates but only hexaspherical *sphere*-coordinates.

6.3. Formulae (6,2) or (6,6) may be considered as defining a mapping of the wave-front manifold on the points, directed spheres, and directed planes of an auxiliary Euclidean x^1, x^2, x^3 space (supplemented by an improper point — see § 6.7 below). This mapping does not treat, however, all the Lie spheres (points, directed spheres, and directed planes) on equal footing but singles out the planes, as every set of finite values of x^1, x^2, x^3, and r (together with an arbitrary value of $\varrho \neq 0$) represents a directed sphere or a point, whereas the planes appear only as limiting cases of spheres for $r \to \infty$.

6.4. We may, however, reverse, so to say, the situation by expressing the same mode of Euclidean mapping in another way, in which points instead of planes play an exceptional rôle. Instead of characterizing a sphere by the coordinates $x^m (m = 1,2,3)$ of its centre and its radius r, we shall do it by means of its curvature $\varkappa = 1/r$ and three parameters defining its tangent plane in A (see Fig. 1). As we are concerned with directed spheres and concordant contacts, the tangent plane π must be also a (suitably) directed plane. We can write, for instance, the equation of this plane in the form

$$n_m \dot{x}^m + d = 0, \tag{6,9}$$

where the unit vector n_m is the (appropriately chosen) normal to π, a unit vector pointing from its negative to its positive side. Then d will be the distance of the origin of coordinates O from π — or, what amounts, to the same, from the sphere — taken positive when O lies on the posi-

tive side of π (as in Fig. 1, where $d>0$ and also $r>0$). From Fig. 1 we see that the coordinates x^m of C may be written in the form

$$x^m = -(r+d)n_m \tag{6,10}$$

and hence, availing oneself of the arbitrariness of ϱ, we replace ϱ by $\varrho\varkappa l$ and get from (6,2)

$$\begin{aligned} \eta^m &= -\varrho l(1+\varkappa d)n_m, \\ \eta^4 &= \varrho l, \\ \eta^5 &= \varrho d(2+\varkappa d), \\ \eta^6 &= \varrho l^2\varkappa. \end{aligned} \tag{6,11}$$

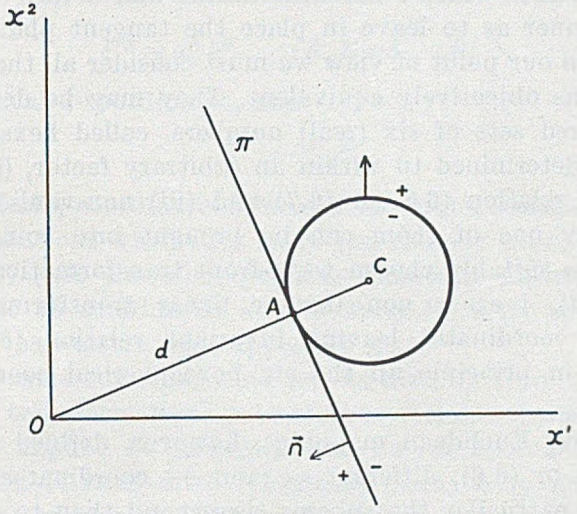

Fig. 1

We may also write the equation of the plane π in the following form

$$u_m x^m + 1 = 0, \tag{6,12}$$

whence, by comparison with (6,9),

$$n_m = u_m d, \tag{6,13}$$

and u_m are the ordinary (non-homogeneous) plane coordinates of π. (6,11) becomes

$$\begin{aligned} \eta^m &= -\varrho l d(1+\varkappa d)u_m, \\ \eta^4 &= \varrho l, \\ \eta^5 &= \varrho d(2+\varkappa d), \\ \eta^6 &= \varrho l^2\varkappa, \end{aligned} \tag{6,14}$$

where, from (6,13),

$$d=\frac{1}{\pm\sqrt{u^m u^m}}, \tag{6,15}$$

and the positive or negative sign has to be taken according as O lies respectively on the positive or negative side of the sphere.

6.5. In particular, the hexaspherical coordinates of a plane (in the Euclidean mapping considered) will have the form

$$\eta^m=-\varrho l n_m=\varrho l, \quad \eta^4=\varrho l, \quad \eta^5=\varrho 2d, \quad \eta^6=0. \tag{6,16}$$

The same expression can be obtained directly from (6,2) by a limiting process in which r becomes infinite and O flies off to infinity in such a manner as to leave in place the tangent plane π.

6.6. From our point of view we must consider all the wave fronts (Lie spheres) as objectively equivalent. They may be defined analytically as ordered sets of six (real) numbers called hexaspherical coordinates, (i) determined to within an arbitrary factor, (ii) subject to the quadratic relation (6,3) or (6,7) and (iii) non-vanishing simultaneously. Every one of them can be brought into coincidence with any other by a suitably chosen wave-front transformation (Lie sphere transformation), i. e., a non-singular linear transformation of the hexaspherical coordinates leaving invariant relation (6,3) or (6,7). Consequently in principle all the six hexaspherical coordinates play the same rôle.

6.7. In our Euclidean mappings, however, defined by means of formulae (6,2) or (6,6), different η- (and ξ-) coordinates get rôles of their own. In particular the spheres correspond then to sets of hexaspherical coordinate with

$$\eta^4=\xi^4\neq 0 \quad \text{and} \quad \eta^6=\xi^5+\xi^6\neq 0, \tag{6,17}$$

the points to

$$\eta^4=\xi^4=0 \quad \text{and} \quad \eta^6=\xi^5+\xi^6\neq 0, \tag{6,18}$$

and the planes to

$$\eta^4=\xi^4\neq 0 \quad \text{and} \quad \eta^6=\xi^5+\xi^6=0. \tag{6,19}$$

There remains still the alternative

$$\eta^4=\xi^4=0 \quad \text{and} \quad \eta^6=\xi^5+\xi^6=0 \tag{6,20}$$

which corresponds in our case to one wave front (Lie sphere) only, as (6,3) and (6,20) involve

$$(\eta^1)^2+(\eta^2)^2+(\eta^3)^2=0 \tag{6,21}$$

and hence $\eta^1=\eta^2=\eta^3=0$. This improper wave-front of the given Euclidean mapping:

$$(0,0,0,0,1,0) \tag{6,22}$$

may be called with equal right improper point or improper plane.

With another signature of the quadratic forms in (6,3) and (6,7), or even with these forms unaltered if one takes into consideration not only real but also imaginary elements, there exists a two-dimensional infinity of improper elements.

6.8. Now, to end this section, we have only to give the analytic expressions for the wave-front transformations (Lie sphere transformations) defined in § 6.6. We shall write them in tensor notation, as follows

$$\xi'^A = a^A_B \xi^B \qquad (A,B=1,..,6) \tag{6,23}$$

for hexaspherical ξ-coordinates, and

$$\eta' = \beta^A_B \eta^B \tag{6,24}$$

for hexaspherical η-coordinates. These linear transformations are wave-front transformations when they leave respectively invariant equations (6,3) and (6,7) (not the corresponding quadratic forms!). To do it, each set of the 36 coefficients α or β must fulfil 21 conditions with one arbitrary parameter, see Appendix *A*.

6.9. The same reasoning that led us directly from the definition of the Möbius transformations in pentaspherical coordinates to formula (5,16) can be applied to hexaspherical coordinates and wave-front transformations. It suffices to replace in (5,16) n by $n+1$. Thus we get the number of essential parameters of the group of Lie's sphere transformations in an n-dimensional Euclidean (or pseudo-Euclidean) space, viz.

$$\frac{(n+2)(n+3)}{2}, \tag{6,25}$$

e. g., 15 for $n=3$.

§ 7. The relation between Euclidean geometry and wave-front transformations

7.0. We have now to find out in what sense Euclidean geometry ceases to be valid *towards the inside*, i. e. in the interior of our sphere S_R of Section 2. On this sphere and outside of it lies our macroscopic standpoint, the standpoint of our measuring instruments. There, we can be sure of the prevailing of Euclidean geometry (in the ordinary sense of physical or „school" geometry). It does not follow, however, that Euclidean geometry must be also valid in the interior of S_R.

On the contrary, we shall show that a wave-front transformation which is practically indistinguishable from a Euclidean motion outside ot S_R does not, in general, coincide with it even approximately within S_R. This will in no sense exhaust what might be said on the importance of wave-front transformations in physics but, at any rate, it may serve as a first introduction.

7.1. At first we shall prove that when R is very large in comparison to l every wave-front transformation carrying approximately points outside of S_R into points outside of it is for these points approximately an equiform Euclidean transformation, i. e., a combination of a Euclidean displacement with a change of scale (a dilatation) [13]. One could be tempted to say shortly that Euclidean geometry may be used with sufficient approximation in all regions of the wave-front manifold in which l may be neglected, but this would make no sense as (i) l does not play any rôle either in „wave-front geometry" or in Euclidean geometry, it appears only in the relation between them defined by a „Euclidean mapping", (ii) the same is true of „points": by itself a wave front is never a point and it can only appear as such in a given Euclidean mapping.

7.2. A more precise formulation of the above theorem would, therefore, read somewhat as follows. If there are given two Euclidean mappings connecting respectively the ordinary rectangular coordinates and the radii x^m, r and $\dot{x}^m, r'$ of directed spheres in two auxiliary Euclidean spaces E_3 and E_3' with two systems of hexaspherical coordinates η^A and $\dot{\eta}^A$ in the wave-front manifold, and a wave-front transformation

$$\dot{\eta}^A = \beta^A_B \eta^B, \tag{7,1}$$

then $\dot{x}^m$ and r' are functions of x^n and r

$$\dot{x}^m = \dot{x}^m(x^n, r), \qquad r' = r'(x^n, r). \tag{7,2}$$

In general these functions are not linear, our theorem states, however, that for

$$r = 0, \quad x^n x^n \geqslant R^2 \gg l^2, \quad x^n x^n \text{ of the order of } R^2 \tag{7,3}$$

$\dot{x}^m$ in (7,2) will be approximately given by linear functions of x^n:

$$\dot{x}^m = b^m_n x^n + a^m, \tag{7,4}$$

---

[13] The group of all Euclidean displacements and dilatations was called bei Felix Klein in his famous *Erlanger Programm* the main group *(Hauptgruppe)*. In his later works he calls it the equiformal group (*Gruppe der äquiformen Transformationen*, cf., e. g., Klein 1910), a denomination introduced by Heffter and Koehler. We prefer to say equiform Euclidean as we shall have also to do with equiform Lorentz groups, consisting of Lorentz transformations and dilatations.

where the matrix of the coefficients b_n^m is proportional to an orthogonal matrix.

7.3. We begin the proof by inserting in (6,24) the expressions (6,2) for η^A and similar expressions with primes for $\dot{\eta}^A$:

$$\varrho' \dot{x}^\mu = \varrho \left[\beta_\nu^\mu x^\nu + \beta_5^\mu \frac{(xx)}{l} + \beta_6^\mu l \right], \tag{7,5}$$

$$\varrho' \frac{(\dot{x}\dot{x})}{l} l = \varrho \left[\beta_\nu^5 x^\nu + \beta_5^5 \frac{(xx)}{l} + \beta_6^5 l \right], \tag{7,6}$$

$$\varrho' l = \varrho \left[\beta_\nu^6 x^\nu + \beta_5^6 \frac{(xx)}{l} + \beta_6^6 l \right]. \tag{7,7}$$

Here $\mu, \nu = 1, \ldots, 4$ and $x^4 = r$. Dividing (7,6) by (7,7) and then both numerator and denominator on the right-hand side by $(xx)/l$, we obtain

$$\frac{l^2}{(\dot{x}\dot{x})} = \frac{\beta_5^6 + \beta_\nu^6 \dfrac{x^\nu l}{(xx)} + \beta_6^6 \dfrac{l^2}{(xx)}}{\beta_5^5 + \beta_\nu^5 \dfrac{x^\nu l}{(xx)} + \beta_6^5 \dfrac{l^2}{(xx)}}, \tag{7,8}$$

and similarly from (7,5) and (7,7)

$$\frac{\dot{x}^\mu l}{(\dot{x}\dot{x})} = \frac{\beta_5^\mu + \beta_\nu^\mu \dfrac{x^\nu l}{(xx)} + \beta_6^\mu \dfrac{l^2}{(xx)}}{\beta_5^5 + \beta_\nu^5 \dfrac{x^\nu l}{(xx)} + \beta_6^5 \dfrac{l^2}{(xx)}}. \tag{7,9}$$

Outside of S for $r = 0$ (or sufficiently small r)

$$(xx) \geqslant R^2 \gg l^2 \tag{7 9'}$$

and $l^2/(xx)$ has to be considered as infinitely small of the second order, whereas $x^\nu l/(xx)$ are infinitely small of the fist order at least. We assume that in general all β_B^A are of the same order of magnitude, though some of them may vanisn or become infinitely small in comparison to other β's.

In order that the left-hand side of (7,8) should be also infinitely small of the second order, we must have

$$\beta_6^6 \neq 0, \quad \beta_\nu^6 = O(1), \quad \beta_5^6 = O(2). \tag{7,10}$$

The assumption concerning the order of magnitude of the β's may be now stated more precisely: β_6^6 does not vanish and all other β's are of the same order of magnitude as β_6^6 or smaller. $\beta = O(n)$ signifies that β/β_6^6 has to be infinitely small of the n-th order.

A similar reasoning as applied to (7,8) yields with (7,9), whose left-hand member has to be infinitely small of the first order,

$$\beta_5^\mu = O(1). \tag{7,11}$$

7.4. At first, let us assume

$$\beta_\nu^6 = \beta_5^6 = \beta_5^\mu = 0. \tag{7,12}$$

The β_B^A matrix has then the form

$$\left|\begin{array}{c|c|c} & 0 & \\ & 0 & \\ \beta_\nu^\mu & 0 & \beta_6^\mu \\ & 0 & \\ \hline \beta_\mu^5 & \beta_5^5 & \beta_6^5 \\ \hline 0\ 0\ 0\ 0 & 0 & \beta_6^6 \end{array}\right| \tag{7,13}$$

In passing, $\beta_5^\mu = 0$ follows from $\beta_\nu^6 = \beta_5^6 = 0$ with $\beta_6^6 \neq 0$ due to (A.7) for $(A,B) = (\mu, 6)$, but at the moment at least this is of no importance for us.

From (7,13) we see that

$$\det \beta_B^A = \beta_5^5 \beta_6^6 \det \beta_\nu^\mu \tag{7,14}$$

7.5. If we put now

$$\frac{\beta_B^A}{\beta_6^6} = b_B^A \tag{7,15}$$

and divide (7,9) by (7,8), we obtain

$$\dot{x}^\mu = b_\nu^\mu x^\nu + b_6^\mu l \tag{7,16}$$

or

$$\dot{x}^m = b_n^m x^n + b_4^m r + b_6^m l \tag{7,17}$$

and an equation for r which does not interest us for the moment.

If we now restrict ourselves to point-to-point transformations, we can leave out the term with r in (7,17) and in conditions (A.7) and (A.8) all the β-coefficients which vanish according to the lemma of Appendix B.

7.6. As shown in Appendix A, we can assume without loss of generality that

$$|\det \beta_B^A| = 1. \tag{7,18}$$

Then the coefficients β will fulfil the conditions (A.7) and (A.8) with $C = 1$. Writing these conditions in full and crossing out all the terms

containing any coefficient enumerated in (7,12), we can find all the restrictions imposed on the β's in consequence of (7,12). Thus we find among others

$$\beta_s^m \beta_s^n = \delta^{mn}, \tag{7,19}$$

$$\beta_5^5 \beta_6^6 = 1. \tag{7,20}$$

The first of these relations shows that the nine β_n^m form, due to (7,18), an orthogonal matrix, hence its determinant is equal to ± 1. This is confirmed by (7,20) together with (7,14) and (7,18). *There are no restrictions for* β_6^6 *and* β_6^m. All that we can say therefore of the coefficients $b_n^m = \beta_n^m/\beta_6^6$ is that they are proportional to the β_n^m which form an orthogonal matrix (so long as the $\dot{x}^m$'s do not increase so much as to invalidate the initial assumptions of our problem), and of the coefficients $b_6^m = \beta_6^m/\beta_6^6$ that they are quite arbitrary (with the same proviso as for b_n^m) as β_6^6 is quite arbitrary. We can put therefore

$$b_6^m l = a^m \tag{7,21}$$

and the three coefficients a^m will be also quite arbitrary (with the same proviso). Finally, as r has already been put equal to zero, (7,17) goes over into (7,4) with the same significance of the coefficients.

7.7. We have yet to free ourselves from the too restrictive assumption (7,12), which is sufficient but not necessary for the validity of conditions (7,10) and (7,11). Instead of (7,12), we write therefore

$$\beta_\nu^6 = \frac{l}{R}\gamma_\nu^6, \quad \beta_5^6 = \frac{l^2}{R^2}\gamma_5^6, \quad \beta_5^\mu = \frac{l}{R}\gamma_5^\mu \tag{7,22}$$

and consider the coefficients γ as of the same (or a smaller) order of magnitude as β_6^6. Thus we could obtain more general formula than (7,4), but the difference between them and the old formulae would disappear with $1/R$ tending to zero. We do not write them down here, as they stand in near connection with a general formula which will be obtained in Part II.

8. Conclusion

8.0. Many of the foregoing formulae invite us, so to say. to put

$$r = ct, \tag{8,1}$$

where c is the velocity of light in vacuo, and thus to pass to four-dimensional space and to „progressive wave fronts". This has been often done before by mathematicians, for instance in both textbooks

mentioned in § 4.0 and also by Cunningham (1910) and Bateman (1910) when they demonstrated for the first time that Maxwell's equations are not only invariant under the 10-parameter Lorentz group but also under the wider 15-parameter Möbius group of conformal point-transformations in four-dimensional space. This group is known to be isomorphic with Lie's group of sphere transformations in three-dimensional space, i. e. with our group of wave-front transformations. Equation (8,1) stands just in connection with the so-called isotropic projection which establishes a one-to-one correspondence between the points of a four-dimensional space and the directed spheres of a three-dimensional space.

But from the point of view of the physicist it is not so sure that by considering c in (8,1) as the velocity of light we do not restrict our theory to photons (and the electromagnetic field). It is hoped that the question of the relation of the present theory with four-dimensional (and five-dimensional) space will be more thoroughly investigated in the forthcoming Part III of this intended series of papers.

8.1. To conclude the present Part I it may be claimed that in it the elementary length has been built in in principle into the foundations of physics. The working out of the details will undoubtedly be possible only conjointly with the building in of the two remaining fundamental constants of microphysics, c and h. It is true that the physical significance of these constants seems to be already known, but their building in must surely undergo essential modifications if it has to be done on the background of the wave-front manifold. In this connection it may be noticed that already the first step of our theory has introduced some features of the relation between microphysics and macrophysics which are generally considered to-day as typical quantum effects. It is hoped that this aspect of the question will be dealt with in greater detail in the forthcoming Part II.

Appendix A

Let us first consider equation (6,23) of the text with given values of the 36 coefficients a as representing a *coordinate transformation*.

$$\xi'^A = a^A_B \xi^B. \tag{A.1}$$

As the ξ's on both sides of this equation are homogeneous coordinates, we can multiply all the a^A_B by an arbitrary non-vanishing factor, getting thus anoter coordinate transformation representing the same *wave-front transformation*. No loss in generality arises by fixing the arbitrary factor in the a's, e. g. by postulating that the absolute

value of the transformation determinant of (A.1), which must be obviously different from zero, be equal to unity [14]:

$$\left| \text{let } \alpha_B^A \right| = 1 \tag{A.2}$$

Then, the coordinate transformation becomes a (six-dimensional) pseudo-orthogonal transformation (with 4 plus signs a 2 minus signs). Indeed. instead of immediately postulating (A.2), we can start with an arbitrary wave-front coordinate-transformation. By its very definition it leaves undisturbed equation (6,7):

$$\langle \xi \xi \rangle \equiv (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2 + (\xi^5)^2 - (\xi^6)^2 \tag{A.3}$$

and therefore, being a non-singular linear homogeneous transformation, it carries over $\langle \xi \xi \rangle$ into

$$\langle \xi' \xi' \rangle = \lambda \langle \xi \xi \rangle, \tag{A.4}$$

where λ is a non-vanishing constant factor (depending on the transformation in question).

It is easily seen that in our case λ is always positive. This follows from the fact that $\langle \xi \xi \rangle$ is an algebraic sum of 4 squares with plus signs and 2 with minus signs. Just so the quadratic form in ξ^A obtained from $\langle \xi' \xi' \rangle$ by expressing the ξ''s in terms of the ξ's by (A.1) will consist of 6 linear combinations of the ξ's, 4 of them being preceded by plus signs and 2 by minus signs. It follows by the well-known law of inertia of real quadratic forms that λ must be positive [15]. If we divide now all the α-coefficients by $\pm\sqrt{\lambda}$ we get a new coordinate transformation for which $\langle \xi \xi \rangle = \text{inv.}$ and hence (A.2) is satisfied. The 15 relations fulfilled then by the α's are well-known and we shall not repeat them here. They become ordinary relations of orthogonality by considering $i\xi^4$ and $i\xi^6$ as fourth and sixth coordinates respectively.

Without fixing the arbitrary factor in the α-coefficients, we can characterize the wave-front transformation matrix α for hexaspherical ξ-coordinates by stating that it is proportional to a pseudo-orthogonal matrix.

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[14] As we are considering only real transformations, the sign of this determinant characterizes the transformation. It cannot be changed by multiplying all the coefficients α by an appropriate factor, say C, as the determinant gets thereby the factor C^6 which is always positive.

[15] The argument breaks down only in the case when there are as many plus signs as minus signs, as for instance for Plücker's line coordinates in three-dimensional space.

All the above considerations apply *mutatis mutandis* to the hexaspherical η-coordinates, whose transformation formulae we write in the form

$$\dot{\eta}^{\mu} = \beta^{A}_{B}\,\eta^{B}, \tag{A.5}$$

but the conditions to be fulfilled by the β's are a little more complicated. As the η-coordinates are connected with the ξ-coordinates by formulae (6,5) of the text, we could use these formulae to compute the β's in terms of the α's and get the required relations from the pseudo-orthogonality relations of the α's. But it will be still easier to take them directly from Weyl's *Analyse des Raumproblems* (1923, Appendix 1). Weyl deals with hexaspherical coordinates (as „lower spherical coordinates") in a four-dimensional Minkowski world, but analytically their transformations are identical with the wave-front transformations of our hexaspherical coordinates. The connection between Weyl's $u_0,\ldots,u_5$ coordinates and our $\eta^1,\ldots,\eta^6$ coordinates are

$$u_{0,1,2,3} = \eta_{1,2,3,4}, \quad u_5 = \sqrt{2}\,\eta^6, \quad u_6 = -\sqrt{2}\,\eta^5. \tag{A.6}$$

Taking this into consideration, we get for the „modified conditions of orthogonality"

$$\beta^{A}_{n}\beta^{B}_{n} - \beta^{A}_{4}\beta^{B}_{4} - 2(\beta^{A}_{5}\beta^{B}_{6} + \beta^{A}_{6}\beta^{B}_{5}) = \dot{\delta}^{AB}\,C^2, \tag{A.7}$$

$$\beta^{n}_{A}\beta^{n}_{B} - \beta^{4}_{A}\beta^{4}_{B} - \tfrac{1}{2}(\beta^{5}_{A}\beta^{6}_{B} + \beta^{6}_{A}\beta^{5}_{B}) = \dot{\delta}_{AB}\,C^2 \tag{A.8}$$

(the sum over $n=1,2,3$ is understood), where all the non-vanishing components of $\dot{\delta}^{AB}$ and $\dot{\delta}_{AB}$ are

$$\dot{\delta}^{11} = \dot{\delta}^{22} = \dot{\delta}^{33} = -\dot{\delta}^{44} = 1, \qquad \dot{\delta}^{56} = \dot{\delta}^{65} = -2, \tag{A.9}$$

$$\dot{\delta}_{11} = \dot{\delta}_{22} = \dot{\delta}_{33} = -\dot{\delta}_{44} = 1, \qquad \dot{\delta}_{56} = \dot{\delta}_{65} = -\tfrac{1}{2}. \tag{A.10}$$

C is the factor by which the coefficients β have to be divided in order to reduce the absolute value of their determinant to unity.

The correctness of the values (A.9) and (A.10) may be directly seen from (A.7) and (A.8) by inserting there $\beta^{A}_{C} = \delta^{A}_{C}$, where δ^{A}_{C} is the ordinary Kronecker symbol; β^{A}_{C} represent then the identical transformation which must evidently verify (A.7) and (A.8) with $C=1$.

Appendix B

Lemma. The coefficients β^{A}_{B} of a wave-front transformation (7,1) connected with two Euclidean spaces E_3 and E'_3 as in § 7.2, and car-

rying over points of E_3 into points of E_3' and vice versa fulfil the relations

$$\beta_n^4 = \beta_5^4 = \beta_6^4 = \beta_4^n = \beta_4^5 = \beta_4^6 = 0 \tag{B.1}$$

$$\beta_4^4 = \pm C \neq 0 \tag{B.2}$$

Proof. (7,1) for $\mu = 4$ reads

$$\dot{\varrho}\,\dot{\eta}^4 = \varrho(\beta_n^4 r_{,}^n + \beta_4^4 r_{,}^4 + \beta_5^4 r_{,}^5 + \beta_6^4 r_{,}^6). \tag{B.3}$$

In order that $\eta^4 = 0$ implies $\dot{\eta}^4 = 0$ for arbitrary values of $r_{,}^n$, η^5 and η^6, we must have

$$\beta_n^4 = \beta_5^4 = \beta_6^4 = 0. \tag{B.4}$$

It follows then from (A 7) for

$$\begin{aligned} (A, B) &= (4, 4) \quad \ldots \quad \beta_4^4 = \pm C, \\ (A, B) &= (n, 4) \quad \ldots \quad \beta_4^n = 0, \\ (A, B) &= (5, 4) \quad \ldots \quad \beta_4^5 = 0, \\ (A, B) &= (6, 4) \quad \ldots \quad \beta_4^6 = 0. \end{aligned} \tag{B.5}$$

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DISTRIBUTIONS AND STATISTICAL MOMENTS OF BOSONS AND FERMIONS WITH SOME OF THEIR APPLICATIONS *

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The present paper contains a sketch of an analytic method of solving combinatory problems concerning the distribution of particles into cells, the particles being either distinguishable or not and the different possibilities of filling the cells being possibly subject to *a priori* given restrictions. This method rests upon the use of so called Laplace's generating functions. The fundamental generating function as well as the characteristic function of a generating function have been defined and Newton's polynomial expansion applied to the generating functions. Distributions of probabilities as well as their statistical moments have been obtained. Also application of the above method to problems in cosmic radiation, the theory of contrast in photographic emulsions, and the theory of number have been given.

1. Formulation of the problem

Let us consider K cells consisting each of k elementary cells, i. e., regions which may be considered from the point of view of quantum mechanics as homogeneous and indivisible (e. g., one of the quantum levels in the case of a discrete spectrum of an operator or an infinitesimal interval in the case of a continuous spectrum). Let N particles be falling on these Kk cells. The possibilities of occupying particular cells by the particles may be restricted by *a priori* given

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exclusion rules. One particle may occupy at most one cell. We assign the same statistical weight to each cell.

Problem I. What is the probability of hitting by the N particles m out of the K cells in a given manner consistent with the given exclusion rules? For instance, each of the m cells has to be hit once (if there are no restrictions concerning the initial state) or each of them an even number of times, etc.

Problem II. What is the probability that n particles will fall on a given group of k cells?

Add to this a convention fixing the kind of statistics to be applied to the incident particles and each problem will split into three subproblems:

(a) classical (distinguishable) particles ... Maxwell-Boltzmann statistics (abbreviated M. B.)

(b) quantum (indistinguishable) particles ... Bose-Einstein statistics (abbreviated B. E.)

(c) quantum particles with Pauli's exclusion principle ... Fermi-Dirac statistics (abbreviated F. D.).

2. The solution of problem I

To avoid complicated expressions of combinatory analysis as well as loose intuitive reasonings we shall use the method of generating functions introduced by Laplace. As the expression „generating function" is used in different senses by different authors, we begin by giving a definition of this function as we shall use it together with some of its properties.

The method of the generating functions is a special, discrete, case of Mellin's transformation based on the use of the complete set of functions $1, z, z^2, z^3, \ldots$ (i. e. on the use of power series)

$$f(z) = \sum_{s=-\infty}^{+\infty} \varphi(s)\, z^s$$

$$\varphi(s) = \frac{1}{2\pi i} \oint_C f(z)\, z^{-s-1}\, dz \left(= \left[\frac{d^s f(z)}{dz^s} \right] \frac{1}{s!} \right)_{z=0}$$

where $z = x + iy$, C is a closed curve in the Gaussian plane surrounding the point $z^0 = 0$, $f(z)$ is called the generating function (abbreviated

gen. f.) and $\varphi(s)$ the determining function. The correspondence between these functions one-to-one as the set of function z^n is closed and complete.

An important property of this transformation is that for

$$f(z)=\sum_s \varphi(s)\, z^s; \quad g(z)=\sum_s \gamma(s)\, z^s$$

we have

$$f(z)\cdot g(z)=\sum_n \Big(\sum_{j=0}^{n} \varphi(n-j)\cdot\gamma(j)\Big) z^n$$

$$f(z)+g(z)=\sum_n \big(\varphi(n)+\gamma(n)\big) z^n$$

so that multiplication goes over in convolution and addition in addition. Due to these properties a great many complicated operations on determining functions can be performed on generating functions in a far simpler manner.

In our case the determining functions are given by combinatory expressions for the number of events, hence we get the required expressions by expanding gen. f. $\varphi(z)$ in a power series in z: $\sum_N C_N z^N$ and picking out that term of the series whose index is equal to the number N of incident particles. In the case of classical statistics the number of events (the thermodynamic probability) is given by $N!\,C_N$, and in the case of quantum statistics — by C_N, the latter result being the consequence of the indistinguishability of the particles.

Putting

$$\{f(z)\}_N=\frac{1}{2\pi i}\oint_C f(z)\, z^{-N-1}\, dz$$

where C is a closed curve in the Gaussian plane around the point $z_0=0$, we get for the number of events

for M. B. statistics ... $N!\,\{f(z)\}_N$

for B. E. and F. D. statistics ... $\{f(z)\}_N$.

Let us now introduce the fundamental generating function giving the distribution of the number of particles per one cell without any *a priori* exclusion rules (the determining function is then equal to one). The fundamental gen. f. equals

for M. B. statistics ... e^z

for B. E. statistics ... $(1-z)^{-1}$ where $|z|<1$.

We obtain the distributions of an arbitrary number of particles into a greater amount, say s, of cells by making use of the multiplicative property of the gen. functions. Thus we get

for M. B. statistics ... $(e^z)^s = \sum_{N=0}^{\infty} s^N \frac{z^N}{N!}$, number of events s^N,

for B. E. statistics ... $(1-z)^{-s} = \sum_{N=0}^{\infty} \binom{s+N-1}{s-1} z^N$ number of events $\binom{s+N-1}{s-1}$.

To find out those distributions which admit only of certain occupations of the cells (exclusion rules), we set in the expansion of the gen. f. all the terms with the r-th powers of z equal to zero, where r runs over all forbidden numbers of occupation, the determining function vanishes then for all values of the variable which are equal to r. Thus, for instance, in the case of the occupation of one cell by at least one particle the gen. f. is given by

$$\text{for M. B. statistics ... } e^z - 1$$

$$\text{for B. E. statistics ... } \frac{1}{1-z} - 1 = \frac{z}{1-z}.$$

The gen. f. of an occupation of m cells by at least one particle per cell in the case of M. B. statistics is

$$(e^z-1)^m = \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} \prod_{l=1}^{m} \frac{z^{n_l}}{n_l!} = \sum_{N=m}^{\infty} \left(\sum_{n_l \ldots} \prod_{l} \frac{1}{n_l!} \right) z^N \qquad \sum_l n_l = N,\ n_l > 0.$$

But on the other hand

$$\left[\frac{d^N}{dz^N} (e^z-1)^m \right]_{z=0} = \sum_l (-1)^l \binom{m}{l} (m-l)^N,$$

hence the number of the distribution in question is given by the formula

$$\sum_{\substack{n_l \ldots \\ \Sigma n_l = N}} \prod_{\substack{l=1 \\ n_l > 0}}^{m} \frac{N!}{n_l!} = \sum_l (-1)^l \binom{m}{l} (m-l)^N. \tag{1}$$

The gen. f. for particles which may occupy each cell once or not at all (Pauli's exclusion principle) is for K cells given by

$$\text{for M. B. and B. E. statistics ... } (1+z)^K = \sum_N \binom{K}{N} z^N$$

and the corresponding number of events are

for $M.$ B. statistics ... $N!\binom{K}{N}$, number of variations of K over N

for B. E. statistics ... $\binom{K}{N}$, number of combinations of K over N.

The gen. f. $(1+z)$ is the fundamental gen. f. for the F. D. distribution. It may be noticed that in the case of Pauli's exclusion principle both the M. B. and B. E. distributions have the same gen. f. but due to the indistinguishability of the particles the numbers of events differ by the factor $N!$ which plays a rôle only of thermodynamic probabilities and cancels out in the calculation of ordinary probabilities [1].

The gen. functions may also serve to investigate mixed distribution, e. g., distributions of N particles into $m+n$ cells, n cells being hit and the remaining m cells filled at random. We get

$$f(z)=e^{zm}(e^z-1)^n, \quad \varphi(N)=\sum_l (-1)^l\binom{n}{l}(m+n-l)^N \tag{2}$$

It may be shown without trouble that, similarly to the power of a sum, expressions (1) and (2) may be expanded respectively into Newton's binomial and polynomial. For instance

$$\sum_l (-1)^l\binom{r+s}{l}\cdot(r+s-l)^N=\sum_j\binom{N}{j}\left\{\sum_l(-1)^l\binom{r}{l}(r-l)^{N-j}\right\}$$
$$\cdot\left\{\sum_l(-1)^l\binom{s}{l}(s-l)^j\right\}.$$

For real $z\ll 1$

$$e^z\sim(1+z)^{+1}\sim(1-z)^{-1}.$$

We proceed now to the solution of our problem. The gen. functions giving the number of all the distributions into K cells with *a priori* given restrictions are of the form $\{f(z)\}^K$ where $f(z)$ is the fundamental gen. f. subject to the given restrictions or its power.

[1] Ordinary, not thermodynamic, probabilities have to be used for the computation of distributions. It can be shown that „overdistinguishability" of the particles, i. e. distinguishability of particles occupying the same elementary cell, is equivalent to indistinguishability. For bosons both „overdistinguishable" as indistinguishable statistics may be used, for fermions there are three possibilities: „overdistinguishability", distinguishability and indistinguishability. This problem will be dealt with in a forthcoming note.

The gen. functions for the number of these distributions into m cells subject not only to the *a priori* given restrictions concerning the occupation numbers but also to the interdictions contained in the formulation of our problem are of the form $\{g(z)\}^m$. As

$$(f(z))^K = \sum_m \binom{K}{m} g(z)^m (f(z) - g(z))^{K-m}$$

we can interpret the terms of this binomial expansion as gen. f. of the events in which m out of the K cells are filled in conformity with the requirements resulting from the gen. f. $g(z)$. At given N and K the probability for m is given by the ratio of the number of favourable events to that of all possible events, namely

$$p(m, K, N; g(z)) = \{f(z)^K\}_N^{-1} \binom{K}{m} \{g(z)^m [f(z) - g(z)]^{K-m}\}_N.$$

To find the moments we may apply the method of *characteristic functions*, called also often generating functions or Laplace's adjuncts as they were also first introduced by Laplace (cf. Gniedienko 1950). We obtain the characteristic function of a given distribution by subjecting the probability $P(m; K, N; g(z))$ to the Fourier transformation

$$\sum_{m=0}^{\infty} e^{lm\zeta} P(m; K, N; g(z)) \tag{3}$$

As the distribution of probability differs from the value of the determining function for a given N by the factor $\{f(z)\}_N^{-1}$ which is independent of m, we may consider the characteristic function (3) as the characteristic function of the determining function. Subjecting the gen. f. to a discrete Fourier transformation, we get the characteristic function of the gen. f.

$$\Phi(z, \zeta, K) = [f(z) - g(z) + g(z) \cdot e^{i\zeta}]^K$$

and consequently the momenta

$$M_l = \{f(z)^K\}_N^{-1} \frac{(-i)^l l!}{(2\pi i)^2} \oint \oint \frac{(f - g + g \cdot e^{i\zeta})^K}{\zeta^{l+1} \cdot z^{N+1}} d\zeta\, dz.$$

For example the gen. f. of the first moment: $M_1 = Kgf^{K-1}$ and of the second moment: $M_2 = K(K-1) g^2 f^{K-2} + M_1$. From the properties of the characteristic function and from the gen. f. itself one can draw conclusions concerning the asymptotic behaviour of the moments and the distributions (e. g., Hadamard's radius of convergence, and the criteria of convergence of power series).

The above method may be simply generalized to the case of multidimensional probabilities $P(m_j; N, K; g_j(z); \ j=1,\dots s)$, where m_j is the number of cells occupied in conformity with the initial exclusions rules and the exigencies arising from gen. f. $g_j(z)$. In order to find the gen. f. for our problem, we substitute Newton's polynomial for Newton's binomial and get

$$f(z)^K = \sum_{m_j \dots} \prod_{j=1}^{s} \frac{K!}{m_j!\left(K-\sum_{l=1}^{s} m_l\right)} \cdot g_j(z)^{m_j} \left[f(z) - \sum_{l=1}^{s} g_l(z)\right]^{K-\sum_{l=1}^{s} m_l}.$$

The characteristic function of the gen. f. is in this case

$$\Phi(z, \zeta_j, j=1,\dots s) = \left[f(z) - \sum_{j=1}^{s} g_j(z) + \sum_{j=1}^{s} g_j(z)\, e^{i\zeta_j}\right]^K.$$

3. The solution of problem II

This problem does not concern the number of cells subject to certain conditions — as in Problem I —, but the number of particles incident on a chosen set of k cells. Consequently the formalism of the gen. functions must undergo some modifications and instead of the polynomial expansions of gen. functions we shall consider gen. functions as products of other gen. functions.

For classical particles (case IIa) we get the binomial expansion

$$\binom{N}{n}\left(\frac{1}{K}\right)^n\left(1-\frac{1}{K}\right)^{N-n}$$

and in the limit when $N, K \to \infty$ at constant $NK^{-1} = \alpha = \text{const.}$, we get the Posisson distribution

$$\frac{\alpha^N}{N!}\, e^{-\alpha}$$

In the case of quantum particles (cases IIb and IIc) the gen. f. has the form

$$\begin{aligned} &\text{for B. E. statistics} \ \dots \ (1-z)^{-Kk} = (1-z)^{-(K-1)k}(1-z)^{-k} \\ &\text{for F. D. statistics} \ \dots \ (1+z)^{Kk} = (1+z)^{(K-1)k}(1+z)^{k}. \end{aligned} \tag{4}$$

Expanded in power series in z these functions are

$$\text{for B. E. statistics} \ \dots \sum_N \binom{N+Kk-1}{N} z^N = \sum_N \left[\sum_n \binom{N-n+Kk-k-1}{N-n}\binom{n+k-1}{n}\right] z^N$$

$$\text{for F. D. statistics} \ \dots \sum_N \binom{Kk}{N} z^N = \sum_N \left[\sum_n \binom{Kk-k}{N-n}\binom{k}{n}\right] z^N;$$

and the distributions read

for B. E. statistics ... $\binom{N+Kk-1}{N}^{-1}\binom{N-n+Kk-k-1}{N-n}\binom{n+k-1}{n}$

for F. D. statistics ... $\binom{Kk}{N}^{-1}\binom{Kk-k}{N-n}\binom{k}{n}$.

Whereas for classical distributions, due to their infinite divisibility, the numbers of elementary cells in a group are irrelevant (Gniedienko 1950), these numbers play an essential part for quantum distributions.

Applying Fowler and Darwin's method (1936) to the gen. functions (4) for $N, K \to \infty$ at $NK^{-1} = a = \text{const.}$, we get the quantum analogue of the Poisson distribution

for B. E. statistics ... $\binom{n+k-1}{n}(1-\vartheta)^k\vartheta^n$ where $\vartheta = \frac{a}{k}\left(1+\frac{a}{k}\right)^{-1}$

for F. D. statistics ... $\binom{k}{n}(1+\vartheta)^{-k}\cdot\vartheta^n$ where $\vartheta = \frac{a}{k}\left(1-\frac{a}{k}\right)^{-1}$.

We see that for finite k's these distributions differ widely from that of Poisson and go over into it only for $k \to \infty$ at $a = \text{const.}$ Whereas quantum distributions depend on k not only through a, their average values and their dispersions depend on k in the same manner as Poisson's distributions:

for B. E. statistics ... $E(n; k, \vartheta) = a$

$\sigma(n; k, \vartheta) = a\left(1+\frac{\vartheta}{1-\vartheta}\right) > a$ Lexis' coefficient > 1

for F. D. statistics ... $E(n; k, \vartheta) = a$

$\sigma(n; k, \vartheta) = a\left(1-\frac{\vartheta}{1+\vartheta}\right) < a$ Lexis' coefficient < 1.

Let us now return to Problem I and assume that N conforms in the classical case to the Poisson distribution with an average of zK^{-1} per group of k observational cells, and in the quantum theoretical case to the quantum analogue of Poisson's distribution. Let us further assume in this case that $\vartheta = z$. Then the gen. f. multiplied by e^{-z} in the classical case and by $(1 \mp \vartheta)^{\pm Kk}$ in the quantum theoretical case gives the probability of m cells being occupied according to the gen. f. $g(z)$ when z or a particles fall on the average on K cells. Thus we have defined the physical meaning of the gen. f. for a real positive value of z (which must be smaller than 1 in the quantum theoretical case).

4. Applications

The main fields application of the above theory lie in the theory of cosmic radiation, the theory of photographic plates particularly the theory of contrast, and in the theory of numbers.

In cosmic radiation one measures the distribution of showers by means of G. M.-counters at a given instant of time. These measurements and their theory constitute a complement to the theory of stochastic processes. Just as in statistical mechanics one either considers the system at a given instant of time or follows the particle in its motion using the ergodic properties of the system.

We shall consider now a special case of Problem I which play an important rôle in the theory of measurements of cosmic radiation, namely: without restrictions concerning the occupation of the cells, what is the probability that m groups of cells consisting each of k elementary cells will be hit each at least l times. We put $k=1$ in Problem Ia, the question is: what is the probability of m cells being hit ($l=1$). The gen. f. giving the distribution is then

$$\binom{K}{m}(e-1)^m$$

and the probability, due to formula (1), is

$$P(m; K, N) = K^{-N}\binom{K}{m}\sum_l (-1)^l\binom{m}{l}(m-l)^\Lambda \tag{5}$$

This formula was given by Schrödinger (1951); it was also obtained independently of one another in two different ways by K. Florek and the author in spring 1951[2,3]. It concerns cosmic particles but it is based on classical statistics.

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[2] Presented at the 37th session of the Wrocław Section of the Polish Physical Society, April 12, 1951.

[3] Formula (5) may be also obtained in another way without the use of gen. functions. N classical particles fall on K cells. One particle can hit only one cell. What is the probability that m cells will be hit? It is obviously given by the ratio of the number of all possible distributions of N particles into m arbitrary chosen cells out of the given K cells, each of the m cells being hit, to the number of all possible distributions of N particles into K cells. The number of all possible distributions is K^N. We choose a certain group of m cells. N particles can be distributed into these m cells in such a manner that no cell remain empty in

$$\sum_{n_i\ldots}\prod_{i=1}^{m}\frac{N!}{n_i!}=\sum_l(-1)^l\binom{m}{l}(m-l)^N;\quad \sum_i^m n_i=N,\quad n_i>0$$

For the expectation we get

$$E(m) = K\left(1-\left(1-\frac{1}{K}\right)^N\right).$$

This is the Nutting-Romer formula (cf. Ingarden and Mikusiński 1952) of the theory of photographic emulsions. For

$$K, N \to \infty \text{ at } NK^{-1} = b = \text{const.}$$

it goes over into Nutting's formula (1913)

$$E\left(\frac{m}{K}\right) = 1-e^{-b}.$$

If N is subject to the Poisson distribution, we obtain from our gen. f.

$$p(m; K, z) = \binom{K}{m}(1-e^{-\frac{z}{K}})^m(e^{-\frac{z}{K}})^{K-m}$$

where z is the average number of particles per observational cell. For $m=K$ this formula goes over into Auger's formula (Cocconi, Loverdo and Tongiorgi 1946). It is also based on classical statistics and in particular it implies the Poisson distribution for N, which do not seems legitimate. In the above formulae K is the number of the G. M.-counters, N — the number of incident cosmic particles and m — the number of counters hit.

For quantum particles problems *Ib, c* yield the gen. functions

for B. E. statistics ... $\binom{K}{m}\left[\sum_{j=0}^{l-1}\binom{j+k-1}{j}z^j\right]^{K-m}\left[\sum_{l=j}^{\infty}\binom{j+k-1}{j}z^j\right]^m$

for F. D. statistics ... $\binom{K}{m}\left[\sum_{j=0}^{l-1}\binom{k}{j}z^j\right]^{K-m}\left[\sum_{j=l}^{k}\binom{k}{j}z^j\right]^m$

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fashions. The choice of m out of K cells can be brought about in $\binom{K}{m}$ fashions. Hence the number of distributions in which N particles cover exactly m cells is

$$\binom{K}{m}\sum_l(-1)^l\binom{m}{l}(m-l)^N$$

and the required probability is

$$p(m; K, N) = K^{-N}\binom{K}{m}\sum_l(-1)^l\binom{m}{l}(m-l)^N.$$

where K is again the number of Geiger-Müller or scintillation counters, k — the number of possible positions of the particles in a counter (which is infinite due to the continuity of the position spectrum), z may be interpreted as the average number of particles in an elementary cell. For $k \to \infty$ at $zk = a = \text{const.}$, we get both for B. E. and F. D. statistics

$$\binom{K}{m}\left[\sum_{s=0}^{l-1}\frac{a^s}{s!}\right]^{K-m}\left[\sum_{s=l}^{\infty}\frac{a^s}{s!}\right]^{m} \tag{6}$$

that is the gen. f. of Problem Ia (for M. B. particles) in which $k=1$ and $a=z$. Thus it has been shown how the quantum problem for G. M.-counters passes into the classical one. For $l=1$ formula (6) goes over in (5).

The above formulae can be applied to measurements of distributions of cosmic particles, as follows[4]. We have

$$P(m; l, a, K) = \sum_N P(m; l, N, K)\, P(N; a)$$

where a is a parameter defining the distribution of the particles (e. g., the mean value for one cell). The quantity $P(m; l, a, K)$, $(m > 0)$, is measured experimentally, $P(m; l, N, K)$ is found theoretically in the present work. There remains $P(N; a)$ to be found, which is equivalent to the solving of a system of linear equations. As the case $m = 0$ is not measured, the distribution is determined but for an arbitrary factor. The parameter a depends evidently on the domain of measurement, which may be varied in different ways. In small domains in comparison with the dimensions of the shower ($r \sim 200$ m) the distribution $P(N; a)$ must lie close to an infinitely divisible distribution with the Kolmogorow characteristic function

$$\psi(t) = \exp\left[ita + \int (e^{itx} - 1 - itx)\frac{1}{x^2}\,dG(x)\right]; \quad G(+\infty) = \sigma(N)$$

the integral being extended to the whole range of variability of the stochastic variable.

We proceed now to the theory of photographic plates and in particular to the problems of contrast. We shall consider grains of different sensibility, the sensibility being characterized by the aver-

[4]) Incidentally, the same is true for all „generalized Schrödinger formulae" giving the probability of m cells being hit each exactly l times.

age number l of light quanta needed to blacken a grain. Let the whole emulsion be uniformly illuminated. The grains being macroscopic entities are subject to classical statistics, the photons — to quantum statistics. Let us divide the domain of observation into N cells of equal surface which is also equal to the cross section of the grains. Let the grains be distributed over these K cells on the plate according to a Poisson distribution, $af(s)$ being the average number per cell of the grains whose sensibility is s, where $\sum_{s=1}^{\infty} f(s)=1$; hence a is the average number of grains of all sensibilities per cell (s decreases with increasing sensibility of the grains). The position spectrum being continuous the distribution of the photons is also of the Poisson type with the mean value b.

We shall calculate the blackening caused by an illumination b when the sensibility distribution of the grains is $f(s)$. This problem was treated by Selwyn (1935), Silberstein (1941), Webb (1941), and others. It is to be noticed that b is the average number of photons per cell and not per grain as in the paper of Silberstein.

Let us consider grains of sensibility l falling at random on K cells; they hit cells which are already occupied by more sensitive grains ($s<1$), as well as n_l yet unoccupied cells. The probability that out of $(K-\sum_{r}^{l-1} n_r)$ cells which were not yet hit n_l cells will be hit is

$$P(n_l; \sum_{r}^{l-1} n_r, af(l), K) = \binom{K-\sum_{r}^{l-1} n_r}{n_l} (1-e^{-af(l)})^{n_l} (e^{-af(l)})^{K-\sum_{r}^{l} n_r}.$$

The probability that n_s new cells will be occupied by grains of sensibility s, for $s=1,2,3,\ldots$ is

$$P(n_s; af(s), K, s=1,2,\ldots) = \prod_{l=1}^{\infty} \frac{(K-\sum_{r}^{l-1} n_r)!}{n_l!\,(K-\sum_{r}^{l} n_r)!} (1-e^{-af(l)})^{n_l} (e^{-af(l)})^{K-\sum_{r}^{l} n_r}.$$

The probability that not less than l photons will fall on m_l out of the n_l cells is

$$P(m_s; n_s, b) = \prod_{l=1}^{\infty} \binom{n_l}{m_l} \left(1-e^{-b}\sum_{j=0}^{l-1}\frac{b^j}{j!}\right)^{m_l} \left(e^{-b}\sum_{j=0}^{l-1}\frac{b^j}{j!}\right)^{n_l-m_l}$$

Combining the distribution probabilities of photons and grains, we obtain for the probability of blackening of m out of K cells at

given $af(s)$ and b

$$P(m; b, af(s), K; s=1,2,\ldots) = \sum_{\substack{m_l\ldots \\ \Sigma m_l = m}} \sum_{n_l\ldots} P(m_s; n_s, b)\, P(n_s; af(s), K; s=1,2,\ldots)$$

$$= \sum_{\substack{m_l\ldots \\ \Sigma m_l = m}} \sum_{n_l\ldots} \prod_{l=1}^{\infty} \frac{K!}{m_l!(K-\sum_r^{\infty} n_r)!\,(n_l-m_l)!} (1-e^{-af(s)})^{n_l} (e^{-af(s)})^{K-\sum_r^{l} n_r} \left(1-e^{-b}\sum_{j=0}^{l-1}\frac{b^j}{j!}\right)^{m_l} \tag{7}$$

$$\times \left(e^{-b}\sum_{j=0}^{l-1}\frac{b_j}{j!}\right)^{n_l-m_l}$$

where it has been already taken into account that

$$\prod_{l=}^{\infty} \frac{(K-\sum_r^{l-1} n_r)!}{m_l!\,(K-\sum_r^{l} n_r)!\,(n_l-m_l)!} = \prod_{l=1}^{\infty} \frac{K!}{m_l!\,(K-\sum_r^{\infty} n_r)!\,(n_l-m_l)!}.$$

Summing up (7) over m we get $\sum_m P=1$.

Let us calculate now the expectation of the blackening $\sum_m mP = E(m)$

Substituting $\sum_{s=1}^{\infty} \sum_m \sum_{\substack{m_l\ldots \\ \Sigma m_l = m}} m_s$ for $\sum_m m \sum_{\substack{m_l\ldots \\ \Sigma m_l = m}}$, and then $\prod_{\substack{l=1 \\ l\neq s}}^{\infty} \sum_{m_l}$

for $\sum_m \sum_{m_l\ldots} \prod_{\substack{l=1 \\ l\neq s}}^{\infty}$ we get:

$$E(m) = \sum_{s=1}^{\infty} \sum_{n_l\ldots} \sum_{m_s} m_s \left\{ \binom{n_s}{m_s} \left(1-e^{-b}\sum_{j=0}^{s-1}\frac{b^j}{j!}\right)^{m_s} \left(e^{-b}\sum_{j=0}^{s-1}\frac{b^j}{j!}\right)^{n_s-m_s} \right\}$$

$$\times \prod_{l=1}^{\infty} \left\{ \binom{K-\sum_r^{l-1} n_r}{n_l} (1-e^{-af(l)})^{n_l} (e^{-af(l)})^{K-\sum_r^{l} n_r} \right\}.$$

Due to the equality

$$\sum_{m_s} m_s \binom{n_s}{m_s} \left(1-e^{-b}\sum_{j=0}^{s-1}\frac{b^j}{j!}\right)^{m_s} \left(e^{-b}\sum_{j=0}^{s-1}\frac{b^j}{j!}\right)^{n_s-m_s} = n_s \left(1-e^{-b}\sum_{j=0}^{s-1}\frac{b^j}{j!}\right)$$

we have

$$E(m)=\sum_{s=1}^{\infty}\left(1-e^{-b}\sum_{j=0}^{s-1}\frac{b^j}{j!}\right)\sum_{n_l\dots}^{\infty}n_s\prod_{l=1}^{\infty}\left\{\binom{K-\sum_{r}^{l-1}n_r}{n_l}(1-e^{-\alpha f(l)})^{n_l}(e^{-\alpha f(l)})\right\}^{K-\sum_{r}^{l}n_r}$$

We substitute now

$$\sum_{n}n\sum_{\substack{n_1\dots n_s\\ \sum_{l=1}^{s}n_l=n}}\prod_{l=1}^{s}\prod_{l=s+1}^{\infty}\sum_{n_l}-\sum_{n}n\sum_{\substack{n_1\dots n_{s-1}\\ \sum_{l=1}^{s-1}n_l=n}}\prod_{l=1}^{s-1}\prod_{l=s}^{\infty}\sum_{n_l}$$

for $\sum_{n_l\dots}n_s\prod_{l=1}^{\infty}$ and making use of the relation $\prod_{l=1}^{s}\binom{K-\sum_{r}^{l-1}n_r}{n_l}=\binom{K}{n}\prod_{l=1}^{s}\frac{n!}{n_l!}$, $n=\sum_{l}^{s}n_l$

we obtain

$$E(m)=\sum_{s=1}^{\infty}\left(1-e^{-b}\sum_{j=0}^{s-1}\frac{b^j}{j!}\right)\Bigg[\sum_{n}n\binom{K}{n}\sum_{\substack{n_l\dots\\ \sum_{l}^{s}n_l=n}}\prod_{l=1}^{s}\frac{n!}{n_l!}(1-e^{-\alpha f(l)})^{n_l}(e^{-\alpha f(l)})^{K-\sum_{r}^{l}n_r}$$

$$-\sum_{n}n\binom{K}{n}\sum_{\substack{n_l\dots\\ \sum_{l=1}^{s-1}n_l=n}}\prod_{l=1}^{s-1}\frac{n!}{n_l!}(1-e^{-\alpha f(l)})^{n_l}(e^{-\alpha f(l)})^{K-\sum_{r}^{l}n_r}\Bigg].$$

As

$$\prod_{l=1}^{s}(1-e^{-\alpha f(l)})^{n_l}(e^{-\alpha f(l)})^{K-\sum_{r}^{l}n_r}=\prod_{l=1}^{s}[(1-e^{-\alpha f(l)})e^{\sum_{l}^{s}\alpha f(l)}]^{n_l}e^{-K\sum_{l}^{s}\alpha f(l)}$$

and

$$\binom{K}{n}\sum_{\substack{n_l\dots\\ \sum_{l}^{s}n_l=n}}\prod_{l=1}^{s}\frac{n!}{n_l!}[(1-e^{-\alpha f(l)})e^{\sum_{l}^{s}\alpha f(l)}]^{n_l}e^{-K\sum_{l}^{s}\alpha f(l)}=\binom{K}{n}e^{-K\sum_{l}^{s}\alpha f(l)}(e^{\sum_{l}^{s}\alpha f(l)}-1)^n,$$

we get finally

$$E(m)=\sum_{s=1}^{\infty}\left(1-e^{-b}\sum_{l=0}^{s-1}\frac{b^l}{l!}\right)K[(e^{\sum_{l=1}^{s}\alpha f(l)}-1)e^{-\sum_{l=1}^{s}\alpha f(l)}-(e^{\sum_{l=1}^{s-1}\alpha f(l)}-1)e^{-\sum_{l=1}^{s-1}\alpha f(l)}]$$

which may be written in the form

$$E(m) = K \sum_{s=1}^{\infty} \left(1 - e^{-b} \sum_{j=0}^{s-1} \frac{b^j}{j!}\right) (1 - e^{-\alpha f(s)})\, e^{-\alpha \sum_{i=1}^{s-1} f(i)} \tag{8}$$

whereas Silberstein got for the average blackening

$$E(m) = K \sum_{s=1}^{\infty} \left(1 - e^{-b} \sum_{j=0}^{s-1} \frac{b^j}{j!}\right) (1 - e^{-\alpha f(s)}) \tag{9}$$

a different formula from ours.

The difference $e^{-\alpha \sum_{i=1}^{s-1} f(i)}$ of (8) and (9) ascribes a predominant influence on the blackening to grains of greater sensibility, which seems to explain the fact that the average number of photons needed for blackening is low.

From the expression for the dispersion the influence of the configuration factor on the granularity of the emulsion has been calculated, but the formula is to intricate to be worth while to be given here.

The applications to the theory of numbers will be given on two examples only.

Example 1. The distribution of an arbitrary number of bosons in one cell may be realized in one way only. Their distribution over an infinite number of cells, the first of which may be occupied once or not at all, the second twice or not at all,... the j-th 2^{j-1} times or not at all, etc. can be also realized in one way only. Hence the two corresponding generating functions must be equal and

$$(1-x)^{-1} = \prod_{j=0}^{\infty} (1 + x^{2^j}) \quad \text{for} \quad |z| < 1$$

giving an expansion in an infinite product.

Example 2. Every integral number $a > 1$ can be represented in a unique way in the canonical form $a = \prod_{i=1}^{\infty} r_i^{a_i}$, where r_i are prime numbers and a_i — positive integral numbers or zero. Taking into account the equality

$$ln\, a = \sum_{i=1}^{\infty} a_i\, ln\, r_i$$

and applying a similar reasoning as in Example 1, we get

$$\sum_{j=1}^{\infty} z^{\ln j} = \prod_{i=1}^{\infty} (1-z^{\ln r_i})^{-1} \quad \text{for} \quad |z|<1$$

or

$$\sum_{j=1}^{\infty} j^{x} = \prod_{i=1}^{\infty} (1-r_i^{x})^{-1} \quad \text{for} \quad Re(x)<0.$$

One can also give a general form to the Distribution Function of McMahon (1916) introduced in his „Combinatory Analysis" and find a uniform analytic treatment of that analysis.

The subject of this work was suggested to me by Dr R. S. Ingarden in the winter 1950/51; it grew considerably during its execution.

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ON A REGULAR FIELD THEORY I (CLASSICAL)

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Regular field equations are derived from a variational principle in a finite space-time domain. The existence and uniqueness of the solutions in the form of a power series expansion is proved. Conservation laws are inferred.

§ 1. Introduction and Summary

From the mathematical viewpoint, the possible reasons for the convergence difficulties encountered throughout the modern quantum field theory are threefold: (i) The point model of the interacting particles, i. e., the singular character of the Green functions of the field equations. (ii) An infinite space-time domain and improper initial conditions at minus or plus infinity (used in most of the recent investigations). (iii) Singular commutation relations between the field variables (connected with the Jordan-Pauli function).

In the opinion of the author it is the point model of interaction that is chiefly responsible for the convergence difficulties, while the second quantization has nothing to do with the well-known divergences (except for trivial ones like an infinite zero-point energy of a field). Thus, the convergence difficulties seem to be classical in character.

The singular character of the interaction may be avoided by introducing a new physical assumption; a non-point-interaction. This may be achieved by introducing a relativistic form-factor discussed previously by the author (Rayski 1951 a, b), by C. Bloch (1952), and by Kristensen and Møller (1952). In consequence of the appearance of a form factor, the integral field equations (being a substitute for the traditional differential field equations) possess regular kernels.

The second possible reason for the convergence difficulties (ii) may be avoided by taking a well defined, finite domain Ω where the system of fields propagates under proper initial conditions. The

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existence and uniqueness of regular solutions of the integral field equations is then easily demonstrated by means of a (somewhat extended) method due to Picard. The solutions are integer functions of the coupling parameter and may be computed by means of iteration.

The regular field equations may be quantized directly without any reference to the canonical formalism. The problem of quantization and the proof of the existence of expectation values of observable densities will be dealt with in Part II.

1. Stationary Action Principle. Let us consider a set of field components [1] $\Phi^{\alpha}(x)$ with given transformation properties (scalars, vectors, tensors, spinors, etc....) α denoting in short any one of the suffixes of the field components. These field components are functions of the variables x_{μ} in a finite domain Ω of the fourdimensional continuum. We assume that Ω is limited by two space-like hypersurfaces σ_1 and σ_2 of sufficient regularity and with a common edge B. We assume further that the set of fields forms an isolated system in Ω (without interaction with our measuring apparatus except on σ_1 and σ_2). The fields in Ω are characterized by an action functional W which is Lorentz invariant and invariant under transformations of phase by constant values (gauge invariance of the first kind). We assume further that the action functional may be split into two parts

$$W = W^{(0)} + W', \tag{1.1}$$

where $W^{(0)}$ is the well known bilinear form in the field components Φ^{α} and their first derivatives Φ^{α}_{μ} which, in the absence of W', yields the Schrödinger-Gordon-Klein equations for the tensor components, and the Dirac equations for the spinor components of the fields. The other term W' describes interaction between different fields. Let us consider the typical case where W' is formed of products of three field components, i. e. is a homogeneous bilinear form in the complex field functions and linear in the components of the real fields. Let us assume for simplicity that W' does not contain the derivatives of the field components. Sometimes it will be convenient to illustrate the general arguments by a simple (though purely academic) example of two coupled scalar fields: a complex ψ, and a real φ. In the frames of the local field theory the interaction part

[1] We follow in general the notations of Schwinger (1948, 1949, 1951) but use natural units $\hbar = c = 1$.

of the action functional may be assumed simply as

$$W' = g \int dx\, \psi^*(x) \varphi(x) \psi(x),$$

where g is the coupling parameter. In order to build up a regular field theory we shall introduce a non-localized interaction by taking the field components appearing in W' at different points $x', x'' x'''$ respectively and multiplying their product by a suitable form-factor $F(x'. x'', x'''; \Omega)$ which in general may depend upon the domain Ω.

$$W = W^{(0)} + W' = \int\limits_{\Omega} dx L^{(0)}(x) + \int\int\int\limits_{\Omega} dx'\, dx''\, dx'''\, L'(x', x'', x'''; \Omega). \quad (1.2)$$

In the case of two scalar fields L' assumes the form

$$L' = gF(x', x'', x'''; \Omega) \psi^*(x') \varphi(x'') \psi(x'''). \quad (1.3)$$

In order to preserve the hermitian character of W' the form-factor must satisfy the condition

$$F^*(x', x'', x'''; \Omega) = F(x''', x'', x'; \Omega). \quad (1.4)$$

Of course, F must be invariant under inhomogeneous Lorentz transformations. Further restrictions upon the form of the form-factor will be imposed later.

The variations of the field components by $\delta\Phi^\alpha$ introduce a change $\delta_\Phi W$ of the action functional:

$$\begin{aligned} \delta_\Phi W = & \int\limits_{\Omega} dx \left(\frac{\partial L^{(0)}(x)}{\partial \Phi^\alpha_\mu(x)} \delta\Phi^\alpha_\mu(x) + \frac{\partial L^{(0)}(x)}{\partial \Phi^\alpha(x)} \delta\Phi^\alpha(x) \right) \\ & + \int\int\int\limits_{\Omega} dx'\, dx''\, dx''' \frac{\partial L'(x', x'', x'''; \Omega)}{\partial \Phi^\alpha(x)} \delta\Phi^\alpha(x), \end{aligned} \quad (1.4')$$

where x in the last term denotes this one of the points x', x'', x''' which appears as argument of the function Φ^α. An integration by parts yields

$$\begin{aligned} \delta_\Phi W = & \int\limits_{\Omega} dx \left(- \frac{\partial}{\partial x_\mu} \frac{\partial L^{(0)}}{\partial \Phi^\alpha_\mu} + \frac{\partial L^{(0)}}{\partial \Phi^\alpha} + \int\int\limits_{\Omega} dy\, dz \frac{\partial L'}{\partial \Phi^\alpha} \right) \delta\Phi^\alpha(x) \\ & + \left(\int\limits_{\sigma_2} d\sigma_\mu - \int\limits_{\sigma_1} d\sigma_\mu \right) \frac{\partial L^{(0)}}{\partial \Phi^\alpha_\mu} \delta\Phi^\alpha(x), \end{aligned} \quad (1.5)$$

where y and z denote the other two of the points x', x'', x''' of the fourdimensional continuum.

The stationary action principle demands that the variations of the action functional should depend merely on the surface of the domain Ω, or, the variation δW should vanish if $\delta\Phi^\alpha$ vanish on the fixed surfaces σ_1 and σ_2. This yields the Euler-Lagrange equations

$$\frac{\partial}{\partial x_\mu}\frac{\partial L^{(0)}_{(x)}}{\partial \Phi^\alpha_\mu(x)} - \frac{\partial L^{(0)}_{(x)}}{\partial \Phi^\alpha(x)} = \int_\Omega\int dy\, dz \frac{\partial L'(x', x'', x'''; \Omega)}{\partial \Phi^\alpha(x)}, \tag{1.6}$$

where x denotes again this one of the points x', x'', x''' that appears as argument of the function Φ^α and y, z denote the other two points. We notice that the field equations depend upon the domain Ω where the system of fields is considered (and is assumed as isolated from other systems and measuring apparatus). Hence the solutions must depend as well on Ω

$$\Phi^\alpha = \Phi^\alpha(x; \Omega). \tag{1.7}$$

This is an essentially new feature of the field theory brought about by the non-local interaction. In the simple case of two coupled scalar fields equations (6) are

$$\begin{aligned}
(\Box - m^2)\,\psi(x) &= -g\int_\Omega\int dx''\, dx'''\, F(x, x'', x'''; \Omega)\,\varphi(x'')\,\psi(x'''),\\
(\Box - \varkappa^2)\,\varphi(x) &= -g\int_\Omega\int dx'\, dx'''\, F(x' x, x'''; \Omega)\,\psi^*(x')\,\psi(x'''),\\
(\Box - m^2)\,\psi^*(x) &= -g\int_\Omega\int dx'\, dx''\, F(x', x'', x; \Omega)\,\psi^*(x')\,\varphi(x'').
\end{aligned} \tag{1.8}$$

2. Field equations. Equations (1.6) and (1.8) are integro-differential equations. We replace them by pure integral equations. Let us introduce the Green function $G^\sigma_{(\alpha)}(x, y)$, connected with the field Φ^α, with the properties

$$\begin{aligned}
(\Box - m^2)\, G^\sigma_{(\alpha)}(x, y) &= -\delta(x - y),\\
G^\sigma_{(\alpha)}(x/\sigma, y) = 0, \qquad & n_\mu \partial_\mu G^\sigma_{(\alpha)}(x/\sigma, y) = 0,
\end{aligned} \tag{2.1}$$

(2.1) applies to the case where Φ^α denotes a tensor component. In the case of spinors $G^\sigma(x, y)$ is defined by

$$(\gamma^\mu \partial_\mu \pm m)\, G^\sigma_{(\alpha)}(x, y) = -\delta(x - y), \qquad G^\sigma_{(\alpha)}(x/\sigma, y) = 0 \tag{2.2}$$

where $\pm$ apply to the case of a spinor field or its conjugate $\Phi^* \gamma^4$. The symbol x/σ denotes a point x lying on a surface σ which is situated in Ω with a common edge, B, with the surfaces σ_1 and σ_2.

Denoting by σ_x another surface of the same type on which the point x_μ is lying, we have e. g. in the case of bosons

$$G^{\sigma}_{(\alpha)}(x,y) = \begin{cases} -\Delta(x-y) & \text{for} \quad \sigma_x > y > \sigma \\ \Delta(x-y) & \text{for} \quad \sigma > y > \sigma_x \\ 0 & \text{elsewhere,} \end{cases} \tag{2.3}$$

where Δ is the well-known Jordan-Pauli function. In the case of fermions we have a similar expression in terms of the well-known function

$$S(x-y) = (\gamma^\mu \partial_\mu - m)\,\Delta(x-y). \tag{2.4}$$

Let us introduce the kernel

$$K^{\sigma}_{(\alpha)}(x,y,z;\Omega) = \int_{\Omega} du\, G^{\sigma}_{(\alpha)}(x,u)\, F(x',x'',x''';\Omega), \tag{2.5}$$

where u denotes this one of the points x', x'', x''' that appears as argument of the function Φ^α while y, z denote the remaining two points. In view of the properties (2.1) or (2.2) of the Green functions, we see that $K^{\sigma}_{(\alpha)}$ vanishes (together with its normal derivative in the case of bosons) for points x on the surface σ and satisfies the following equation

$$(\Box - m^2)\, K^{\sigma}_{(\alpha)} = -F \quad \text{or} \quad (\gamma^\mu \partial_\mu + m)\, K^{\sigma}_{(\alpha)} = -F \tag{2.6}$$

in the cases of bosons or fermions respectively. In the case of two scalar fields equations (2.6) are

$$\begin{aligned} (\Box - m^2)\, K^{\sigma}_{\psi^*}(x,x''\,x''') &= -F(x,x'',x''';\Omega) \\ (\Box - \varkappa^2)\, K^{\sigma}_{\varphi}\,(x,x',x''') &= -F(x'\,x,x''';\Omega) \\ (\Box - m^2)\, K^{\sigma}_{\psi}\,(x,x'\,x'') &= -F(x',x'',x\,\Omega). \end{aligned} \tag{2.6'}$$

Let us assume now that the function F is continuous and limited in the domain Ω. Consequently, the solutions K^σ of the inhomogeneous equations (2.6) or (2.6′) are limited

$$|K^{\sigma}_{(\alpha)}| \leqslant M \quad \text{in } \Omega \quad \text{for any } \alpha \tag{2.7}$$

and the Lagrange equation (1.6) are equivalent to the integral equations

$$\Phi^\alpha(x) = \Phi^\alpha_\sigma(x) + \int_{\Omega}\int\int dx'\,dx''\,dx'''\, G^{\sigma}_{(\beta)}(x,u)\, \frac{\partial L'(x'\,x'',x''';\Omega)}{\partial \Phi^\beta(u)} \tag{2.8}$$

which may be written also in terms of the kernels K^σ. Φ^α_σ denotes a solution of the interaction-free Schrödinger-Gordon-Klein (or Dirac) equation, Φ^β denotes the complex conjugate of Φ^α (in the case of spinors the corresponding equation is somewhat more complicated). u denotes again this one of the points x', x'', x''' that appears as argument of the function Φ^β. From the properties (2.1) or (2.2) it is seen that Φ^α coincides with the free wave Φ^α_σ (together with the normal derivative in the case of bosons) on the surface σ. With the aid of (2.5) the integral equation (2.8) may be written in the case of two coupled scalar fields, as follows

$$\begin{aligned}
\psi(x) &= \psi_\sigma(x) + g\int_\Omega\int dy\,dz\,K^\sigma_\psi(x,y,z,\varphi(y)\psi(z),\\
\varphi(x) &= \varphi_\sigma(x) + g\int_\Omega\int dy\,dz\,K^\sigma_\varphi(x,y,z)\psi^*(y)\psi(z),\\
\psi^*(x) &= \psi^*_\sigma(x) + g\int_\Omega\int dy\,dz\,K^\sigma_\psi(x,y,z)\psi^*(y)\varphi(z).
\end{aligned} \tag{2.8'}$$

A proof of the existence of the solution of the set of integral eqations is given in the Appendix.

3. Conservation laws and correspondence. The conservation laws follow from the invariance of the action functional W with respect either to infinitesimal phase transformations or to infinitesimal rigid displacements and rotations of the domain Ω. The general variation of the action functional is

$$\delta W = \delta_\Phi W + \delta_\sigma W, \tag{3.1}$$

where $\delta_\Phi W$ is the variation produced by the variations of the field comqonents $\delta_0\Phi^\alpha$ at fixed space-time points, and $\delta_\sigma W$ is the variation produced by the infinitesimal variations of the boundaries σ_1 and σ_2 of the domain Ω. Taking account of the Lagrange equations we have from (1.5)

$$\delta_\Phi W = \delta_\Phi W^{(0)} = \Big(\int_{\sigma_2} d\sigma_\mu - \int_{\sigma_1} d\sigma_\mu\Big)\frac{\partial L^{(0)}}{\partial\Phi^\alpha_\mu}\delta_0\Phi^\alpha. \tag{3.2}$$

We notice that the variations of the field components affect only the interaction-free part of the action functional which is identical with that in the local field theory. Therefore we obtain the same expression for the charge as in the traditional theory. Perfoming infinitesimal variations of the phase of the complex field comqonents we find as usually

$$\sum_\alpha \varepsilon^\alpha \int_{\sigma_2} d\sigma_\mu \frac{\partial L^{(0)}}{\partial\Phi^\alpha_\mu}\Phi^\alpha = \sum_\alpha \varepsilon^\alpha \int_{\sigma_1} d\sigma_\mu \frac{\partial L^{(0)}}{\partial\Phi^\alpha_\mu}\Phi^\alpha, \tag{3.3}$$

where $\varepsilon^\alpha = \pm 1$, or 0 for complex field components their complex conjugates, ane the real field components respectively. Thus, apart from a constant factor, we may call the conserved quantity

$$Q = \int_\sigma d\sigma_\mu \sum_\alpha \varepsilon^\alpha \frac{\partial L^{(0)}}{\partial \Phi^\alpha_\mu} \Phi^\alpha \tag{3.4}$$

the charge or the system, and

$$j_\mu(x) = \sum_\alpha \varepsilon^\alpha \frac{\partial L^{(0)}}{\partial \Phi^\alpha_\mu(x)} \Phi^\alpha(x) \tag{3.5}$$

the charge and current density four-vector. The charge is conserved on the boundaries σ_1 and σ_2 of the domain Ω. The analogy with the local field theory is merely formal owing to the fact that the Lagrange equations (1.6) and their solutions (1.7) depend upon the domain of integration. This domain may be taken arbitrarily small so that we may write formally

$$\lim_{\Omega \to 0} \partial_\mu j_\mu = 0,$$

but this „continuity equation" is an empty statement since the whole field formalism becomes meaningles in the limit $\Omega \to 0$.

In order to infer the energy-momentum and the angular momentum conservation laws we carry out an infinitesimal rigid displacement and rotation of the domain Ω by

$$\delta x_\mu = \varepsilon_\mu - \varepsilon_{\mu\nu} x_\nu \tag{3.6}$$

where ε_μ, $\varepsilon_{\mu\nu}$ are small arbitrary quantities. The change of the action functional under the displacement (3.6) is

$$\delta_\sigma W = \delta_\sigma W^{(0)} + \delta_\sigma W' \tag{3.7}$$

where

$$\delta_\sigma W^{(0)} = \left(\int_{\sigma_1+\delta\sigma_1}^{\sigma_2+\delta\sigma_2} - \int_{\sigma_1}^{\sigma_2} \right) dx\, L^{(0)}(x) = \oint d\sigma_\mu\, \delta x_\mu L^{(0)}(x) \tag{3.7'}$$

$$\delta_\sigma W' = \int_{\sigma_1+\delta\sigma_1}^{\sigma_2+\delta\sigma_2} \int\int dx'\, dx''\, dx'''\, L'(x', x'', x'''; \Omega + \delta\Omega) - \int_{\sigma_1}^{\sigma_2} \int\int dx'\, dx''\, dx'''\, L'(x', x'', x'''; \Omega). \tag{3.7''}$$

At the same time we perform the variations $\delta_0\Phi$ in such a way that the new field components assume the same values on the surfaces $\sigma_2+\delta\sigma_2$ and $\sigma_1+\delta\sigma_1$ as the old ones did on the surfaces σ_2 and σ_1. The action functional is invariant under translations and rotations of the boundaries together with the field components whence the whole variation vanishes

$$\delta W=\delta W^{(0)}+\delta W'=\delta_\Phi W^{(0)}+\delta_\sigma W^{(0)}+\delta_\sigma W'=0 \tag{3.8}$$

where $\delta_\Phi W^{(0)}$ is given by (3.2). By taking ε_μ arbitrary and $\varepsilon_{\mu\nu}=0$, we infer from (3.8) a tensor

$$T_{\mu\nu}=T^{(0)}_{\mu\nu}+T'_{\mu\nu} \tag{3.9}$$

with the property

$$P_\nu(\sigma_1)=\int\limits_{\sigma_1} d\sigma_\mu\, T_{\mu\nu}=\int\limits_{\sigma_2} d\sigma_\mu\, T_{\mu\nu}=P_\nu(\sigma_2), \tag{3 10}$$

so that P_ν may be interpreted as the energy and momentum four-vector. In the same way, by taking $\varepsilon_{\mu\nu}$ arbitrary and $\varepsilon_\mu=0$, we find a tensor

$$M_{\lambda\mu\nu}=M^{(0)}_{\lambda\mu\nu}+M'_{\lambda\mu\nu} \tag{3.11}$$

with the property

$$I_{\mu\nu}(\sigma_1)=\int\limits_{\sigma_1} d\sigma_\lambda\, M_{\lambda\mu\nu}=\int\limits_{\sigma_2} d\sigma_\lambda\, M_{\lambda\mu\nu}=I_{\mu\nu}(\sigma_2), \tag{3.12}$$

where $I_{\mu\nu}$ may be interpreted as the angular momentum.

We notice that the variations $\delta_\Phi W^{(0)}$ and $\delta_\sigma W^{(0)}$ are (formally) identical with those of the local field theory so that we may refer the reader to the literature (W. Pauli 1941, J. Schwinger 1951) for details of the construction of the interaction-free part of the energy-momentum and angular momentum tensors $T^{(0)}_{\mu\nu}$ and $M^{(0)}_{\lambda\mu\nu}$. In the following we shall limit ourselves to the construction of the interaction parts $T'_{\mu\nu}$ and $M'_{\lambda\mu\nu}$ which will follow from a discussion of (3.7″).

In order to compute $T'_{\mu\nu}$ and $M'_{\lambda\mu\nu}$, we must know the dependence of L' on the domain Ω, that is, the dependence of the form-factor $F(x',x'',x''';\Omega)$ on Ω. This dependence may be fixed unambiguously by invoking the requirement of correspondence with the local field theory. In order to secure the correspondence let us introduce a parameter λ (with the dimension of a length) into the form-factor F in such a way that the form-factor goes over into a product of two Dirac delta functions in the limit $\lambda\to 0$:

$$\lim_{\lambda\to 0} F(x',x'',x''';\Omega)=\delta(x'-x'')\,\delta(x''-x'''). \tag{3.13}$$

The form-factor must be invariant under displacement of the origin of the coordinate system, and therefore it is natural to introduce first a Lorentz invariant distribution function $R(x, y z; \lambda)$ independent of Ω, with the properties

$$R(x, y, z; \lambda) = R^*(z, y, x; \lambda), \qquad \lim_{\lambda \to 0} R(x, y, z; \lambda) = \delta(x)\,\delta(y)\,\delta(z). \tag{3.14}$$

With the aid of this distribution function (which should be regarded as more fundamental than F) we may define the form-factor F either by

$$F_1(x', x'', x''') = \int dx\, R(x' - x, x'' - x, x''' - x; \lambda) \tag{3.15'}$$

which is independent of Ω, or by

$$F_2(x', x'', x'''; \Omega) = \int_{\Omega} dx\, R\, x' - x, x'' - x, x''' - x; \lambda) \tag{3.15''}$$

which refers directly to the domain Ω. More generally, we may consider a linear combination

$$F(x', x'', x'''; \Omega) = a_1 F_1(x', x'', x''') + a_2 F_2(x', x'', x'''; \Omega). \tag{3.16}$$

With (3.15) or (3.16) the invariance of the form-factor under translations is automatically secured in contradistinction to the paper of Kristensen and Møller. It is easily seen that for $\lambda \to 0$ the Lagrange equations go over into the local ones if the coefficients of (3.16) satisfy the condition

$$a_1 + a_2 = 1. \tag{3.17}$$

We shall show that the requirement of correspondence of the energy-momentum density tensor and of the angular momentum density tensor with the local ones yields another condition for the coefficients of (3.16), namely

$$3a_1 + 4a_2 = 1, \tag{3.18}$$

so that a_1 nad a_2 are fixed unambiguously

$$a_1 = 3, \quad a_2 = -2 \tag{3.19}$$

and we have no other choice for the form-factor F. Thus, the two requirements of correspondence (that of the field equations and that of the conserved quantities) are independent. In order to satisfy both requirements a simple form-factor (3.15') or (3.15'') is not sufficient but we have to take their linear combination (3.16). In the limit of the domain Ω being the whole space-time F_1 and F_2 become identical and the formalism simplifies considerably.

In order to determine $T'_{\mu\nu}$ and $M'_{\lambda\mu\nu}$, we compute (3.7″)

$$\begin{aligned}\delta_\sigma W' = &\oint d\sigma'_\mu\, \delta x'_\mu \int_\Omega\int dx''\, dx'''\, L'(x', x'', x'''; \Omega)\\ &+ \oint d\sigma''_\mu\, \delta x''_\mu \int_\Omega\int dx'\, dx'''\, L'(x', x'', x'''; \Omega)\\ &+ \oint d\sigma'''_\mu\, \delta x'''_\mu \int_\Omega\int dx'\, dx''\, L'(x', x'', x'''; \Omega)\\ &+ \int_\Omega\int\int dx'\, dx''\, dx'''\, \delta_\sigma L'(x', x'', x'''; \Omega).\end{aligned} \tag{3.20}$$

In order to avoid non-essential complications, we restrict ourselves again to the simple case of two coupled scalar fields, where L' is given by (1.3) with F given by (3.16). The last term in (3.20) becomes

$$a_2 g \oint d\sigma_\mu\, \delta x_\mu \int_\Omega\int\int dx'\, dx''\, dx'''\, R(x' - x, x'' - x, x''' - x)\, \psi^*)x')\, \varphi(x'')\, \psi(x''')$$

so that $\delta_\sigma W'$ may be written as

$$\begin{aligned}\delta_\sigma W' = \oint d\sigma_\mu\, \delta x_\mu \Big\{ &\int_\Omega\int dx''\, dx'''\, L'(x, x'', x'''; \Omega)\\ &+ \int_\Omega\int dx'\, dx'''\, L'(x', x, x'''; \Omega)\\ &+ \int_\Omega\int dx'\, dx''\, L'(x', x'', x; \Omega)\\ &+ a_2 g \int_\Omega\int\int dx'\, dx''\, dx'''\, R(x' - x, x'' - x, x''' - x)\, \psi^*(x')\, \varphi(x'')\, \psi(x''')\Big\}.\end{aligned} \tag{3.21}$$

If, for example, we consider a variation $\varepsilon_{\mu\nu} = 0$ and ε_μ arbitrary then we may write

$$\oint d\sigma_\mu\, \delta x_\mu \ldots = \oint d\sigma_\mu\, \varepsilon_\nu\, \delta_{\mu\nu} \ldots,$$

whence the interaction part of the energy-momentum tensor follows:

$$\begin{aligned}T'_{\mu\nu}(x) = g\, \delta_{\mu\nu} \Big\{ &\int_\Omega\int dx''\, dx'''\, F(x, x'', x'''; \Omega)\, \psi^*(x)\, \varphi(x'')\, \psi(x''')\\ &+ \int_\Omega\int dx'\, dx'''\, F(x', x, x'''; \Omega)\, \psi^*(x')\, \varphi(x)\, \psi(x''')\\ &+ \int_\Omega\int dx'\, dx''\, F\, x', x''\, x; \Omega)\, \psi^*(x')\, \varphi(x'')\, \psi(x)\\ &+ a_2 \int_\Omega\int\int dx'\, dx''\, dx'''\, R(x' - x, x'' - x, x''' - x\ \psi^*(x')\, \varphi(x'')\, \psi(x''')\Big\},\end{aligned} \tag{3.22}$$

where, e. g.,

$$F(x, x'', x'''; \Omega) = a_1 \int dx'\, R(x-x', x''-x', x'''-x') + a_2 \int\limits_{\Omega} dx'\, R(x-x', x''-x', x'''-x') \tag{3.22'}$$

and so on...

By taking $\varepsilon_\mu = 0$ and $\varepsilon_{\mu\nu}$ arbitrary we find the angular momentum density tensor $M_{\gamma\mu\nu}$ connected with $T'_{\mu\nu}$ in the usual way:

$$M_{\lambda\mu\nu}(x) = x_\mu T'_{\lambda\nu}(x) - x_\nu T'_{\lambda\mu}(x). \tag{3.23}$$

The interaction part of the energy-momentum density tensor is symmetrical. The interaction-free part $T^{(0)}_{\mu\nu}$ may be symmetrized in the usual way so that the whole $T_{\mu\nu} = T^{(0)}_{\mu\nu} + T'_{\mu\nu}$ is symmetrical $T_{\mu\nu} = T_{\nu\mu}$.

We see that expression (3.22) is rather complicated, since it contains seven terms altogether, three of which appear multiplied by a_1 and four by a_2. In the limit of localizability ($\lambda = 0$) each of those seven terms goes over into the usual expression for $T'_{\mu\nu}$ of the local theory. Thus we find

$$\lim_{\lambda \to 0} T'_{\mu\nu}(x) = (3a_1 + 4a_2)\, g\, \delta_{\mu\nu}\, \psi^*(x)\, \varphi(x)\, \psi(x) \tag{3.24}$$

wherefrom the condition (3.18) follows.

Going over to the limit of the whole space-time manifold (as the domain Ω) F_2 becomes identical with F_1 so that the form-factor as well as the Lagrange equations and the expressions for the energy-momentum density tensor simplify considerably. For example, $T'_{\mu\nu}$ becomes

$$\begin{aligned} &\lim_{\substack{\sigma_1 \to -\infty \\ \sigma_3 \to +\infty}} T'_{\mu\nu}(x) \\ &= g\, \delta_{\mu\nu} \int\int\int dx'\, dx''\, dx'''\, \{R(x-x', x''-x', x'''-x')\, \psi^*(x)\, \varphi(x'')\, \psi(x''') \\ &\quad + R(x'-x'', x-x'', x'''-x'')\, \psi^*(\varphi')\, \varphi(x)\, \psi(x''') \\ &\quad + R(x'-x''', x''-x''', x-x''')\, \psi^*(x')\, \varphi(x'')\, \psi(x) \\ &\quad - 2R(x'-x, x''-x, x'''-x)\, \psi^*(x')\, \varphi(x'')\, \psi(x''')\}. \end{aligned} \tag{3.25}$$

It may be added that the particular position assumed by the free variable x in the separate terms in (3.25) is not essential. This follows from the fact that the variations are identical in the case of rigid displacements. However, the special form (3.22) or (3.25) with respect to the positions assumed by x appears the most natural if one regards the common domain Ω as a special case of four independent domains $\Omega, \Omega', \Omega'', \Omega'''$ of the variables x, x', x'', x'''.

The results of this section may be summarized as follows. The regular field theory satisfies the conservation laws of quantities which may be interpreted as charge, energy and momentum, and angular momentum. The conservation laws refer to, and only to, the surfaces σ_1 and σ_2 enclosing the domain Ω. The two requirements: (i) that the field equations should go over into the local ones, and (ii) that the conserved quantities should become identical with the usual ones in the limit $\lambda \to 0$ (λ is a fundamental length) are independent from each other. Only a special form of the form-factor (3.16) with F_1 and F_2 given by (3.15) and a_1 and a_2 given by (3.19) secures the correspondence with the local field theory, The form-factor (3.16), the Lagrange equations (1.6), and the expression for the energy-momentum tensor density simplify considerably in the limiting case of the surfaces σ_1 and σ_2 tending respectively to minus and plus infinity.

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Appendix. An investigation of the problem of existence of regular solutions.

Let us introduce the natural units $c = \hbar = \lambda = 1$ whereby all the field quantities become dimensionless. We assume $g > 0$. In order to simplify the notations we omit the arguments x, y, z of the field quantities and of the kernels, and omit the symbols $dx\,dy\,dz$. The integral field equations (2.8) may be written simply

$$\psi = \psi_\sigma + g \iiint K_\psi \varphi \psi\,, \quad \varphi = \varphi_\sigma + g \iiint K_\varphi \psi^* \psi\,. \tag{A.1}$$

We assume that the following (limited) integrals exist

$$\begin{gathered} \iiint |K| \leqslant M, \quad \iiint |K \psi_\sigma| \leqslant M, \quad \iiint |K \varphi_\sigma| \leqslant M, \\ \iiint |K \varphi_\sigma \psi_\sigma| \leqslant M, \quad \iiint |K \psi_\sigma^* \psi_\sigma| \leqslant M, \end{gathered} \tag{A.2}$$

where K stands either for K_ψ or K_φ.

We try to solve equations (A.1) by means of iteration. We form the sequences ψ_n, φ_n for $n = 0, 1, 2, \ldots$ with $\psi_0 = \psi_\sigma$, $\varphi_0 = \varphi_\sigma$ defined by

$$\psi_n = \psi_0 + g \iiint K_\psi \varphi_{n-1} \psi_{n-1}\,, \quad \varphi_n = \varphi_0 + g \iiint K_\varphi \psi_{n-1}^* \psi_{n-1}\,. \tag{A.3}$$

Putting

$$\Delta_n \psi = \psi_n - \psi_0 \tag{A.4}$$

we get

$$\Delta_n\psi = g\int\int\int K_\psi(\Delta_{n-1}\varphi+\varphi_0)(\Delta_{n-1}\psi+\psi_0) \tag{A.5}$$

or

$$\begin{aligned}\Delta_n\psi = g\int\int\int K_\psi\,\Delta_{n-1}\varphi\,\Delta_{n-1}\psi + g\int\int\int K_\psi\varphi_0\,\Delta_{n-1}\psi\\ + g\int\int\int K_\psi\,\Delta_{n-1}\varphi\cdot\psi_0 + g\int\int\int K_\psi\varphi_0\psi_0.\end{aligned} \tag{A.5'}$$

On account of $\Delta_0\psi = \Delta_0\varphi = 0$, we have

$$|\Delta_1\psi| \leqslant g\int\int\int |K_\psi\varphi_0\psi_0| \leqslant gM \tag{A.6}$$

and, in the same way,

$$|\Delta_1\varphi| \leqslant gM. \tag{A.6'}$$

Introducing a number $N > M$ we have also

$$|\Delta_1\psi| < gN, \quad |\Delta_1\varphi| < gN. \tag{A.7}$$

From (A.5') and (A.7) we find

$$|\Delta_2\psi| < g^3MN^2 + 2g^2MN + gM = gM(1+gN)^2. \tag{A.8}$$

Now, it is also

$$|\Delta_2\psi| < gN \tag{A.9}$$

if

$$M(1+gN)^2 \leqslant N. \tag{A.10}$$

Inequality (A.10) is a restriction upon the value of the coupling constant g

$$g^2 \leqslant \frac{1}{M}\,\frac{N-M}{N^2}. \tag{2.11}$$

It may be easily seen that the same procedure applies to all higher terms, so that we have quite generally

$$|\Delta_n\psi| < gN, \quad |\Delta_n\varphi| < gN \tag{A.12}$$

if g satisfies (A.11).

Let us compute the differences

$$\begin{aligned}\psi_{n+1}-\psi_n = g\int\int\int K_\psi(\varphi_n\psi_n - \varphi_{n-1}\psi_{n-1})\\ = g\int\int\int K_\psi[(\varphi_n-\varphi_{n-1})\psi_n + \varphi_{n-1}(\psi_n-\psi_{n-1})].\end{aligned} \tag{A.13}$$

In particular, we have

$$\begin{aligned}|\psi_2-\psi_1| \leqslant g^2N\int\int\int |K_\psi(\psi_1+\varphi_0)| \leqslant g^2N\int\int\int |K(\Delta_1\psi+\psi_0+\varphi_0)|\\ < g^2N^2(gN+2),\end{aligned}$$

$$|\psi_3-\psi_2| < g^3N^2(gN+2)\int\int\int |K(\psi_2+\varphi_1)|$$

$$< g^3N^2(gN+2)\int\int\int |K(\Delta_2\psi+\Delta_1\varphi+\psi_0+\varphi_0|$$

$$< g^3N^2(gN+2)(2gNM+2M) < g^2N^2(gN+2)\cdot 2gN(gN+1).$$

By repeating this procedure we find

$$|\psi_{n+1}-\psi_n| < g^2N^2(gN+2)[2gN(gN+1)]^{n-1}, \tag{A.14}$$

from where a sufficient condition for the convergence of the iteration procedure to a solution of the field equations follows:

$$2gN(gN+1) < 1. \tag{A.15}$$

From (A.11) we find that the maximum permissible value of g is obtained for $N=2M$. By introducing this value into (A.15), we find

$$g < \frac{\sqrt{3}-1}{4}\,\frac{1}{M} \tag{A.16}$$

Thus, the following theorem has been proved:

If the conditions (A.2) are satisfied, then the integral field equations (2.8) possess solutions which may be represented in the form of a power series of the coupling constant $g>0$ for sufficiently small values of g (given by (A.16)).

The first of conditions (A.2) constitutes a condition upon the form function regularizing the kernels. Further conditions (A.2) would be satisfied automatically if the „initial waves", i. e. the solutions of the interaction free field equations ψ_σ and φ_σ, were limited. For the sake of a subsequent quantization we cannot assume a limited ψ_σ and φ_σ but have to consider the general solutions of the interaction-free equations. It seems to be sufficient and necessary to require that (A.2) is satisfied if ψ_σ and φ_σ are replaced by the two fundamental solutions of the interaction-free equations, namely Δ and $\Delta^{(1)}$ or Δ^+ and Δ^-.

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LETTERS TO THE EDITOR

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Battery Pulse-Supply for G-M counters

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February 15, 1952

A battery pulse-supply for G—M counters is presented allowing to obtain small overall dimensions and weight of the apparatus. Working conditions for measurements with the G—M counters often require that the whole apparatus, including high voltage supply, pulse amplifier and registrator, should be transportable, light, and of small dimensions. The circuit shown on fig. 1 has such properties. All the tubes are of the popular type D-series. The high tension is obtained on the coil (80 H) by pulse-breaking of the anode current of the tube DL 21,

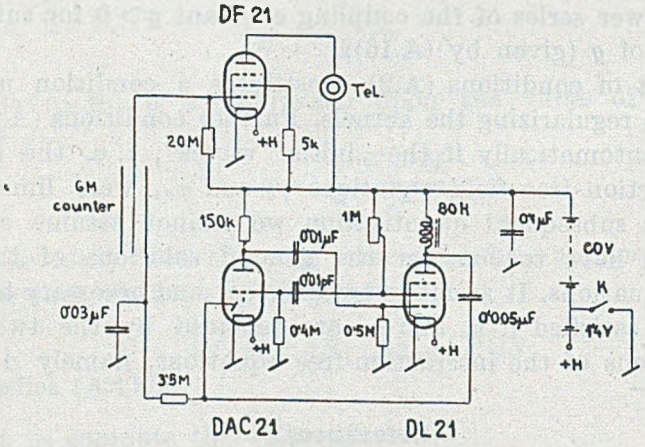

Fig. 1

which works in conjunction with the tube DAC 21 as a relaxation circuit (frequency of relaxations about 60 c/s). The detection occurs on the diodic part of the tube DAC 21. The potentiometer 1 MΩ enables a linear change of DC voltage between the electrodes of the counter. The G—M counter is of the type used in the laboratory of Prof. M. Mięsowicz, working on the voltage range from 960 to 1260 volts. To cover this wide working range of the counter, the potentiometer (1 MΩ) gives the change of voltage from 800 to 1400 volts. Changing the fre-

quency of the relaxation, or by using a coil having a higher relation of the inductance to its self-capacity, it is possible to obtain in our circuit a much higher DC voltage (up to 3000 volts).

The apparatus is independent of line voltage and easily transportable owing to the use of the battery supply, which at the same time gives steady work of the apparatus.

Sur les spectres ramaniens des mélanges de pyridine et de l'acide acétique

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Juin 13, 1952

Par l'étude des spectres ramaniens des mélanges de pyridine et de l'acide acétique on a mis en évidence les changement des fréquences propres des molécules aussi que l'apparition des fréquences nouvelles. Cette méthode mettant en évidence les modifications des fréquences permet d'étudier plus profondement les effets des associations moléculaires. Les méthodes usuelles, utilisant les mesures de tension de vapeur, de parachore, de réfraction moléculaire ne donnent que les effets globaux.

De ce point de vue on a étudié les spectres ramaniens des mélanges de pyridine et de l'acide acétique contenant 61,0, 48,3, 41,4, 30.4, et 22,3% de pyridine par mole. Le fait le plus marquant aparaissant dans les spectres ramaniens de ces mélanges est la présence d'une raie nouvelle $\Delta\nu = 1005$ cm^{-1}. Même dans e mélange contenant 61,0% de pyridine la raie nouvelle es plus intense que les raies plus intenses de pyridine $\Delta\nu = 990{,}5$ et 1030,7 cm^{-1}. Dans les solutions de pyridine plus diluées l'intensité de ces deux raies diminue tandis que la raie nouvelle reste la plus intense du spectre. En outre dans les spectres ramaniens de ces mélanges on observe une raie nouvelle de fréquence $\Delta\nu = 881$ cm^{-1}.

En comparant les spectres ramaniens de pyridine, de l'acide acétique et des mélanges on observe aussi les déplacement suivants des raies:

Les raies de pyridine $\Delta\nu = 650{,}9$, 990,5, 1030,7 cm^{-1} et la raie nouvelle $\Delta\nu = 1005$ cm^{-1} ont dans les mélanges des fréquences augmenteés. L'augmentations passe par un maximum dans un mélange de concentration voisine de 40% de pyridine.

Dans les mélanges la fréquence de la raie de pyridine $\Delta\nu = 3056{,}4$ augmente d'une façon manifeste et celles des raies $\Delta\nu = 1217{,}6$ et 1594,4 cm^{-1} sont diminuées.

La fréquence de la raie de l'acide acétique $\Delta\nu = 2943{,}5$ cm^{-1} est diminuée dans les solutions et celles des raies $\Delta\nu = 3046{,}9$ et 3126,3 cm^{-1} sont augmentées.

Une publication plus étendue va apparaitre dans Acta Physica Polonica.

The Existence of two OH-groups in H_2SO_4 Molecules Verified by the Scattering of Thermal Neutrons

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August 15, 1952

The scattering of slow neutrons by molecular gases is well accounted for by the theory of Sachs and Teller [1]. This theory states that a neutron being scattered by a proton bound in a molecule may cause a rotation of the whole molecule. The cross section for the scattering of a neutron by this bound proton is

$$\sigma = \sigma_{\infty}(\mu_1 \mu_2 \mu_3 \mu)^{1/2}(T_0 + T_1 + \ldots),$$

where $\sigma_{\infty} = 4\sigma_H$ (σ_H is the cross section for the scattering of slow neutrons by free protons; $\sigma_H = 20 \cdot 10^{-24}$ cm^2), T_0, T_1,... are terms depending on the energy of the neutrons and the molecules as well as the Maxwellian velocity distribution of the neutrons, $\mu = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3)$, and μ_1, μ_2, μ_3 are the characteristic values of a tensor constructed in the following manner:

We define a so called mass tensor of a molecule (M) by

$$(M^{-1})_{ii} = \left(\frac{r_j^2}{I_k} + \frac{r_k^2}{I_j} + \frac{1}{M_0}\right), \qquad (M^{-1})_{ij} = -\frac{r_i r_j}{I_k},$$

where $i, j, k = 1, 2, 3$; r_1, r_2, r_3 are the coordinates of the proton as referred to the principal axes of intertia: I_1, I_2, I_3 are the characteristic values of the inertia tensor of the molecule, M_0 is the mass of the molecule.

We define the dimensionless tensor $n = m M^{-1}$, where m is the mass of the proton. The characteristic values of this tensor are n_1, n_2, n_3.

Now, we define μ_1, μ_2, μ_3 by

$$\mu_1 = \frac{1}{1+n_1}, \qquad \mu_2 = \frac{1}{1+n_2}, \qquad \mu_3 = \frac{1}{1+n_3}.$$

Comparison with experiment shows that the theory of Sachs and Teller may be also applied to the scattering of slow neutrons by some liquids. In this case, however, one must replace $(T_0 + T_1 + \ldots)$ in the equation of Sachs and Teller by 1, so that

$$\sigma = \mu_{\infty}(\mu_1 \mu_2 \mu_3 \mu)^{1/2}.$$

For the H_2 molecules for example: $n_1 = 0{,}5$, $n_2 = 1$, $n_3 = 1$

$$\sigma_{H_2} = 48{,}6 \cdot 10^{-24} \text{ cm}^2$$

and the value obtained by Carrol [2] is $\sigma_{H_2} = 48 \cdot 10^{-24}$ cm^2.

Similarly for H_2O molecules: $n_1 = 0$, $n_2 = 0{,}535$, $n_3 = 1$

$$\sigma_{H_2O} = 81{,}1 \cdot 10^{-24} \text{ cm}^2$$

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[1] R. G. Sachs and E. Teller, Phys. Rev. **60**, 18 (1941).

[2] H. Carrol, Phys. Rev. **60**, 702 (1941).

and the value obtained by Rossel [3] is $\sigma_{H_2O} = 85{,}5 \cdot 10^{-24}$ cm², and by the author [4] $\sigma_{H_2O} = 81{,}9 \cdot 10^{-24}$ cm².

In his work on the influence of the electrolytic dissociation and the hydratation of H_2SO_4 molecules for the scattering of thermal nentrons the author obtained the value $(1{,}26 \pm 0{,}03)$ cm^{-1} for the absorption coefficient of thermal neutrons in the pure H_2SO_4 [4]. It seems to be possible to obtain this value theoretically from the theory of Sachs and Teller.

The structural formula for the H_2SO_4 molecule is

$$\begin{matrix} \text{H-O} \\ \text{H-O} \end{matrix} \!\!>\!\!\text{S}\!\!<\!\!\begin{matrix} \text{O} \\ \text{O} \end{matrix}$$

There are many chemical reasons for the existence of two OH groups in the H_2SO_4 molecule. The hydrogen atoms of these OH groups have probably the possibility of rotation arround the SO axis and the thermal neutron scattered by these hydrogen nuclei may produce this rotation.

The characteristic values of the tensor n belonging to these OH group are $n_1 = 0$, $n_2 = 0$, $n_3 = 1$, therefore

$$\sigma = 51{,}2 \cdot 10^{-24}/\text{proton}.$$

The theoretically obtained cross section for the whole molecule is then

$$\sigma_{H_2SO_4} = (51{,}2 + 51{,}2 + 1{,}5 + 4 \cdot 3{,}8) \cdot 10^{-24}\ \text{cm}^2 = 119{,}1 \cdot 10^{-24}\ \text{cm}^2$$

as $\sigma_S = 1{,}5 \cdot 10^{-24}$ cm² and $\sigma_O = 3{,}8 \cdot 10^{-24}$ cm².

The absorption coefficient is

$$\mu = \frac{\sigma \varrho}{M}$$

where ϱ is the density of H_2SO_4 and M the mass of the H_2SO_4 molecule. Inserting $\sigma = \sigma_{H_2SO_4}$, we get

$$\mu = 1{,}29\ \text{cm}^{-1}$$

in agreement with the experimentally obtained value.

A Remark on the Dependence of the Cross-section for Pair Production by Photons on the Atomic Number

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October 10, 1952

In allmost all papers concerning electron pair creation by photons in the Coulomb field of the nucleus only terms proportional to the square of the atomic number Z are calculated. It seems that only Jäger and Hulme [1] proceeded further

[3] J. Rossel, Helv. Phys. Acta **20**, 105 (1947).

[4] J. A. Janik, Acta Phys. Polonica **11**, 146 (1952).

[1] Jäger J. C, and Hulme H. R., Proc. Roy. Soc., **153**, 443 (1936) see also Groshev L. V., Trud. Fiz. Inst. III, 118 (1945).

and got also the Z^4-term. Their formula for the total cross-section for pair creation has the form

$$\sigma = a\left(\frac{Z}{137}\right)^2 + b\left(\frac{Z}{137}\right)^4$$

where a and b are constants.

The author tried to calculate the terms proportional to higher powers of Z than the second one using Dysons [2] formalism of quantum electrodynamics, and neglecting radiation reaction and screening. The S-matrix for this process including terms proportional to Z and Z^2 is

$$U(\infty) = 1 - \int\int d^4x_1\, d^4x_2\, P(H^e(x_1)\, H^i(x_2))$$

$$+ \frac{i}{2}\int\int\int d^4x_1\, d^4x_2\, d^4x_3\, P(H^e(x_1)\, H^e(x_2)\, H^i(x_3)) = 1 + U_1 + U_2,$$

where

$$H^i(x) = -ie\,\bar{\psi}(x)\,\gamma^\mu\,\psi(x)\,A^\mu(x), \quad H^e = -ie\,\bar{\psi}(x)\,\gamma^\mu\,\psi(x)\,A^\mu_{\mathrm{ext}}(x)$$

$$A^\mu(x) = A_{\nu\hat{k}}\, e^{-ik_\nu x^\nu}, \quad A_{\nu\hat{k}} = e_\nu\, b_{\vec{k}}$$

and e_ν is the polarization vector, $b_{\hat{k}}$ — the annihilation operator of photons of momentum $\vec{k}$ and

$$A^i_{\mathrm{ext}} = 0, \quad A^0_{\mathrm{ext}} = \frac{Ze}{r}$$

are the components of the external Coulomb potential.

Feynman graphs for U_1 are given in Fig. 1, a, b, and for U_2 in Fig. 1, c, d, e.

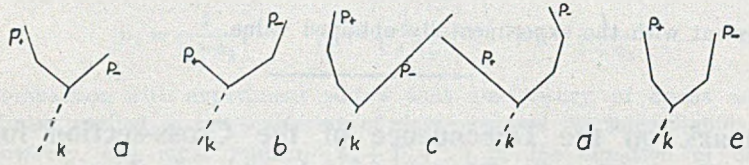

Fig. 1

Computing $U_1 + U_2$, one gets in the extreme relativistic case

$$U_1 + U_2 = e_l\,\{A(p^\lambda_- - k^\lambda)\,\bar{\psi}_{p+}\,\gamma^{4\lambda l}\,\psi_{p-}$$

$$+ B(p^\lambda_+ - k^\lambda_+)\,\bar{\psi}_{p+}\,\gamma^{l\lambda 4}\,\psi_{p-} + C(p^\lambda_+ - k^\lambda)\,B^\mu\,\bar{\psi}_{p+}\,\gamma^{l\lambda 4\mu 4}\,\psi_{p-} \qquad (1)$$

$$+ D(p^\lambda_- - k^\lambda)\,D^\mu\,\bar{\psi}_{p+}\,\gamma^{4\mu 4\lambda l}\,\psi_{p-} + E\,F^{\lambda\mu}\,\psi_{p+}\,\gamma^{4\lambda l\mu 4}\,\psi_{p-}$$

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[2] Dyson J. P., Phys. Rev. **75**, 486 (1949) and **75**, 1736 (1949).

where $\gamma^{\lambda\mu} = \gamma^\lambda \gamma^\mu \gamma^4$ etc. $i, k = 1, 2, 3$; $\lambda, \mu = 1, 2, 3, 4$, $\bar{\psi}_{p+}$, ψ_{p-} are creation and annihilation operators for electrons with the corresponding momenta $\vec{p}_+$, $\vec{p}_-$ and

$$A = \frac{4\pi^2 e^3 Z b \hat{k}}{(\vec{p}_+ + \vec{p}_- - \vec{k})^2 k^\mu p_-^\mu} i, \quad B = \frac{4\pi^2 e^3 Z b \hat{s}}{(\vec{p}_+ + \vec{p}_- - \vec{k})^2 k^\mu p_+^\mu} i,$$

$$C = -\frac{2\pi Z^2 e^5 b \hat{s}}{p_- k^\mu p_+^\mu} i, \quad D = +\frac{2\pi Z^2 e^5 b \hat{s}}{p_+ k^\mu p_-^\mu} i, \quad E = \frac{4\pi^2 Z^2 e^5}{p_+} i.$$

$$B^\mu = \oint d\Omega \frac{s^\mu}{(\vec{s} + \vec{p}_-)^2 + \varkappa^2} \frac{1}{(\vec{s} - \vec{p}_+ + \vec{k})^2 + \varkappa^2}, \qquad s^\mu = (\vec{s}, ip_-), \quad |\vec{s}| = p_-.$$

D^μ ditto changing $\vec{p}_+ \rightleftarrows \vec{p}_-$,

$$F^{\mu\nu} = \oint d\Omega \frac{1}{(\vec{q} - \vec{p}_+)^2 + \varkappa^2} \frac{1}{(\vec{q} - \vec{p} + \vec{k})^2 + \varkappa^2} \frac{(k^\mu - q^\mu) q^\nu}{|\vec{k} - \vec{q}|\, p_+}. \qquad |\vec{q}| = p_+,$$

where $d\Omega = \sin\vartheta\, d\vartheta\, d\Phi$. Here an auxiliary mass $\varkappa$ has been added to make the integrals convergent. The integrals B^μ and C^μ can be reduced to ellipti cand elementary integrals.

Squaring (1) and summing over both directions of polarization of the photons one gets terms proportional to Z^2 and Z^4, whereas the terms proportional to Z^3 vanish. The terms proportional to Z^2 give exactly the Bethe-Heitler formula [3].

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[3] Bethe H. A. and Heitler W., Proc. Roy. Soc. **146**, 83 (1934).

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