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NOTE ON ELECTRODYNAMICS WITHOUT POTENTIALS

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The Hamiltonian of Dirac's new electrodynamics formulated without potentials is investigated.

Dirac (1951) has formulated his new classical electrodynamics in terms of the potentials A_β. He has given a Lagrangian and a Hamiltonian in which A_β play the role of field variables.

Infeld and Plebański (1952) have shown that for a fairly general class of electrodynamics a formulation can be given which makes no use of potentials but instead works with field strengths throughout. They consider the six components of a skewsymmetric tensor

$$f_{\alpha\beta} \equiv -f_{\beta\alpha} \qquad (\alpha, \beta = 0, 1, 2, 3). \tag{1}$$

as field variables and assume a Lagrangian density which depends on $f_{\alpha\beta}$ and their first derivatives. By varying $f_{\alpha\beta}$ they get Euler-Lagrange equations which are equivalent to those of the theories which introduce potentials. They obtain as a special case Dirac's new electrodynamics, the Proca field and the pure Maxwell field. It is the purpose of the present note to examine the Hamiltonian formulation of the field theories working with field strengths $f_{\alpha\beta}$ instead of potentials. We take for definiteness the following Lagrangian density of Dirac's new electrodynamics

$$\mathcal{L} \equiv \frac{1}{4} f_{\alpha\beta} f^{\alpha\beta} - k\sqrt{f_{\alpha\varrho}\, {}^{e} f^{\alpha\sigma}{}_{,\sigma}}, \qquad k = \frac{mc^2}{e},$$

that is

$$\mathcal{L} \equiv -F - \varrho. \tag{2}$$

It differs only in sign from that used by Infeld and Plebański (1952). By varying $f_{\alpha\beta}$ Infeld and Plebański get the equations

$$f_{\alpha\beta} = \frac{\partial}{\partial x^\alpha} \frac{k^2 f_{\beta\sigma}{}^{\sigma}}{\varrho} - \frac{\partial}{\partial x^\beta} \frac{k^2 f_{\alpha\sigma}{}^{\sigma}}{\varrho}, \tag{3}$$

which are precisely Dirac's new equations

$$f_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}, \tag{4}$$

$$A_a A^a = k^2, \tag{4'}$$

$$f^{a\beta}{}_{,\beta} = \lambda A^a, \tag{4''}$$

if we only put

$$A_a = \frac{k^2 f_{a\sigma,}{}^{\sigma}}{\varrho}, \qquad \lambda = \frac{\varrho}{k^2}. \tag{5}$$

Since the six $f_{a\beta}$ play now the role of field variables, we introduce canonical momenta conjugate to them

$$p^{a\beta} = \frac{\partial \mathcal{L}}{\partial f_{a\beta,o}}. \tag{6}$$

We shall call $f_{a\beta,o}$ the velocities. In the canonical formulation the time axis x_0 is preferred. For the sake of simplicity we do not intend to introduce the parameter formalism (which has been introduced, e.g., by Dirac 1951). We see at once that in view of the fact that derivatives of $f_{a\beta}$ appear only in the combination $f_{a\varrho,}{}^{\varrho}$, the velocities $f_{mn,o}$ where $m,n = 1, 2, 3$, do not appear at all. It follows that the momenta conjugate to f_{mn} vanish identically. Therefore we have a situation in which three Φ equations in the sense of Dirac (1950) exist. We proceed to construct the Hamiltonian

$$H \equiv \int d^3x \left(\frac{1}{2} p^{a\beta} f_{a\beta,o} - \mathcal{L}\right)$$

$$\equiv \int d^3x \left(\frac{1}{2} p^{mn} f_{mn,o} + p^{mo} f_{mo,o} - \frac{1}{4} f_{a\beta} f^{a\beta} + k \sqrt{f_{a\varrho,}{}^{\varrho} f^{a\sigma}{}_{,\sigma}}\right). \tag{7}$$

The velocities $f_{mo,o}$ can be expressed in terms of the spatial derivatives of the variables $f_{a\beta}$ and momenta p_{mo}. The velocities $f_{mn,o}$ remain undeterminate but they are multiplied in the Hamiltonian by p^{mn} which vanish in the weak sense. Using strong equations, the Hamiltonian can be put into the form

$$H \equiv \int d^3x \left(\frac{1}{2} p^{mn} f_{mn,o} - \frac{1}{4} f_{a\beta} f^{a\beta} + \sqrt{k^2 - p_{no} p^{no}}\, f_{or,}{}^{r} + p_{so,r} f^{sr}\right). \tag{8}$$

The Poisson brackets (P.B.) between the field variables and their canonical conjugates are as usual

$$[f_{\mu\nu}, p^{\alpha\beta'}] = (\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu})\delta(\vec{x} - \vec{x}'). \tag{9}$$

The Hamiltonian (8) gives the proper equations of motion (i.e. equivalent to Lagrange equations) for $f_{\mu\nu}$ and p_{mo} but for p_{mn} which should be permanently zero during the motion we obtain

$$\dot{p}_{mn} = [p_{mn}, H] = f_{mn} - p_{mo,n} + p_{no,m}. \tag{10}$$

To have $\dot{p}_{mn} = 0$ we must put

$$f_{mn} - p_{mo,n} + p_{no,m} = 0. \tag{11}$$

We have here three χ equations. But examining their P.B. with the $\Phi's$, we see that they do not vanish:

$$[f_{mn} - p_{mo,n} + p_{no,m}, p^{rs'}] = (\delta^{r}_{m}\delta^{s}_{n} - \delta^{r}_{n}\delta^{s}_{m})\delta(\vec{x} - \vec{x}'). \tag{12}$$

So our Φ and χ are not first class. Dirac (1950) has developed the general theory to deal with such cases. On applying it we see that we have now to eliminate the p^{mn} and f_{mn} entirely from the theory, i.e. to put p^{mn} strongly equal to zero and to express f_{mn} by the $p_{mo,n} - p_{no,m}$ from (11). The present example is a good illustration of Dirac's method, but it is so simple that one could be inclined to make the elimination at once. The acquaintance with Dirac's Canadian papers (Dirac 1950, 1951) is rather necessary for understanding the procedure.

After the elimination we have only three field variables f_{mo} and three conjugate momenta p^{mo} between which the P.B. are

$$[f_{mo}, p^{no'}] = \delta_m^n \, \delta(\vec{x} - \vec{x}'). \tag{13}$$

The Hamiltonian is now (after splitting off a divergence term)

$$H \equiv \int d^3x \left\{ -\frac{1}{2} f_{no} f^{no} + \frac{1}{4} (p_{ro,n} - p_{no,r})(p^{ro,n} - p^{no,r}) \right.$$
$$\left. - \sqrt{k^2 - p_{no} p^{no}} \, f_{ro,}{}^{r} \right\}. \tag{14}$$

Notice that in consequence of our elimination spatial derivatives of momenta appear in (14). This suggest to make a canonical transformation

$$A^n = -p^{no}, \qquad B_n = f_{no}, \tag{15}$$

where the A^n are the field variables and B_n the conjugate momenta. The P.B. between them are

$$[A^n, B_r'] = \delta_r^n \, \delta(\vec{x} - \vec{x}').$$

If we introduce the abbreviation

$$F_{rs} \equiv A_{s,r} - A_{r,s},$$

the Hamiltonian becomes

$$H \equiv \int d^3x \left(\frac{1}{4} F_{rs} F^{rs} - \frac{1}{2} B_r B^r - \sqrt{k^2 - A_s A^s} \, B_{r,}{}^{r} \right).$$

which is precisely the Hamiltonian given by Dirac (1951) for his new electrodynamics.

We see therefore: when starting with field strengths instead of potentials as field variables we can work out the Hamiltonian formalism for Dirac's new electrodynamics, but then second class Φ and χ equations appear in the theory. We must get rid of them before we can pass to quantum theory. While performing the necessary elimination we come back to a formulation precisely equivalent to that of Dirac.

The special form of Lagrangian density here investigated is not essential; the Proca field is even simpler to handle.

I wish to express my hearty thanks to Professor L. Infeld for suggesting this problem and many stimulating discussions.

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М. Суффчинский, *К электродинамике без потенциалов.*

Исследован гамильтониан новой электродинамики Дирака, сформулированной без потенциалов. Получена эквивалентность с первоначальной формулировкой Дирака.

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RELATIVISIERUNG DER THEORIE DER STOCHASTISCHEN PROZESSE

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(Eingegangen am 28. Oktober 1952)

Es werden die Grundintegralgleichungen der stochastischen Prozesse und zwar die Smoluchowski-Kolmogorow-Chapmanschen, so wie die, die Erhaltung der Wahrscheinlichkeit ausdrückenden, Integralgleichungen auf eine Lorentz-invariante Form gebracht. Es werden dann aus diesen ihnen äquivalente relativistische Differentialgleichungen abgeleitet. Aus der Diskussion der von Kolmogorow angesetzten Bedingungen geht die Unhaltbarkeit dieser Bedingungen im relativistischen Falle hervor. Es werden die Kolmogorowschen Gleichungen auf eine mit der Relativitätstheorie verträgliche Form gebracht und es wird gezeigt, dass diese mit den früher genannten relativistischen Differentialgleichungen übereinstimmen.

1. Einleitung

In den letzten Jahrzehnten (1900—1952) entwickelte sich stark ein neuer Teil der Wahrscheinlichkeitsrechnung, der über die Rahmen der klassischen Theorie hinauswuchs und welcher — im Grunde genommen — den neuen Fragen der Physik und Technik entsprungen ist. In den Arbeiten der Physiker: M. Smoluchowski (1913, 1915, 1927), Planck (1917) und Fokker (1914) wurden die Grundlagen der neuen Theorie geschaffen. Es handelte sich dabei um Wahrscheinlichkeitsaussagen, die ausser der stochastischen Variablen noch von einem oder mehreren Parametern abhängen. Am wichtigsten für die Physik sind offensichtlich die Wahrscheinlichkeitsschemata, wo der Parameter als Zeit gedeutet werden kann, also die zeitlich verlaufenden Prozesse. Dank den Arbeiten von Bachelier (1900, 1910), Markow (1907, 1911, 1912), Kolmogorow (1931, 1933), Feller (1937, 1939), Khintschin (1933, 1938), Chapman und anderen wurde die Theorie beträchtlich ausgebaut, wobei man ihr den Namen: „Theorie der stochastischen Prozesse", verliehen hat. Man könnte sie die Kinematik der Wahrscheinlichkeitsrechnung nennen, während man die klassische Wahrscheinlichkeitsrechnung als Statik bezeichnet.

Wir betrachten im folgenden die bedingte Wahrscheinlichkeit, dass ein Ereignis x welches der Menge E angehört zur Zeit t stattfindet, vorausgesetzt dass zur Zeit t_0 das Ereignis x_0 stattfand. Wir haben also

$$P(x \varepsilon E, t; x_0, t_0) \qquad (t \geqslant t_0).$$

Einen solchen Wahrscheinlichkeitsprozess nennen wir mit Kolmogorow einen stochastisch-definiten Prozess, wobei wir voraussetzen, dass dieser Prozess von

der Vorgeschichte für $t < t_0$ unabhängig ist. Man kann immer den Einfluss der Vorgeschichte durch Einführung neuer Parameter vermeiden (z.B. in der Mechanik durch Einführung der Impulskoordinaten). Wir werden uns im folgenden der Wahrscheinlichkeitsdichte statt der Wahrscheinlichkeit selbst, also

$$\varrho(x,t;x_0,t_0)$$

bedienen.

Es liegt nahe, nach der Beziehung der Theorie der stochastischen Prozesse zur Relativitätstheorie zu fragen. Diese Frage wurde mir von Prof. Dr. Ingarden, im Sommer 1952 gestellt. Die vorliegende Arbeit ist ein Versuch das Problem der Relativisierung der stochastischen Prozesse zu lösen.

Es ist zweckmässig das ganze Problem in zwei Teile zu trennen und zwar:

1) Das Wahrscheinlichkeitsproblem in dem die Wahrscheinlichkeitsdichte zwar von den Ortskoordinaten und der Zeit, ferner aber noch von einer Variablen, die eine andere Grösse z.B. die Teilchenanzahl representiert und welche eben als stochastische Variable erscheint, abhängt, während Zeit- und Ortskoordinaten als blosse Parameter auftreten. In diesem Fall, der von Wichtigkeit für die kosmische Strahlung ist, ist es sinnvoll die Theorie der stochastischen Prozesse mit einem Parameter auf die Prozesse mit vier Parametern, die mit den Prinzipien der Relativitätstheorie im Einklang ist, zu erweitern.

2) Das Wahrscheinlichkeitsproblem, wo die in der Wahrscheinlichkeitsdichte auftretenden Ortskoordinaten die stochastischen Variablen bilden, während die Zeit ein Parameter bleibt. Ein Prozess dieser Art tritt z.B. in der Physik beim Übergang eines Teilchens von einem Ort x_0 zur Zeit t_0 zu einem anderen Ort x zur Zeit t auf, wobei nach der Wahrscheinlichkeit dieses Übergangs gefragt wird.

Wir beschränken uns im folgenden auf die unter 2) erwähnten Prozesse, während die Lösung des unter 1) genannten Problems keine grösseren Schwierigkeiten zu bieten scheint.

Wir legen unserer Theorie, wie üblich, die Smoluchowski-Kolmogorow-Chapmanschen Integralgleichungen zu Grunde, welche auch die Erhaltung der Wahrscheinlichkeit ausdrücken:

$$\varrho(x,t;x_0,t_0) = \int\limits_{\tau=const} \varrho(x,t;\xi,\tau)\,d\xi\,\varrho(\xi,\tau;x_0,t_0) \quad (t>\tau>t_0), \tag{1}$$

$$\int\limits_{\tau=const} \varrho(\xi,\tau;x_0,t_0)\,d\xi = 1 \quad (\tau \geqslant t_0), \tag{2}$$

wo $x=(x_1,x_2,x_3)$, $d\xi = d\xi_1 d\xi_2 d\xi_3$, $\varrho(x,t;x_0,t_0)$ die Übergangswahrscheinlichkeitsdichte $(x_0,t_0 \to x,t)$ bedeutet und das Integral über den ganzen Raum zu erstrecken ist.

Als dritte Gleichung fügen wir

$$\varrho(x,t) = \int_{\tau=const} \varrho(x,t;\,\xi,\tau)\,d\xi\,\varrho(\xi,\tau) \qquad (t \geqslant \tau) \tag{3}$$

hinzu, wo $\varrho(x,t)$ die Wahrscheinlichkeitsverteilungsfunktion zur Zeit t am Ort x bedeutet. Diese Gleichung können wir als Definition von $\varrho(x,t)$ durch $\varrho(x,t;\,\xi,\tau)$ betrachten. Aus ihr folgt unmittelbar wiederum die Gleichung (1).

Die Gleichungen (1)—(3) sind rein „probabilistische" Gleichungen, in denen keine „dynamischen" Elemente, die das betrachtete physikalische System charakterisieren, auftreten. Ihnen müssen also weitere, ihrem Ursprung nach „dynamische" Gleichungen beigefügt werden, die das Verhalten des Systems näher beschreiben.

Unter gewissen Bedingungen, welche die Anfangsgeschwindigkeit des Mittelwertes und der Streuung der Übergangswahrscheinlichkeitsverteilung betreffen, wurden von Kolmogorow (l. c.) aus (1) und (2) zwei Differentialgleichungen abgeleitet.[1]

Es ist unser Ziel die Intergralgleichungen auf eine Lorentz-invariante Form zu bringen und aus diesen dann ihnen äquivalente Differentialgleichungen abzuleiten. Wir bringen dann auch die Kolmogorowschen Gleichungen auf eine dem Relativitätsprinzip genügende Form und vergleichen sie mit den erhaltenen relativistischen Differentialgleichungen. Dann stellt sich heraus dass die Kolmogorowschen Gleichungen sich auf die relativistischen Differentialgleichungen zurückführen lassen. Auf diese Weise wird die erste Etappe der Relativisierung der stochastischen Prozesse abgeschlossen. Auf die „dynamischen" Gleichungen wird in dieser Arbeit nicht eingegangen. Sie sollen das Thema einer späteren Mitteilung bilden.

2. Relativistische Bedingungen

Da wir $\varrho(x,t;\,x_0,t_0)$ als die Wahrscheinlichkeitsdichte der Verschiebung eines Teilchens vom Raumzeitpunkt $(x_0,\,t_0)$ in den Raumzeitpunkt $(x,\,t)$ ansehen können, ist es augenscheinlich, dass für

$$|\,x - x_0\,| > c\,|\,t - t_0\,|$$

$$\varrho(x,t;\,x_0,\,t_0) = 0 \tag{4}$$

sein muss.

In der Fig. 1 bedeuten (a) und (b) die Lichtkegel, die sich bzw. auf die Raumzeitpunkte $(x_0,\,t_0)$ und $(x,\,t)$ beziehen. Das eingezeichnete Gebiet Ω entspricht dem gemeinsamen Gebiet der genannten Kegel. In der Gleichung (1) tritt die

[1] Diese Differentialgleichungen wurden von Fokker (l. c.) und Planck (l. c.) eingeführt.

Integration über den ganzen dreidimensionalen Raum auf, die der über die in Fig. 1 gezeichnete Hyperebene $\tau = \text{const.}$ erstreckten Integration entspricht. Da die subintegrale Funktion ausserhalb Ω verschwindet, so kommt bei der Integration nur dieser Teil von $\tau = \text{const.}$, der innerhalb Ω verläuft, in Betracht.

Wir betrachten die Wahrscheinlichkeit als eine Lorentz-invariante Grösse. Dann transformiert sich die Wahrscheinlichkeitsdichte wie die vierte Komponente eines Vierervektors.

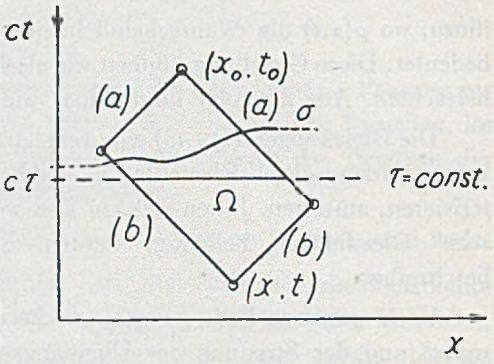

Fig. 1

3. Relativistisch-invariante Form der Grundintegralgleichungen

Es ist offensichtlich, dass die Gleichungen (1)—(3) ihrer Form nach weder Lorentz-in- noch ko-variant sind. Bei einer Lorentz-Transformation geht die Hyperebene $\tau = \text{const.}$ in eine andere Hyperebene über, welche aber gegenüber dem neuen Koordinatensystem nicht durch die Gleichung $\tau' = \text{const.}$ dargestellt wird. Ferner ist $\varrho(x,t;x_0,t_0)$, wie erwähnt, nicht invariant.

Um diese Mängel zu beseitigen ersetzen wir die τ-Hyperebene durch eine beliebige, raumartige Hyperfläche σ, die in (1) „zwischen" den Punkten (x_0, t_0) und (x, t), in (2) „unterhalb" (x_0, t_0) verläuft (s. Fig. 1). Es ist zu erwähnen, dass der Begriff: der Punkt (x_0, t_0) sei „früher" und der Punkt (x, t) sei „später" in bezug auf jeden beliebigen Punkt der Hyperfläche σ, der innerhalb des Gebietes Ω liegt, seinen Sinn weiterhin behält; es ist folglich ein der Lorentz-Transformation gegenüber invarianter Begriff. Wegen $\varrho(x,t;x_0,t_0) = 0$ für $|x - x_0| > c|t - t_0|$ kommen in unseren Erwägungen in (1) nur die Ω-Punkte, in (2) nur die innerhalb des (a)-Kegels gelegenen Punkte in Betracht.

Da $\varrho(x,t;x_0,t_0)$ sich wie die vierte Komponente eines Vierervektors transformiert, erweitern wir den Wahrscheinlichkeitsdichte-Begriff zu einem Wahrscheinlichkeitsvierervektor-Begriff $I_\nu(x,t;x_0,t_0)$ $(\nu = 1, 2, 3, 4)$. Entsprechend geht auch die Bedingung (4) in $I_\nu = 0$ für $|x - x_0| > c|t - t_0|$ über. Wir beschäftigen uns einstweilen nicht mit der Deutung der Grössen I_i $(i = 1, 2, 3)$, wir erwähnen nur dass sie einen dreidimensionalen Wahrscheinlichkeitsstrom bilden, auf dessen Deutung wir noch näher bei Erörterung der Kolmogorowschen Gleichung eingehen werden. Dabei ist $I_4(x,t;x_0,t_0)$ gleich $i\varrho(x,t;x_0,t_0)$ zu setzen. Dann liegt es nahe das Produkt

$$\varrho(\xi,\tau;x_0,t_0)\,d\xi$$

als ein skalares Produkt zweier Vierervektoren

$$(\xi,\tau;x_0,t_0)\,dS_\nu(\xi,\tau)$$

zu betrachten, wo dS_ν ein in der Richtung der positiven ct-Achse gerichtetes Hyperflächenelement bedeutet und die Komponenten

$$(-id\xi_2 d\xi_3 d\xi_4, -id\xi_3 d\xi_1 d\xi_4, -id\xi_1 d\xi_2 d\xi_4, -id\xi_1 d\xi_2 d\xi_3), \quad d\xi_4 = icdt$$

hat. Im Falle der τ-Hyperebene sind die Komponenten gleich, $(0, 0, 0, -id\xi)$. Wir haben also statt (1) und (2)

$$\varrho(x,t;x_0,t_0) = \int_\sigma \varrho(x,t;\xi,\tau) I_\nu(\xi,\tau;x_0,t_0)\, dS_\nu(\xi,\tau) \tag{5}$$

und

$$\int_\sigma I_\nu(\xi,\tau;x_0,t_0)\, dS_\nu(\xi,\tau) = 1\,.$$

Damit (5) Lorentz-kovariant ist, stellen wir ihr zur Seite weitere drei Gleichungen, nämlich

$$I_i(x,t;x_0,t_0) = \int_\sigma I_i(x,t;\xi,\tau) I_\nu(\xi,\tau;x_0,t_0)\, dS_\nu(\xi,\tau), \quad (i = 1, 2, 3).$$

Wir haben also im Ganzen folgendes System von fünf Gleichungen

$$I_\mu(x,t;x_0,t_0) = \int_\sigma I_\mu(x,t;\xi,\tau) I_\nu(\xi,\tau;x_0,t_0)\, dS_\nu(\xi,\tau)\,, \tag{6}$$

wo $\mu = 1, 2, 3, 4$, und σ „später" als $(x_0 t_0)$ und „früher" als (x,t) ist, und

$$\int_\sigma I_\nu(\xi,\tau;x_0,t_0)\, dS_\nu(\xi,\tau) = 1\,, \tag{7}$$

wo σ „nicht früher" als (x_0,t_0) ist.

Auf dieselbe Weise verallgemeinern wir den Begriff $\varrho(x,t)$ zu $I_\nu(x,t)$ $(\nu = 1, 2, 3, 4)$. Dann bekommt unsere Definitionsgleichung (3) die Form

$$I_\mu(x,t) = \int_\sigma I_\mu(x,t;\xi,\tau) I_\nu(\xi,\tau)\, dS_\nu(\xi,\tau) \tag{8}$$

für σ „nicht später" als (x,t) ist.

Beim Einführen der Vierervektoren $I_\nu(x,t;x_0,t_0)$ und $I_\nu(x,t)$ ist folgendes zu beachten: die Komponente $I_4 = i\varrho$, i ungeachtet, muss ihr Vorzeichen durchweg behalten. Es kommt darauf hinaus, dass das Vektorfeld $I_\nu(x,t;x_0,t_0)$ in einer Zeitrichtung — und zwar in unserem Fall in der positiven Zeitachsenrichtung — gerichtet ist. I_4 lässt sich als

$$\frac{\delta P}{\delta\omega}\, dx_4 \qquad \delta P\text{-Wahrscheinlichkeit}$$

ausdrücken, wo $\delta\omega$ ein vierdimensionales Raumelement bedeutet. Folglich können wir schreiben

$$I_\nu = \frac{\delta P}{\delta\omega}\, dx_\nu\,. \tag{9}$$

Aus dieser Schreibform ist es ersichtlich, dass δP proportional der Länge des I_ν-Vektors ist. Wählen wir ϱ immer positiv, d.h. das Vektorfeld in der positiven Richtung der ct-Achse gerichtet, dann folgt dass $\delta P > 0$, da $\delta\omega > 0$ ist. Wegen der Bedingung (4) ist das Vektorfeld $I_\nu(x,t;x_0,t_0)$ zeitartig und auf den (a)-Kegel beschränkt. Aus der Zeitartigkeit des Vektorfeldes schliessen wir, dass

$$\left(\frac{dx_i}{dt}\right)^2 < c^2 \qquad (i=1,2,3).$$

Die Wahl der Richtung des Vektorfeldes gegenüber der Zeitachse ist belanglos. Es ist aber von Wichtigkeit dass die Richtung des Vektorfeldes mit der Normale der σ-Flächen übereinstimmt, sonst werden nämlich die Grundgleichungen nicht erfüllt. Falls die Strömung und der σ-Normalvektor in der Richtung der negativen Zeitachse zeigen, dann bleibt zwar δP weiterhin positiv, doch $\varrho(x,t;x_0,t_0)$ wechselt das Vorzeichen.

4. Relativistisch-invariante Differentialgleichungen

Wir wollen nun die Integralgleichungen (6)—(8) in Differentialform bringen. Wir führen die abkürzende Bezeichnungen ein:

$$(x,ict) = x_\lambda \quad (\xi,ic\tau) = \xi_\lambda \quad (x_0,ict_0) = x_{0\lambda}.$$

Wir berechnen die funktionelle Ableitung aus dem Ausdruck (6) in einem beliebigen Raumzeitpunkte des Ω-Gebietes mit Ausnahme der Punkte x_λ und $x_{0\lambda}$. Da die linke Seite von (6) $I_\mu(x_\lambda;x_{0\lambda})$ von der Gestalt des σ unabhängig ist, haben wir für $\delta\omega(\xi_{0\lambda}) \to 0$

$$\lim \frac{1}{\delta\omega}\,\delta\left\{\int_\sigma I_\mu(x_\lambda;\xi_\lambda)\,I(\xi_\lambda;x_{0\lambda})\,dS_\nu(\xi_\lambda)\right\} = 0$$

und folglich

$$\frac{\partial I_\mu(x_\lambda;\xi_\lambda) I_\nu(\xi_\lambda;x_{0\lambda})}{\partial \xi_\nu} = 0, \qquad \xi_\lambda \in [\Omega - x_{0\lambda} - x_\lambda], \tag{10}$$

wo wir statt $\xi_{0\lambda}$ die ξ_λ's eingesetzt haben.

Ganz analog verläuft die Berechnung der funktionellen Ableitung für (7) und wir bekommen so die Differentialgleichung

$$\frac{\partial I_\nu(\xi_\lambda,x_{0\lambda})}{\partial \xi_\nu} = 0 \qquad \xi_\lambda \text{ „nicht früher“ als } x_{0\lambda}. \tag{11}$$

Diese Gleichung hat die Form einer Kontinuitätsgleichung in den ξ-Variablen. Auf Grund von (11) nimmt (10) folgende Gestalt an

$$I_\nu(\xi_\lambda;x_{0\lambda})\frac{\partial I_\mu(x_\lambda;\xi_\lambda)}{\partial \xi_\nu} = 0\,. \tag{12}$$

Die Gleichungen (11) und (12) bilden das gesuchte System, welches den Integralgleichungen (6) und (7) äquivalent ist.

Aus den Gleichungen (8) folgen zwar die Differentialgleichungen

$$\frac{\partial I_\mu(x_\lambda;\xi_\lambda)}{\partial \xi_\nu} I_\nu(\xi_\lambda) = 0 \qquad x_\lambda \text{ „nicht früher“ als } \xi_\lambda$$

da

$$\frac{\partial I_\nu(\xi_\lambda)}{\partial \xi_\nu} = 0$$

ist, aber dieselben Gleichungen folgen auch sofort aus (12), indem man sie mit $I_\varrho(x_{o\lambda})\,dS_\varrho(x_{o\lambda})$ multipliziert und dann über eine raumartige, „nichtspätere“ als ξ_λ, sonst aber beliebige σ-Fläche über $x_{0\lambda}$ integriert.

Wir haben also fünf Gleichungen, während zur Angabe der vier Grössen $I\,(x_\lambda;x_{o\lambda})$ acht Gleichungen nötig sind. Unser Gleichungsystem bedarf also einer Ergänzung durch „dynamische“ Gleichungen.

5. Die Bedingungen der Anwendbarkeit der Kolmogorowschen Gleichungen

Kolmogorow leitete l. c. aus den Integralgleichungen (1) und (2) zwei Differentialgleichungen ab, wobei gewisse Integralbedingungen der ϱ-Funktion gestellt worden sind. Wir führen diese Bedingungen an:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|\xi - X| < \delta} (\xi_i - x_i)\varrho(\xi, t+\Delta t; x, t)\,d\xi = a_i(x,t) \qquad (i = 1,2,3). \tag{13}$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|\xi - X| < \delta} (\xi_i - x_i)(\xi_k - x_k)\varrho(\xi, t+\Delta t; x, t)\,d\xi = b_{ik}(x,t) \qquad (k = 1,2,3),$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|\xi - X| < \delta} \prod_{i=1}^{3} (\xi_i - x_i)^{\nu_i} \varrho(\xi, t+\Delta t; x, t)\,d\xi = 0$$

$$(n > 2,\ \nu_i = 0,1,2,\ldots), \quad \delta > 0, \text{ beliebig.}$$

Sie besagen, dass der Mittelwert und die Streuung zur Zeit $(t + \Delta t)$

$$E_i(\xi, t+\Delta t; x, t) = x_i + a_i(x,t)\,\Delta t$$

bzw.

$$\sigma_{ik}^2(\xi, t+\Delta t; x, t) = b_{ik}(x,t)\,\Delta t - a_i a_k \Delta t^2$$

beträgt, während die Geschwindigkeit der übrigen Momente klein ist von der Ordnung $O(\Delta t)$.

Unter diesen Bedingungen erhalten wir folgende Differentialgleichungen

$$\frac{\partial \varrho(\xi,\tau;x,t)}{\partial t} + a_i(x,t)\frac{\partial \varrho(\xi,\tau;x,t)}{\partial x_i} + \frac{1}{2} b_{ik}(x,t)\frac{\partial^2 \varrho(\xi,\tau;x,t)}{\partial x_i \partial x_k} = 0, \tag{14}$$

$$\frac{\partial \varrho(\xi,\tau;x,t)}{\partial \tau} + \frac{\partial}{\partial \xi_i}[a_i(\xi,\tau)\varrho(\xi,\tau;x,t)] - \frac{1}{2}\frac{\partial^2}{\partial \xi;\partial \xi_k}[b_{ik}(\xi,\tau)\varrho(\xi,\tau;a)] = 0.$$

Wir wollen den Sinn von $a_i(x,t)$ plausibel machen. Wir bilden die funktionelle Ableitung

$$\lim_{\delta\omega(x_\lambda)\to 0} \frac{1}{\delta\omega}\,\delta\left\{\int_\sigma \xi_\mu I_\nu(\xi_\lambda;x_{0\lambda})\,dS_\nu(\xi_\lambda)\right.,$$

wo x_λ verschieden von $x_{0\lambda}$ und σ „später" als $x_{0\lambda}$ ist. Wir erhalten wegen (11) die Gleichung

$$\lim_{\delta\omega(x_\lambda)\to 0} \frac{1}{\delta\omega}\,\delta\left\{\int_\sigma \xi_\mu I_\nu(\xi_\lambda;x_{0\lambda})\,dS_\nu(\xi_\lambda) = I_\mu(x_\lambda;x_{0\lambda}),\right. \tag{15}$$

wo x_λ „später" als $x_{0\lambda}$ ist. Wir wollen nun die funktionelle Ableitung im singulären Punkte $x_{0\lambda}$ finden. Es ist

$$\lim_{\delta\omega(x_\lambda)\to 0} \frac{1}{\delta\omega}\left\{\int_\sigma \xi_\mu I_\nu(\xi_\lambda;x_{0\lambda})\,dS_\nu(\xi_\lambda) - \int_\sigma \xi_\mu I_\nu(\xi_\lambda;x_{0\lambda})\,dS_\nu(\xi_\lambda)\right\}, \tag{16}$$

wo σ_0 den $x_{0\lambda}$-Punkt enthält. Da $I_\nu(\xi_\lambda;\, x_{0\lambda}) = 0$ für alle σ_0-Punkte ausser für $\xi_\lambda = x_{0\lambda}$ ist und

$$\int_{\sigma_0} I_\nu(\xi_\lambda;\, x_{0\lambda})\,dS_\nu(\xi_\lambda) = I_\nu(x_{0\lambda};x_{0\lambda})\,dS_\nu(x_{0\lambda}) = 1$$

(s. Gleichung (2)), so ist

$$\int_{\sigma_0} \xi_\mu I_\nu(\xi_\lambda;x_{0\lambda})\,dS_\nu(\xi_\lambda) = x_{0\mu}.$$

Wir sehen also, dass (16) in

$$\lim_{\substack{\delta\omega\to(x_{0\lambda})\\ \delta\omega\to 0}} \frac{1}{\delta\omega}\left\{\int_{\sigma^1} (\xi_\mu - x_{0\mu}) I_\nu(\xi_\lambda;x_{0\lambda})\,dS_\nu(\xi_\lambda)\right\}$$

übergeht. Wir führen folgende Bezeichnungen ein

$$\delta\omega(x_{0\lambda}) = \delta S(x_{0\lambda},\sigma_0)\,\delta n,$$

$$\frac{1}{\delta n}\left\{\int_{\sigma^1} (\xi_\mu - x_{0\mu}) I_\nu(\xi_\lambda;x_{0\lambda})\,dS_\nu(\xi_\lambda)\right\} = \frac{1}{c}\,\mathcal{L}_\mu(x_{0\lambda};\sigma_0),$$

$$\substack{\delta n\to 0\\ \sigma^1\to\sigma_0}$$

wo $\delta S(x_{0\lambda},\sigma_0)$ das Flächenelement von σ_0 am Ort $x_{0\lambda}$ und δn das Differential der Normalen zur σ_0-Fläche bedeuten. Mit diesen Bezeichnungen geht (16) weiter in

$$\lim_{\delta\omega\to 0} \frac{1}{\delta S(x_{0\lambda},\sigma_0)}\,\frac{1}{c}\,\mathcal{L}_\mu(x_{0\lambda},\sigma_0)$$

über. Der Ausdruck (16) gibt aber laut (15) den Wert von $I_\mu(x_0;x_0)$ an (hier wurde der Gaussche Satz angewandt, trotzdem die σ_0-Fläche eine Singularität für I_ν aufweist). Es folgt

$$\lim_{\delta\omega \to 0} \frac{1}{\delta S(x_{0\lambda}, \sigma_0)} \frac{1}{c} \mathcal{L}_\mu(x_{0\lambda}, \sigma_0) = I_\mu(x_{0\lambda}; x_{0\lambda}). \tag{17}$$

Man sieht sofort, dass falls im Limes die $\mathcal{L}_\mu(x_0, \sigma_0)$ von Null verschieden und endlich sind, dann strebt $I_\mu(x_{0\lambda}; x_{0\lambda})$ der Unendlichkeit zu. Führt man statt $\frac{1}{\delta S(x_{0\lambda}, \sigma_{0\lambda})}$ eine dreidimensionale Diracsche δ^3-Funktion ein, welche ausser von x_0 auch von σ_0 abhängt, so bekommt (17) die Gestalt

$$\frac{1}{c} \delta^3(0; \sigma_0) \mathcal{L}_\mu(x_{0\lambda}, \sigma_0) = I_\mu(x_{0\lambda}; x_{0\lambda}).$$

In dem von Kolmogorow behandelten Falle ist die σ_0-Fläche die τ_0-Hyperebene und σ' entsprechend die $(t_0 + \Delta t)$-Hyperebene. Dann geht auch δn in $c\Delta t$ über und folglich

$$\mathcal{L}_i(x_{0\lambda}, t_0) \to a_i(x_{0\lambda}) \qquad (i = 1, 2, 3.),$$

$$\mathcal{L}_4(x_{0\lambda}, t_0) \to a_4(x_{0\lambda}) = \frac{ic}{\Delta t} \int_{t_0 + \Delta t} \Delta t \, \varrho(\xi_\lambda; x_{0\lambda}) \, d\xi = ic,$$

$$\delta^3(0; \sigma_0) \to \delta^3(0) \delta_{t_0 t}.$$

Wir haben also

$$\frac{1}{c} \delta^3(0) a_\mu(x_{0\lambda}) = I_\mu(x_{0\lambda}; x_{0\lambda})$$

oder

$$\frac{1}{c} \varrho(x_{0\lambda}; x_{0\lambda}) a_\mu(x_{0\lambda}) = I_\mu(x_{0\lambda}; x_{0\lambda}). \tag{18}$$

Die Grössen $\mathcal{L}_\mu(x_{0\lambda}; \sigma_{0\lambda})$ sind zwar Lorentz-invariant definiert, sie hängen aber von der Art der σ-Fläche ab, obgleich sie auf ihr nicht „senkrecht" stehen. Dasselbe betrifft die δ^3-Funktion.

Ähnlich berechnen wir die funktionelle Ableitung

$$\lim_{\delta\omega(x_\lambda) \to 0} \frac{1}{\delta\omega} \delta \left\{ \int_\sigma (\xi_\mu - x_{0\mu})(\xi_\varkappa - x_{0\varkappa}) I_\nu(\xi_\lambda; x_{0\lambda}) \, dS_\nu(\xi_\lambda) \right.$$

$$= (x_\mu - x_{0\mu}) I_\nu(x_\lambda; x_{0\lambda}) + (x_\varkappa - x_{0\varkappa}) I_\mu(x_\lambda; x_{0\lambda}),$$

wo x_μ „später" ist als $x_{0\mu}$. Im Grenzfall $x_\mu \to x_{0\mu}$ führen wir, analog wie oben, die Bezeichnung ein

$$\lim_{\substack{\delta\omega \to 0 \\ \delta n \to 0}} \frac{1}{\delta n} \int_{\substack{\sigma' \\ \sigma' \to \sigma_0}} (\xi_\mu - x_{\mu t})(\xi_\varkappa - x_{0\varkappa}) I_\nu(\xi_\lambda; x_{0\lambda}) \, dS_\nu(\xi_\lambda) = \frac{1}{c} \beta_{\mu\varkappa}(x_{0\lambda}, \sigma_0),$$

wo $\beta_{\mu\kappa}(x_{0\lambda}; \sigma_0)$ im Kolmogorow-Fall in die $b_{il}(x_{0\lambda})$ $(i, l = 1, 2, 3)$ übergehen. und

$$b_{i4} = \frac{ic}{\Delta t} \int_{t + \Delta t} (\xi_i - x_{0i}) \Delta t \varrho(\xi_\lambda; x_{0\lambda}) \, d\xi = ic \, \Delta t \, a_i(x_{0\lambda}) = O(\Delta t)$$

$$b_{44} = -\frac{c^2}{\Delta t}\int_{t+\Delta t} \Delta t^2\, \varrho(\xi_\lambda; x_{0\lambda})\, d\xi = -c^2 \Delta t = O(\Delta t).$$

Es ist also in diesem Fall

$$\lim_{\delta\omega \to 0} \frac{1}{\delta S}\frac{1}{c}\left[b_{\mu\varkappa}(x_{0\lambda}) - (x_\mu - x_{0\mu})\, a_\varkappa(x_{0\lambda}) - (x_\varkappa - x_{0\varkappa})\, a_\mu(x_{0\lambda})\right] = 0$$

und wenn a_ν endlich sind muss $b_{\mu\varkappa}$ höherer Ordnung klein sein als δS. Wir haben bei dieser Erörterung aus der Endlichkeit der Lichtgeschwindigkeit einen ständigen Gebrauch gemacht. Die Erörterung hätte sich anders gestaltet wenn wir, wie es in der klassischen Mechanik üblich ist, $c = \infty$ und $c\Delta t =$ endliche von Null verschiedene Grösse, gesetzt hätten.

Zu demselben Ergebnis kann man auch auf einem anderen Wege gelangen. Wir beschränken uns einstweilen auf eine Raumkoordinate und erweitern dann unmittelbar die Resultate auf mehrere Raumkoordinaten.

Wir haben

$$E(x, t_0 + \Delta t; x_0, t_0) = x_0 + a(x_0, t_0)\Delta t,$$

$$\sigma^2(x, t_0 + \Delta t; x_0, t_0) = b(x_0, t_0)\Delta t - a^2(x_0, t_0)\Delta t^2.$$

Im relativistischen Fall muss, laut (4),

$$a(x_0, t_0) \leqslant c,$$

$$\sigma^2(x, t_0 + \Delta t; x_0, t_0) \leqslant c^2 \Delta t^2$$

sein. Also

$$b\Delta t - a^2\Delta t^2 \leqslant c^2\Delta t^2,$$

woraus folgt, dass $b(x_0, t_0)$ von der Ordnung $O(\Delta t)$ klein sein muss. Im klassischen Falle, wo $c \to \infty$, $c\Delta t = L =$ endliche Zahl, $L \neq 0$, haben wir für begrenztes $a < M$ und $\Delta t \to 0$

$$b\Delta t - a^2\Delta t^2 \leqslant L^2, \qquad b\Delta t \leqslant L^2,$$

also ist b beliebiger Werte fähig.

Wir erweitern jetzt diese Auseinandersetzung auf den dreidimensionalen Fall. Wir haben

$$\sigma_{ik}^2 = b_{ik}\Delta t - a_i a_k \Delta t^2 \qquad (i, k = 1, 2, 3)$$

und bilden den Trägheitstensor

$$T = \begin{pmatrix} \sigma_{22} + \sigma_{33}, & -\sigma_{12}, & -\sigma_{13} \\ -\sigma_{21}, & \sigma_{11} + \sigma_{33}, & -\sigma_{23} \\ -\sigma_{31}, & -\sigma_{32}, & \sigma_{11} + \sigma_{22} \end{pmatrix}$$

$(i, k = 1, 2, 3)$. Die Streuung in der durch den Dreiereinheitsvektor n_i bestimmten Richtung ist durch

$$T_{ik} n_i n_k = r^2(n_j)$$

gegeben. Die grösste Streuung findet in der Richtung der kleinen Achse des Trägheitsellipsoides statt. Wir nennen diese Streuung r^2_{max}. Dann muss die Ungleichung

$$r^2_{max} \leqslant c^2 \Delta t^2$$

bestehen, woraus

$$T_{ik} n_i n_k = \varkappa^2(n_j) \leqslant r^2_{max} \leqslant c^2 \Delta t^2$$

folgt. Mit den abkürzenden Bezeichnungen

$$B = (b_{22} + b_{33}) n_1^2 + (b_{11} + b_{33}) n_2^2 + (b_{11} + b_{22}) n_3^2 - 2b_{12} n_1 n_2 - \ldots,$$
$$A = (a_2^2 + a_3^2) n_1^2 + (a_1^2 + a_3^2) n_2^2 + (a_1^2 + a_2^2) n_3^2 - 2a_1 a_2 n_1 n_2 - \ldots,$$

nimmt diese Ungleichung die Form an

$$B\Delta t - A\Delta t^2 \leqslant c^2 \Delta t^2.$$

Für $\Delta t \to 0$ strebt B gegen Null für einen willkürlich orientierten n_i-Vektor, da B eine nicht-negative Quadratform ist. Wegen

$$b_{22} = -b_{33}, \qquad b_{11} = -b_{33}, \qquad b_{22} = -b_{11}, \qquad b_{ik} = 0 \qquad (i \neq k)$$

folgt allgemein

$$b_{ik} = 0 \quad \text{für} \quad i, k = 1, 2, 3. \tag{19}$$

Damit ist unsere frühere Behauptung auf einem anderen Wege bewiesen. Die Kolmogorow-Bedingung für die Streuung lässt sich nicht mit den Prinzipien der Relativitätstheorie vereinbaren; diese Bedingung verlangt dass das erörterte Teilchen sich mit Überlichtgeschwindigkeit bewegt.

6. Der Zusammenhang der Kolmogorowschen Gleichung mit den relativistischen Differentialgleichungen

Setzen wir im Einklang mit den oben angeführten Erwägungen (vgl.. Gl. (19)) $b_{ik} = 0$, dann fallen in den Kolmogorowschen Differentialgleichungen die b_{ik} enthaltenden Glieder weg und (14) geht über in

$$\frac{\partial \varrho(\xi, \tau; x, t)}{\partial t} + a_i(x, t) \frac{\partial \varrho(\xi, \tau; x, t)}{\partial x_i} = 0,$$
$$\frac{\partial \varrho(\xi, \tau; x, t)}{\partial \tau} + \frac{\partial}{\partial \xi_i} [a_i(\xi, \tau) \varrho(\xi, \tau; x, t)] = 0. \tag{20}$$

Setzen wir in diesen Gleichungen, nach dem Vorbilde von (18),

$$I_i(\xi, \tau; x, t) = \frac{1}{c} a_i(\xi, \tau) \varrho(\xi, \tau; x, t) \qquad (i = 1, 2, 3)$$
$$I_4(\xi, \tau; x, t) = i\varrho(\xi, \tau; x, t) \qquad (a_4(\xi, \tau) = ic)$$

und multiplizieren die erste Gleichung nacheinander mit $\frac{i}{c} \varrho(x, t; x_0, t_0)$ und $\frac{a_j(\xi, \tau)}{c^2} \varrho(x, t; x_0 t_0)$, $j = 1, 2, 3$, so erhalten wir statt (20)

$$I_4(x,t;x_0,t_0)\frac{\partial I_\nu(\xi,\tau;x,t)}{\partial ict}+I_i(x,t;x_0,t_0)\frac{\partial I_\nu(\xi,\tau;x,t)}{\partial x_i}=0$$

$$\nu=1,2,3,4$$

$$\frac{\partial I_4(\xi,\tau;x_0,t_0)}{ic\partial\tau}+\frac{\partial I(\xi,\tau;x_0,t_0)}{\partial\xi_i}=0,$$

d.h. die relativistischen Gleichungen (11)-(12).

Diese Gleichungen bilden einen Rahmen für jede statistisch-relativistische Theorie, ebenso wie die Gleichungen (14) für jede statistisch-nichtrelativistische Theorie, die den zeitlichen Ablauf der Ereignisse behandelt.

Wie erwähnt, reichen diese Gleichungen zur Angabe der I_ν nicht aus. So wie in den Kolmogorowschen Gleichungen die $a_i(x,t)$ und $b_{ik}(x,t)$ die „dynamischen" Eigenschaften des beobachteten Systems charakterisieren und als gegebene Funktionen auftreten, treten auch in den relativistischen Gleichungen Elemente auf, die nicht aus den stochastischen Gleichungen abgeleitet werden können, (z.B. das singuläre Vektorfeld der Randwerte $I_\nu(x_\lambda;x_\lambda)$). Diese Gleichungen besagen, dass für Zwecke unserer Wahrscheinlichkeitsaussagen die Kenntnis des Beobachtungsortes und der Beobachtungszeit des Teilchens unentbehrlich ist. In Abhängigkeit von der Gestalt des gegebenen Feldes, bekommen wir verschiedene Lösungen. Ist z.B. das Feld konstant, d.h. der vierdimensionale Raum isotrop, so folgt, dass $I_\nu(\xi_\lambda;x_\lambda)=I_\nu(\xi_\lambda-x_\lambda)$ ist. Dann bewegt sich das Teilchen mit unveränderlicher dreidimensionaler Geschwindigkeit längs einer Geraden (eine δ^3-artige Lösung). Im anderen Falle macht man z.B. spezielle Symetrieannahmen über das Feld, welche der Reversibilität der Vorgänge in der klassischen Mechanik entsprechen, und so bekommt man die klassische Bewegung eines Teilchens (auch eine δ^3-artige Lösung). Diese Lösung in (3) eingesetzt gibt uns die statistische Mechanik im Koordinatenraum, wo die Geschwindigkeit als Funktion des Ortes auftritt. Wir gehen auf diese Fragen nicht näher ein, da diese in der nächsten Mitteilung, die der Verbindung der „dynamischen" Probleme mit der stochastischen Theorie gewidmet wird, behandelt werden.

Es ist interessant aus den relativistischen Gleichungen durch einen Grenzübergang die nichtrelativistischen Kolmogorowschen Gleichungen zu erhalten. Fasst man beim Übergang von den Intergralgleichungen (6)—(8) zu den Differentialgleichungen (11)—(12), d.h. beim Bilden der Viererdivergenzen, den Umstand ins Auge, dass das Differential cdt nicht eine infinitesimale sondern eine endliche Grösse ist, wenn c der Unendlichkeit zustrebt, so müssen auch weitere Glieder in der Entwicklung der subintegralen Funktion berücksichtigt werden. Auch diese Ableitung der Formeln wird ausführlich in der nächsten Mitteilung erörtert.

7. Schlussbemerkungen

Der weitere Ausbau dieser Theorie muss die fundamentalen Eigenschaften der dynamischen Systeme berücksichtigen, z.B. dass diese Systeme ergodisch sind, oder dass die Gleichung $(p_\nu-A_\nu)^2=m^2c^2$, wo p_ν den Impuls, als Koordi-

natenfunktion gemeint, A das Feld und m die Ruhemasse bedeuten, immer erfüllt ist. Bekäme man daraus die Gleichungen der Hydrodynamik, so könnte man, im nichtrelativistischen Grenzfall, für wirbelfreie Flüssigkeit das Bernoullische Integral und die stochastischen Gleichungen zu der klassischen Wellenmechanik ohne Spin nach der WKB-Methode zusammenfügen. (Den Gedanken, die WKB-Gleichung, die die Energie betrifft, nicht als Hamilton-Jacobische Gleichung sondern als Bernoullisches Integral zu deuten, verdanke ich Herrn Dr J. Plebański aus Warschau). Es scheint sachgemäss, die Hamilton-Jacobische Gleichung als einen Spezialfall des Bernoullischen Integrals zu betrachten und zwar im Grenzfall wo die Flüssigkeitmasse auf einen Punkt zusammengedrängt ist. Erweitert man das Schema, nach dem Gedanken von J. Plebański, auf Wirbelflüssigkeiten, so bekommt man die klassische Wellenmechanik mit Spin.

Tut man einen Schritt weiter bis zur Quantisierung der stochastisch-dynamischen Theorie, so muss man die Arbeit von Fényes (1952) ins Auge fassen, der, als erster, die Theorie der stochastischen Prozesse, vor allem die Diffusionsprozesse, mit der nichtrelativistischen Wellenmechanik zu verbinden suchte.

Ähnlich könnte man im relativistischen Falle verfahren, wo man die Gleichungen von Dirac bzw. Proca, ohne die zweite Quantisierung durchzuführen, bekommen könnte.

Ich äussere meinen besten Dank für wertvolle Ratschläge und Diskussionen Herrn Prof. Dr. R.S. Ingarden, so wie auch Frl. Prof. Dr. H. Pidek und Herrn Mgr. A. Rybarski.

КРАТКОЕ СОДЕРЖАНИЕ

Я. Лопушанский, *Релятивисткая теориа стохастических процессов.*

Приведены к лоренц-инвариантному виду основные интегральные уравнения стохастических процессов, в частности уравнения Смолуховского—Колмогорова — Чапмэна и уравнения сохранения вероятности. Затем получены эквивалентные им релятивистские дифференциальные уравнения. Из рассмотрения условий Колмогорова следует, что они не могут быть сохранены в релятивистском случае. Уравнения Колмогорова приведены к релятивистскому виду и показано, что они переходят тогда в релятивистски-инвариантные дифференциальные уравнения, о которых говорилось выше.

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QUANTIZATION OF A CERTAIN CLASS OF NON-LOCAL FIELD THEORIES

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In this paper the problem of canonical quantization of non-local field theories is reduced, for a certain class of form-functions, to the problem of canonical quantization of field theories containing higher (ev. infinite) order derivatives of the fields. For finite order derivatives the interaction representation is introduced and general commutation rules between field quantities at two arbitrary points of space-time are constructed. The S-matrix is calculated by means of generalized Green's functions. Finally the connection with direct (integral) methods of calculating the S-matrix for non-local theories is discussed.

Introduction. Non-local field theories have proved to be very convenient for a consequent description of processes involving elementary particles since they are free of the divergent quantities occuring in any local theory. This improvement has been obtained by introduction of certain form-factors giving account of the structure of the particles. These form-factors are to a large extent arbitrary functions and there exists, so far, no theory explaining their nature. It may be hoped, however, that a future theory of elementary particles will at the same time be a theory of the form-factor, i.e., a theory of the structure of the particles. The word "structure" is used here in a rather symbolical sense. One may also consider the introduction of non-locality from a different point of view, namely as change in the nature of forces acting between the particles. Thus, a theory of extended particles may be viewed upon as a theory of point particles interacting by means of modified forces.

It is difficult at the present stage to attack the problem of constructing a theory of the form-factor, since on the one hand we have not enough experimental knowledge, on the other there are no sufficiently strong theoretical arguments for a radical change in the theory. One may, however, prepare for the new theory by leaving the structure of the particles unexplained and constructing a quantum theory valid for arbitrary form-factors.

It was shown by the author (Rzewuski 1951) that it is possible to construct Heisenbergs S-matrix for non-local field theories by use of Feynman's (1949) semi-classical version of quantum field theory. Thus the problem of describing collision problems and bound states is solved, the practical calculations being not much more complicated than in local theories. The construction of an S-matrix for non-local field theories was later carried out by Rayski (1951) by means of

the method of Yang and Feldman (1950). Møller (1952) applies the same method to a somewhat more general class of form-functions.

However satisfactory these results may be, they give only an integral description of the phenomena, since they deal only with observable quantities on two space-like surfaces σ_1 and σ_2 on which measuring experiments preparing the initial and registering the final state are carried out. (For most practical problems these surfaces may be moved to $-\infty$ and $+\infty$ respectively). From the experimental point of view this seems to suffice as matrix elements for all processes may then in principle be calculated. From the theoretical point of view, however, one should like to have a differential description of the phenomena by means of a state vector depending on a certain number of commuting observables on a space-like surface σ. The change of this state vector from one surface σ to another infinitesimally near surface $\sigma + d\sigma$ would be described by Schrödingers equation thus giving an account of the unobservable part of the process occuring between σ_1 and σ_2. Such a description seems to be especially important in view of the eventual applications to a future theory of the form-factor.

It is clear that a differential description is impossible with non-local theories in their original and most general form, that is in the form of integral or integro-differential equations. In fact, all quantities deduced from such a theory show besides an explicit dependence on certain observables attached to points of a space-like surface σ also an implicit dependence on these observables at points lying outside of σ. For a certain class of non-local theories, however, there exist equations of a purely differential (and, therefore, local) character equivalent to the original integro-differential equations and it is the aim of the present paper to show how for this class a differential description is possible. The problem reduces, as may be easily seen, to the problem of quantization of differential equations of arbitrary order.

A canonical formalism for higher order differential equations was developed by deWet (1949[1] cf. deWet's paper for further references) and leads to commutation relations for the conjugate field variables taken at two points with space-like distance. We shall develop this formalism further by deriving Schrödingers and Heisenbergs equations, introducing the interaction representation, establishing general commutation relations for field quantities at two arbitrary points of space-time and finally constructing Heisenbergs S-matrix. We obtain thus a complete differential description of those non-local quantum field theories which are expressible in form of differential equations of finite order. One should like, of course, to extend the formalism to equations of infinite order covering thus a larger class of non-local theories. This problem is not investigated in this paper, but a preliminary discussion is given in the last section.

[1] An alternative treatment of higher order equations was used e.g. by Pais and Uhlenbeck (1950-cf. Pais and Uhlenbecks paper for further references) and consists in reducing higher order equations to a set of second order equations. All the solutions of this set must be treated on equal footing which makes the negative energy difficulties especially drastic,

1. General remarks. We shall consider throughout this paper (with the exception of the second section concerning arbitrary fields) the simple case of one neutral scalar field φ interacting with a complex scalar field ψ, ψ^*. The considerations are, of course, immediately applicable to other types of fields. The integro-differential equations describing a non-local interaction of the fields φ, ψ and ψ^* are of the form (cf. Rzewuski 1952)

$$(\Box - \varkappa^2)\psi(1) = g\int P(1,2)\tilde{\varphi}(2)\tilde{\ }(2)\,dx_2,$$

$$(\Box - \varkappa^2)\psi^*(1) = g\int P^*(1,2)\tilde{\gamma}^*(2)\tilde{\mu}(2)\,dx_2$$

$$(\Box - \lambda^2)\varphi(1) = g\int F(1,2)\tilde{\varphi}^*(2)\tilde{\psi}(2)\,dx_2, \tag{1.1}$$

where $\tilde{\psi}(1) = \int dx_2\psi(2)P^*(2,1)$, $\psi^*(1) = \int dx_2\psi^*(2)P(2,1)$

$$\tilde{\varphi}(1) = \int dx_2\varphi(2)F(2,1). \tag{1.2}$$

P and F are arbitrary functions of the two points 1 and 2 of space-time. For physical reasons they must be invariant with respect to translations and rotations of space-time and, therefore, functions of the invariant $(x_\mu^1 - x_\mu^2)^2$. In the limiting case of local interactions they go over into $\delta(x_\mu^1 - x_\mu^2)$-functions. g is a coupling constant and $dx_2 = dx_1^2 dx_2^2 dx_3^2 dx_0^2$. Equations (1.1) may be derived from the following action integral

$$W = \int\left\{\psi^*{}_\nu\psi_\nu + \varkappa^2\psi^*\psi + \frac{1}{2}\varphi_\nu^2 + \frac{\lambda^2}{2}\varphi^2\right\}dx +$$

$$+ g\iiiint\varphi(2)F(2,1)\psi^*(3)P(3,1)\psi(4)P^*(4,1)\,dx_1\,dx_2dx_3dx_4 \tag{1.3}$$

It contains multiple integrals which makes quantization in the differential sense impossible.

Equations (1.1) may be written in form of pure integral equations

$$\psi(1) = \psi^0(1) + g\iint R_\psi(1,2)P(2,3)\tilde{\varphi}(3)\tilde{\psi}(3)\,dx_2dx_3,$$

$$\psi^*(1) = \psi^{0*}(1) + g\iint R_\psi(1,2)P^*(2,3)\tilde{\varphi}^*(3)\tilde{\psi}(3)\,dx_2dx_3$$

$$\varphi(1) = \varphi^0(1) + g\iint R_\varphi(1,2)F(2,3)\tilde{\varphi}^*(3)\tilde{\psi}(3)\,dx_2dx_3 \tag{1.4}$$

where the functions φ^0, ψ^0, ψ^{0*}, R_ψ and R_φ are general solutions of the following equations

$$(\Box - \varkappa^2)\psi^0 = (\Box - \varkappa^2)\psi^{0*} = (\Box - \lambda^2)\varphi^0 = 0,$$
$$(\Box - \varkappa^2)R_\psi(x) = \delta(x), \quad (\Box - \lambda^2)R_\varphi(x) = \delta(x). \tag{1.5}$$

If we choose for R_ψ and R_φ e.g. the retarded or advanced solutions of (1. 5) we get accordingly the interpretation of φ^0 and ψ^0 as boundary values of φ and ψ at $\pm\infty$ respectively. The interpretation of φ^0 and ψ^0 as boundary values will become important for calculations of the elements of the S-matrix.

Multiplication of (1.4) with P^*, P and F resp. and integration gives on account of (1.2)

$$\tilde{\psi}(1) = \tilde{\psi}^0(1) + g\int K(1,2)\tilde{\varphi}(2)\tilde{\psi}(2)\,dx_2,$$

$$\tilde{\psi}^*(1) = \tilde{}^0{}^*(1) + g\int K^*(1,2)\tilde{\varphi}(2)\tilde{\psi}^*(2)\,dx_2,$$

$$\tilde{\varphi}(1) = \tilde{\varphi}^0(1) + g\int L(1,2)\tilde{\psi}^*(2)\tilde{\psi}(2)\,dx_2, \tag{1.6}$$

where

$$K(1,2) = \iint P^*(3,1)R_\psi(3,4)P(4,2)\,dx_3dx_4,$$

$$L(1,2) = \iint F(3,1)R_\varphi(3,4)F(4,2)\,dx_3dx_4. \tag{1.7}$$

Equations (1.6) may be immediately written in form of differential equations if there exist functions $K^{-1}(k^2)$, K^{*-1} (k^2) and L^{-1} (k^2) such that

$$K^{-1}(-\Box)K(x) = \delta(x),\quad K^{*-1}(-\Box)K^*(x) = \delta(x),\quad L^{-1}(-\Box)L(x) = \delta(x). \tag{1.8}$$

Indeed, the differential operation defined by (1.8) carried out on both sides of (1.6) yields

$$K^{-1}(-\Box)\tilde{\psi} = g\tilde{\varphi}\tilde{\psi},\qquad K^{-1*}(-\Box)\tilde{\psi}^* = g\tilde{\varphi}\tilde{\psi}^*,\qquad L^{-1}(-\Box)\tilde{\varphi} = g\tilde{\varphi}^*\tilde{\varphi}. \tag{1.9}$$

The contributions from $K^{-1}(-\Box)\tilde{\psi}^0$, $K^{*-1}(-\Box)\tilde{\psi}^{0*}$ and $L^{-1}(-\Box)\tilde{\varphi}^0$ vanish on account of (1.5).

Equations (1.9) are of a more general character then (1.6) since their solutions may be written in the form (1.6), $\tilde{\psi}^0$, $\tilde{\varphi}^0$ being general solutions of the unperturbed equations (1.9). To maintain the equivalence with the non-local theory as described by equations (1.1), however, we have to admit only those of the solutions of (1.9) for which $\tilde{\psi}^0$ and $\tilde{\varphi}^0$ satisfy equations (1.5).

The circumstances stressed above become especially clear in the momentum representation. In this representation

$$\psi^0(x) = \frac{1}{(2\pi)^4}\int \psi^0(k)\,\delta(k_\mu^2 + \varkappa^2)\,e^{ik_\mu x_\mu}dk,$$

$$\varphi^0(x) = \frac{1}{(2\pi)^4}\int \varphi^0(k)\,\delta(k_\mu^2 + \lambda^2)\,e^{ik_\mu x_\mu}dk,$$

$$R_\psi(x) = \frac{1}{(2\pi)^4}\int \frac{e^{ik_\mu x_\mu}}{-k_\mu^2 - \varkappa^2}dk,\qquad R_\varphi(x) = \frac{1}{(2\pi)^4}\int \frac{e^{ik_\mu x_\mu}}{-k_\mu^2 - \lambda^2}dk, \tag{1.10}$$

$$P(x) = \frac{1}{(2\pi)^4}\int P(k_\mu^2)e^{ik_\mu x_\mu}dk,\qquad F(x) = \frac{1}{(2\pi)^4}\int F(k_\mu^2)e^{ik_\mu x_\mu}dk.$$

The integrand in the integral representing R_ψ and R_φ has two poles at $k_0 = \pm\sqrt{\vec{k}^2 + \varkappa^2}$ or $k_0 = \pm\sqrt{\vec{k}^2 + \lambda^2}$ respectively. According to the choice of the integration path in the complex k_0-plane we get various types of solutions of (1.1). The Fourier transforms of $K(x)$, $L(x)$, $\tilde{\psi}^0(x)$, $\tilde{\varphi}^0(x)$ become now

$$K(x) = \frac{1}{(2\pi)^4}\int \frac{|P(k_\mu^2)|^2}{-k_\mu^2 - \varkappa^2} e^{ik_\mu x_\mu} dk, \qquad L(x) = \frac{1}{(2\pi)^4}\int \frac{[F(k_\mu^2)]^2}{-k_\mu^2 - \lambda^2} e^{ik_\mu x_\mu} dk, \tag{1.11}$$

$$\tilde{\psi}^0(x) = P^*(-\varkappa^2)\psi^0(x), \qquad \tilde{\varphi}^0(x) = F(-\lambda^2)\varphi^0(x). \tag{1.12}$$

The functions K and L differ from R_ψ and R_φ by a factor $|P(k_\mu^2)|^2$ or $[F(k_\mu^2)]^2$ respectively in the integrand of their Fourier transforms. This factor may be chosen in such a way as to secure convergence of the theory. On the other hand $\tilde{\psi}^0$ and $\tilde{\varphi}^0$ differ from ψ^0 and φ^0 only by a constant factor, which for physical reasons may be normalized to unity.

The differential operators occuring in (1.9) are of course

$$K^{-1}(-\Box) = \frac{\Box - \varkappa^2}{|P(-\Box)|^2}, \qquad L^{-1}(-\Box) = \frac{\Box - \lambda^2}{[F(-\Box)]^2}, \tag{1.13}$$

which clearly shows that solutions of the first three equations (1.5) are also special solutions of the homogeneous equations corresponding to (1.9).

Now the functions $K(k_\mu^2)$ and $L(k_\mu^2)$ are in general absolutely arbitrary apart from the essential factor $(-k_\mu^2 - \varkappa^2)^{-1}$ or $(-k_\mu^2 - \lambda^2)^{-1}$ and from the conditions required by invariance with respect to translations and rotations of space-time and certain normalization conditions. We shall limit ourselves in our considerations to such integral equations (1.1) for which $K^{-1}(k_\mu^2)$ and $L^{-1}(k_\mu^2)$ are polynomials

$$K^{-1}(k_\mu^2) = \sum_{i=o}^{s} \varkappa_i (k_\mu^2)^i, \qquad L^{-1}(k_\mu^2) = \sum_{i=o}^{s} \lambda_i (k_\mu^2)^i. \tag{1.14}$$

It may be mentioned that this is already sufficient to obtain full convergence of the theory.

Thus we have reduced the problem of quantizing a certain class of integral equations corresponding to the special choice (1.14) of K^{-1} and L^{-1} to the problem of quantizing higher order differential equations. In the next section we derive the formalism for arbitrary Lagrangians containing higher order derivatives of the fields. In the remaining sections we shall develop the theory further for the special equations given by (1. 9) and (1.14).

2. Canonical formalism. In this section we shall use a generalization to higher order Lagrangians of a method of Schwinger (1951) in which the quantization is deduced from one variational principle.

Denoting by $q^a_{\nu_1 \ldots \nu_i}$ the i-th derivative

$$\partial_{\nu_1 \ldots \nu_i} q^a = \frac{\partial^i q^a}{\partial x_{\nu_1} \ldots \partial x_{\nu_i}} \tag{2.1}$$

of the field q^a, where the superscript a indicates the transformation properties and the multiplicity of the fields, we may start our considerations with an action integral of the form

$$W = \int_{\sigma_1}^{\sigma_2} \mathcal{L}[\, q^a, q^a_{\nu_1}, \dots q^a_{\nu_1 \dots \nu_s}\,]\, dx \tag{2.2}$$

in which the integration is extended over the region of space-time contained between two space-like surfaces σ_1 and σ_2.
Defining the momenta

$$p^a_{\nu_1 \dots \nu_k} = \sum_{i=o}^{s-k} (-1)^i \partial_{\mu_1 \dots \mu_i} \frac{\partial \mathcal{L}}{\partial q^a_{\nu_1 \dots \nu_k \mu_1 \dots \mu_i}} \qquad (k = 1, 2, \dots s) \tag{2.3}$$

satisfying the obvious identities

$$p^a_{\nu_1 \dots \nu_k} = \frac{\partial \mathcal{L}}{\partial q^a_{\nu_1 \dots \nu_k}} \partial_\mu p^a_{\mu \nu_1 \dots \nu_k}, (k = 1, 2, \dots s-1),\ p^a_{\nu_1 \dots \nu_s} = \frac{\partial \mathcal{L}}{\partial q^a_{\nu_1 \dots \nu_s}}, \tag{2.4}$$

we may write the Euler-Lagrange equations following from (2.2) in the form

$$\frac{\partial \mathcal{L}}{\partial q^a} - \partial_\mu p^a_\mu = 0. \tag{2.5}$$

The change in W produced by an infinitesimal translation and rotation of space-time

$$\delta x_\mu = \varepsilon_\mu - \varepsilon_{\mu\nu} x_\nu, \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} \tag{2.6}$$

and by an independent change $\delta_0 q^a$ in the field variables q^a is easily seen to be

$$\delta W = F(\sigma_2) - F(\sigma_1), \quad F(\sigma) = \int_\sigma d\sigma_\nu \left\{ \mathcal{L}\, \delta x_\nu + \sum_{i=0}^{s-1} p^a_{\nu \nu_1 \dots \nu_i}\, \delta_0 q^a_{\nu_1 \dots \nu_i} \right\} \tag{2.7}$$

if the fields q^a satisfy the field equations (2.5).

To introduce canonical variables we shall now restrict the surfaces σ to be hyperplanes (with normals n_μ) and decompose every vector in two components normal and parallel to σ

$$a_\mu = -n_\mu a_n + a^t_\mu, \quad \partial_\mu = n_\mu \partial_n + \partial^t_\mu, \quad n^2_\mu = -1. \tag{2.8}$$

Following relations may be easily verified

$$a_n = n_\mu a_\mu, \qquad \partial_n = -n_\mu \partial_\mu, \tag{2.9}$$

$$a^t_\mu = (\delta_{\mu\nu} + n_\mu n_\nu) a_\nu, \quad \partial^t_\mu = (\delta_{\mu\nu} + n_\mu n_\nu) \partial_\nu. \tag{2.10}$$

Introducing this decomposition into (2.7) we obtain for the operator $F(\sigma)$ the following expression

$$F(\sigma) = \int_\sigma d\sigma_\nu \{ \mathcal{L}\, \delta x_\nu + \binom{0}{0} p^a_\nu \delta_0 q^a + \binom{1}{0} p^a_{\nu n} \delta_0 \partial_n q^a + \binom{1}{1} p^a_{\nu \nu_1} \delta_0 \partial^t_{\nu_1} q^a +$$

$$+ \binom{2}{0} p^{\alpha}_{\nu nn} \delta_0 \partial_{nn} q^{\alpha} + \binom{2}{1} p^{\alpha}_{\nu n \nu_1} \delta_0 \partial^{t}_{\nu_1} \partial_n q^{\alpha} + \binom{2}{2} p^{\alpha}_{\nu \nu_1 \nu_2} \delta_0 \partial^{t}_{\nu_1 \nu_2} q^{\alpha} + \dots \tag{2.11}$$

$$+ \binom{s-1}{0} p^{\alpha}_{\nu (s-1,n)} \delta_0 \partial_{(s-1,n)} q^{\alpha} + \binom{s-1}{1} p^{\alpha}_{\nu (s-2,n) \nu_1} \delta_0 \partial^{t}_{\nu_1} \partial_{(s-2,n)} q^{\alpha} + \dots$$

$$+ \binom{s-1}{s-1} p^{\alpha}_{\nu \nu_1 \dots \nu_{s-1}} \delta_0 \partial^{t}_{\nu_1 \dots \nu_{s-1}} q^{\alpha} \},$$

where

$$a_{(i,n)} = n_{\nu_1} \dots n_{\nu_i} a_{\nu_1 \dots \nu_i}, \qquad \partial_{(i,n)} = (-1)^i n_{\nu_1} \dots n_{\nu_i} \partial_{\nu_1 \dots \nu_i}. \tag{2.12}$$

In (2.11) integration by parts may be carried out with respect to the tangential derivatives. It yields some contributions to $F(\sigma)$ from the infinitely remote points of σ which we assume to vanish. There remains

$$F(\sigma) = \int_{\sigma} d\sigma_{\nu} \{ \mathcal{L} \delta x_{\nu} + \binom{0}{0} p^{\alpha}_{\nu} \delta_0 q^{\alpha} + \binom{1}{0} p^{\alpha}_{\nu n} \delta_0 \partial_n q^{\alpha} - \binom{1}{1} \partial^{t}_{\nu_1} p^{\alpha}_{\nu \nu_1} \delta_0 q^{\alpha}$$

$$+ \binom{2}{0} p^{\alpha}_{\nu nn} \delta_0 \partial_{nn} q^{\alpha} - \binom{2}{1} \partial^{t}_{\nu_1} p^{\alpha}_{\nu n \nu_1} \delta_0 \partial_n q^{\alpha} + \binom{2}{2} \partial^{t}_{\nu_1 \nu_2} p^{\alpha}_{\nu \nu_1 \nu_2} \delta_0 q^{\alpha} + \dots$$

$$\binom{s-1}{0} p^{\alpha}_{\nu (s-1,n)} \delta_0 \partial_{(s-1,n)} q^{\alpha} - \binom{s-1}{1} \partial^{t}_{\nu_1} p^{\alpha}_{\nu (s-2,n) \nu_1} \delta_0 \partial_{(s-2,n)} q^{\alpha} + \dots$$

$$+ (-1)^{s-1} \binom{s-1}{s-1} \partial^{t}_{\nu \dots \nu_{s-1}} p^{\alpha}_{\nu \nu_1 \dots \nu_{s-1}} \delta_0 q \} \tag{2.13}$$

In this formula we may collect terms belonging to the same $\delta_0 \partial_{(i,n)} q^{\alpha}$. Introducing now the s canonical fields

$$\partial_{(i,n)} q^{\alpha} = q^{\alpha}_{(i,n)} \qquad (i = 0, 1, \dots s-1), \qquad q^{\alpha}_{(0,n)} = q^{\alpha} \tag{2.14}$$

and the s conjugate canonical momenta

$$\pi^{\alpha}_{(r,n)} = \sum_{i=0}^{s-r} (-1)^i \binom{i+r-1}{i} \partial^{t}_{\nu_1 \dots \nu_i} p^{r}_{(r,n) \nu_1 \dots \nu_i} \qquad (r = 1, 2, \dots s) \tag{2.15}$$

and noticing that $d\sigma_{\mu} = n_{\mu} d\sigma$, we may write the operator (2.13) in the simple form

$$F(\sigma) = \int_{\sigma} d\sigma \left\{ \mathcal{L} n_{\nu} dx_{\nu} + \sum_{r=1}^{s} \pi^{\alpha}_{(r,n)} \delta_0 q^{\alpha}_{(r-1,n)} \right\}. \tag{2.16}$$

We shall now introduce the total change $\delta q^{\alpha}_{\nu_1 \dots \nu_i}$ of $q^{\alpha}_{\nu_1 \dots \nu_i}$ as produced by the change $\delta_0 q^{\alpha}_{\nu_1 \dots \nu_i}$ of the functional form and the change caused by the infinitesimal transformation (2.6) of the coordinates

$$\delta q^{\alpha}_{\nu_1 \dots \nu_i} = \delta_0 q^{\alpha}_{\nu_1 \dots \nu_i} + q^{\alpha}_{\lambda \nu_1 \dots \nu_i} \delta x_{\lambda} + \varepsilon_{\delta \varrho} S^{\alpha\beta}_{\delta \varrho \nu_1 \dots \nu_i \mu_1 \dots \mu_i} q^{\beta}_{\mu_1 \dots \mu_i}. \tag{2.17}$$

Multiplying (2.17) by $n_{\nu_1} \dots n_{\nu_i}$ we obtain

$$\delta q^{\alpha}_{(i,n)} = \delta_0 q^{\alpha}_{(i,n)} + q^{\alpha}_{\lambda (i,n)} \delta x_{\lambda} + (-1)^i \varepsilon_{\delta \varrho} S^{\alpha\beta}_{\delta \varrho (i,n) \mu_1 \dots \mu_i} q^{\beta}_{\mu_1 \dots \mu_i}. \tag{2.18}$$

Introducing this into (2.16) and defining the tensor

$$f_{\nu \delta \varrho} = \sum_{r=0}^{s-1} (-1)^r \{ \pi^{\alpha}_{\nu (r,n)} S^{\alpha\beta}_{\delta \varrho (r,n) \mu_1 \dots \mu_r} + \pi^{\alpha}_{\varrho (r,n)} S^{\alpha\beta}_{\delta \nu (r,n) \mu_1 \dots \mu_r} +$$

$$\pi^{\alpha}_{\delta (r,n)} S^{\alpha\beta}_{\varrho \nu (r,n) \mu_1 \dots \mu_r} \} q^{\beta}_{\mu_1 \dots \mu_r} \tag{2.19}$$

where $\pi^{\alpha}_{\nu (r,n)}$ is given by the equation

$$n_{\nu} \pi^{\alpha}_{\nu (r,n)} = \pi^{\alpha}_{(r+1,n)} \tag{2.20}$$

and $f_{\nu\delta\varrho}$ is antisymmetric in its first two indices, we get

$$F(\sigma) = \int_{\sigma} d\sigma_\nu \left\{ T_{\nu\mu}\, \delta x_\mu + \sum_{r=0}^{s-1} \pi^a_{\nu(r,n)}\, \delta q^a_{(r,n)} \right\} \tag{2.21}$$

with

$$T_{\nu\mu} = \delta_{\nu\mu}\, \mathcal{L} - \sum_{r=0}^{s-1} \pi^a_{\nu(r,n)}\, q^a_{\mu(r,n)} - \partial_\lambda f_{\lambda\nu\mu}\,. \tag{2.22}$$

The conservation laws for the energy-momentum, angular momentum, and current tensor densities follow from (2.21) and (2.22) and the invariance of the action integral with respect to translations and rotations of space-time and to gauge transformations of the first kind in the usual way (cf. Schwinger 1951 and for higher order Lagrangians Rzewuski 1952). We shall not consider these questions here.

Going over to the quantum aspects of the theory it shall be our first task to derive commutation rules between the conjugate variables. For this purpose we consider transformations for which $\delta x_\mu = 0$. In this case (2.21) becomes

$$F_{\delta q}(\sigma) = \int_{\sigma} d\sigma \sum_{r=1}^{s} \pi^a_{(r,n)}\, \delta q^a_{(r-1,n)}\,. \tag{2.23}$$

We may change the Lagrangian in (2.2) by adding a divergence without affecting the Euler-Lagrange equations. According to Schwinger this corresponds merely to a unitary transformation of the observables on σ. Choosing the divergence to be of the form

$$-\partial_\nu \sum_{r=0}^{s-1} \pi^a_{\nu(r,n)}\, q^a_{(r,n)}\,, \tag{2.24}$$

we obtain another equivalent form for the operator $F(\sigma)$, namely

$$F_{\delta\pi}(\sigma) = -\int_{\sigma} d\sigma \sum_{r=1}^{s} \delta\pi^a_{(r,n)}\, q^a_{(r-1,n)}\,. \tag{2.25}$$

Now the change in an arbitrary operator G being a function of the fields on σ produced by the infinitesimal transformation (2.23) or (2.25) is

$$i\,\delta_q\, G = [G, F_{\delta q}], \qquad i\,\delta_\pi\, G = [G, F_{\delta\pi}]. \tag{2.26}$$

Taking G to be $q^a_{(i,n)}$ $(i = 0, 1, \ldots s-1)$ or $\pi^a_{(i,n)}$ $(i = 1, 2, \ldots s)$ respectively and assuming that $\delta q^a_{(r,n)}$ and $\delta\pi^a_{(r,n)}$ commute or anticommute with all other fields according to their transformation properties, we get in full analogy with Schwingers work the following commutation or anticommutation relations for the canonical variables

$$\left[\overset{a}{q}_{(i,n)}(x), \overset{\beta}{\pi}_{(k,n)}(x') \right]_{\pm} = i\delta^{a\beta}\, \delta_{k-1,i}\, \delta_\sigma(x - x'),$$

$$\left[\overset{a}{q}_{(i,n)}(x), \overset{\beta}{q}_{(k,n)}(x') \right]_{\pm} = \left[\overset{a}{\pi}_{(i,n)}(x), \overset{\beta}{\pi}_{(k,n)}(x') \right]_{\pm} = 0 \tag{2.26a}$$

valid for points x and x' whose distance is space-like. $\delta_\sigma(x-x')$ denotes here the generalization to arbitrary space-like planes of the three-dimensional $\delta(\vec{x}-\vec{x}')$ function. The commutation rules (2.26) were deduced by deWet (1949) by means of a slightly different method.

All considerations of Schwinger (1951) concerning commutation rules between other observables and the equations of Heisenberg and Schrödinger may be applied directly to the operator (2.21). We shall quote here only those of the results which are of importance for further applications.

Heisenbergs equations may be deduced from the fact that an operator G which does not depend explicitly on the space-time coordinates satisfies

$$0 = \delta_x G - i[G, F_{\delta x}]. \tag{2.27}$$

We restrict ourselves to translations, in which case [cf. (2.21)]

$$F_{\delta x} = \int_\sigma d\sigma_\nu \, T_{\nu\mu} \, \delta x_\mu = P_\mu \, \delta x_\mu, \quad P_\mu = \int_\sigma d\sigma_\nu T_{\nu\mu}. \tag{2.28}$$

If besides, the operator G is a function of the field variables at one point x only, the relation (2.27) becomes

$$[G(x), P_\nu] = \frac{1}{i} \partial_\nu G(x). \tag{2.29}$$

Multiplication by n_ν gives

$$[G(x), P_n] = i \, \partial_n G(x). \tag{2.30}$$

Schrödingers equations may be obtained by considering the change in the state vector $\Psi(q', \sigma)$ as produced by an alteration of σ ; q' stands for the eigenvalues of the set of canonical variables $q^a_{(r,n)}$ on σ. We have

$$\Psi(q', \sigma + d\sigma) = [1 - i F_{\delta x}]\Psi(q', \sigma). \tag{2.31}$$

Considering again only translations, we may write

$$dx_\mu = \varepsilon_\mu = -n_\mu \varepsilon_n + \varepsilon^t_\mu. \tag{2.32}$$

Since ε^t_μ produces no change in the hyperplane σ, we may consider only translations along the normal n_μ. We introduce a parameter τ numerating the planes σ and choose it in such a way that

$$\delta\tau = -n_\mu \delta x_\mu, \qquad \delta x_\mu = n_\mu \delta\tau, \qquad n_0 > 0. \tag{2.33}$$

Ψ becomes now a function of τ and we have from (2.31)

$$\frac{d\Psi(q', \tau)}{d\tau} = -i P_n \cdot \Psi(q', \tau). \tag{2.34}$$

Having thus derived the commutation relations and Heisenbergs and Schrödingers equations, we shall go over to the special case of field equations considered in section 1.

3. The interaction representation and the general commutation rules. The field equations (1.5) with the particular form (1.14) of the differential operator may be derived from the following action integral

$$W=\int_{\sigma_1}^{\sigma_2} dx\left\{\sum_{i=0}^{s}\left[\varkappa_i \psi^*_{\nu_1\ldots\nu_i}\psi_{\nu_1\ldots\nu_i}+\frac{\lambda_i}{2}\varphi^2_{\nu_1\ldots\nu_i}\right]-g\varphi\psi^*\psi\right\}=\int_{\sigma_1}^{\sigma_2} dx\{\mathcal{L}^0+\mathcal{L}'\}. \quad (3.1)$$

The momenta (2.3) are now

$$p^{\psi}_{\nu_1\ldots\nu k}=\sum_{i=0}^{s-k}(-1)^i\varkappa_{k+i}\,\Box^i\psi^*_{\nu_1\ldots\nu_k},\quad p^{\psi^*}_{\nu_1\ldots\nu_k}=\sum_{i=0}^{s-k}(-1)^i\varkappa_{k+i}\,\Box^i\psi_{\nu_1\ldots\nu_k},$$

$$p^{\varphi}_{\nu_1\ldots\nu_k}=\sum_{i=0}^{s-k}(-1)^i\lambda_{k+i}\,\Box^i\varphi_{\nu_1\ldots\nu_k}. \quad (3.2)$$

The Euler-Lagrange equations (2.5) take the form

$$\sum_{i=0}^{s}(-1)^i\varkappa_i\,\Box^i\psi=g\varphi\psi,\qquad \sum_{i=0}^{s}(-1)^i\varkappa_i\,\Box^i\psi^*=g\varphi\psi^*,$$

$$\sum_{i=0}^{s}(-1)^i\lambda_i\Box^i\varphi=g\psi^*\psi. \quad (3.3)$$

These are exactly equations (1.5), (1.14) the functions φ, ψ, ψ^* having the meaning of $\tilde{\varphi}$, $\tilde{\psi}$, $\tilde{\psi}^*$ of section 1.

The canonical fields are $\psi_{(r,n)}$, $\psi^*_{(r,n)}$, $\varphi_{(r,n)}$ $(r=0,1,\ldots s-1)$. The corresponding momenta may be calculated from (2.15) and (3.2)

$$\pi^{\psi}_{(r,n)}=\sum_{i=0}^{s-r}\sum_{k=0}^{s-r-i}(-1)^{k+i+r}\binom{i+r-1}{i}\varkappa_{k+i+r}(\partial^t_{\nu\nu})^i\Box^k\partial_n\psi^*_{(r-1,n)},$$

$$\pi^{\psi^*}_{(r,n)}=\sum_{i=0}^{s-r}\sum_{k=0}^{s-r-i}(-1)^{k+i+r}\binom{i+r-1}{i}\varkappa_{k+i+r}(\partial^t_{\nu\nu})^i\Box^k\partial_n\psi_{(r-1,n)}, \quad (3.4)$$

$$\pi^{\varphi}_{(r,n)}=\sum_{i=0}^{s-r}\sum_{k=0}^{s-r-i}(-1)^{k+i+r}\binom{i+r-1}{i}\lambda_{k+i+r}(\partial^t_{\nu\nu})^i\Box_k\partial_n\varphi_{(r-1,n)}.$$

Since we have to do with Bose fields the commutation relations (2.26) with the minus sign have to be used.

The energy-momentum four-vector p_μ appearing in Heisenbergs and Schrödingers equations takes now the form

$$P_\mu=P^0_\mu+\int_\sigma d\sigma_\mu:\mathcal{L}', \quad (3.5)$$

where P^0_μ is that part of this vector which does not contain the interaction constant g.

To go over to the interaction representation it is convenient to use the following identities derived in appendix I.

$$\sum_{i=0}^{s}(-1)^i\varkappa_i\,\Box^i\,\psi^* \equiv \sum_{i=0}^{s}(-1)^i\varkappa_i\,(\partial^t_{\nu\nu})^i\psi^* - \partial_n\pi^\nu_n,$$

$$\sum_{i=0}^{s}(-1)^i\varkappa_i\,\Box^i\,\psi \equiv \sum_{i=0}^{s}(-1)^i\varkappa_i\,(\partial^t_{\nu\nu})^i\psi - \partial_n\pi^{\nu^*}_n, \tag{3.6}$$

$$\sum_{i=0}^{s}(-1)^i\lambda_i\,\Box^i\psi \equiv \sum_{i=0}^{s}(-1)^i\lambda_i\,(\partial^t_{\nu\nu})^i\varphi - \partial_n\pi^\varphi_n$$

We start with Heisenberg's representation in which the state vector is independent of σ. The corresponding Schrödinger equation is [cf. (2.34)]

$$\frac{d\Psi^H(q',\tau)}{d\tau} = 0. \tag{3.7}$$

We perform now a unitary transformation

$$\Psi^{in}(\tau) = U(\tau)\,\Psi^H, \quad F^{int}(x) = U(\tau)F^H(x)U^{-1}(\tau), \quad UU^* = 1, \tag{3.8}$$

where $F(x)$ denotes an arbitrary operator constructed from field variables at the point x. The surface σ corresponding to the particular value of τ considered in (3.8) is passing through this point.

With U defined by the equation

$$i\frac{dU}{d\tau} = \int_\sigma d\sigma'\,\mathcal{L}\cdot U \tag{3.9}$$

we get the following Schrödinger equation in the interaction representation

$$i\frac{d\Psi^{int}}{d\tau} = \int_\sigma d\sigma\,\mathcal{L}'\cdot\Psi^{int} \tag{3.10}$$

The new field equations may be obtained if we know the transformation properties of the derivatives. To derive them we differentiate the second of equations (3.8) with respect to x_μ remembering that $u(\tau)$ depends on x_μ by means of the parameter τ : $d\tau = -n_\mu\,\delta x_\mu$.

$$\partial_\mu F^{int}(x) = U\partial_\mu F^H(x)U^{-1} + n_\mu\,i\int_{\sigma'}[\mathcal{L}'(x'),\ F^{int}(x)]\,d\sigma' \tag{3.11}$$

with σ' passing through the point x. Decomposition into the normal and tangential parts gives

$$\partial_n F^{int} = U\partial_n F^H U^{-1} + i\int_{\sigma'}[\mathcal{L}'(x'),\ F^{int}(x)]\,d\sigma',\ \ \partial^t_\mu F^{int} = U\,\partial^t_\mu F^H\,U^{-1}. \tag{3.12}$$

Now let us consider the operator

$$\sum_{i=0}^{s}(-1)^i\,\varkappa_i\,(\partial^t_{\nu\nu})^i\,\psi^{int} - \partial_\nu\pi^{\nu^* int}_n = U\sum_{i=0}^{s}(-1)^i\,\varkappa_i(\partial^t_{\nu\nu})^i\,\psi^H\,U^{-1} +$$

$$- U\partial_\mu \pi_n^{\psi^{*H}} U^{-1} - i\int_{\sigma'} [\mathcal{L}'(x'),\ \pi_\nu^{\psi^{*int}}(x)]\, d\sigma'. \tag{3.13}$$

According to the commutation rules (2.26), we have

$$[\mathcal{L}'(x'),\ \pi_\nu^{\psi^{*int}}(x)] = -g\varphi^{int}\,\psi^{int}\,\delta_\sigma(x - x'). \tag{3.14}$$

The equations satisfied by the fields in the interaction representation are, therefore, on account of (3.3) and (3.6),

$$\sum_{i=0}^{s} (-1)^i \varkappa_i (\partial_{\nu\nu}^t)^i \psi^{int} - \partial_n \pi_n^{\psi^{*int}} = U\left\{\sum_{i=0}^{s} (-1)^i \varkappa_i (\partial_{\nu\nu}^t)^i \psi^H + \right.$$

$$\left. - \delta_n \pi_n^{\psi^{*H}} - g\varphi^H \psi^H \right\} U^{-1} = 0 \tag{3.15}$$

Similar equations hold for ψ^{*int} and φ^{int}.

We have used here the obvious fact that the commutation rules (2.26) remain unchanged by the unitary transformation (3.8). The relations (3.4) between the canonical variables which lead to the identities (3.6) remain also unchanged. This is evident from the fact that the $\psi_{(r,n)}$ $(r = 0, 1, \ldots s - 1)$ commute with $\mathcal{L}'$ (The highest normal derivatives occuring on the right hand sides of (3.4) are $\partial_n \psi_{(s-1,n)}$, $\partial_n \psi^*_{(s-1,n)}$ and $\partial_n \varphi_{(s-1,n)}$). Thus we may use the same relations (3.6) to go over to the equations

$$\sum_{i=0}^{s} (-1)^i \varkappa \Box^i \psi = 0, \qquad \sum_{i=0}^{s} (-1)^i \varkappa_i \Box^i \psi^* = 0,$$

$$\sum_{i=0}^{s} (-1)^i \lambda_i \Box^i \varphi = 0 \tag{3.16}$$

satisfied by the fields in the interaction representation (from now on we leave out the superscript "int").

For fields satisfying the unperturbed equations (3.16) it is possible to generalize the commutation rules (2.26) valid for two points x and x' with a space-like distance to general commutation rules valid for two arbitrary points x and x'. For this purpose it is necessary to express the solutions of (3.16) by their boundary values. We shall carry out the calculations for one of the fields only. Introducing a function $\Delta_\varkappa(x - x')$ satisfy Suiəqı equation

$$\sum_{i=0}^{s} (-1)^i \varkappa_i \Box^i \Delta_\varkappa(x) = 0 \tag{3.17}$$

and the differential operator

$$X_\mu = (\vec{\partial}_\mu - \overleftarrow{\partial}_\mu) \sum_{i=1}^{s} (-1)^{i-1} \varkappa_i \sum_{k=0}^{i-1} \vec{\Box}^{\,i-1-k} \overleftarrow{\Box}^{\,k}, \tag{3.18}$$

where the arrows indicate the direction in which the differentiations are to be carried out, we may express $\psi(x)$ at the point x by its values and the values of its $2s-1$ derivatives on the hyperplane σ', which in general does not pass through the point x.

$$\psi(x,\tau') = \int_{\sigma'} \Delta_\varkappa(x-x') X'_\mu \psi(x')\, d\sigma'_\mu. \tag{3.19}$$

To prove this we notice first that $\psi(x,\tau')$ is independent of τ' that is of the special choice of the plane σ'. Indeed the difference

$$\psi(x,\tau'+\delta\tau') - \psi(x,\tau') = \int_{\sigma'}^{\sigma'+\delta\sigma'} \partial'_\mu \quad [\Delta_\varkappa(x-x') X'_\mu \psi(x')]\, dx' \tag{3.20}$$

if we neglect contributions from the time-like surface (of infinitesimal height) connecting the planes σ' and $\sigma'+d\sigma'$ at infinity. The right hand side of (3.20) vanishes on account of

$$\partial'_\mu[\Delta_\varkappa(x-x') X'_\mu \psi(x')] = \Delta_\varkappa(x-x') \sum_{i=1}^{s} (-1)^{i-1} \varkappa_i(\vec{\Box}'^i - \overleftarrow{\Box}'^i)\psi(x') = 0. \tag{3.21}$$

Thus, we may leave out the variable τ and write henceforth simply $\psi(x)$ in place of $\psi(x,\tau')$. Since $\psi(x)$ does not depend on τ', we may choose the plane σ' to pass through the point x. This yields the identity

$$\psi(x) \equiv \int_{\sigma} \Delta_\varkappa (x-x') X'_\mu \psi(x')\, d\sigma'_\mu \tag{3.22}$$

for x and x' on the same σ'. To satisfy this identity we must put on $\Delta_\varkappa$ and its normal derivatives the following boundary conditions

$$(\partial_n)_i \Delta_\varkappa(x) = 0 \quad \text{for} \quad x_\mu^2 > 0, \quad (i = 0, 1, \ldots 2s-2)$$

$$\varkappa_s \int_{\sigma} (\partial'_n)^{2s-1} \Delta_\varkappa (x-x')\, d\sigma' = 1. \tag{3.23}$$

It is easily seen that with these conditions the right hand side of (3.22) becomes

$$\int_{\sigma} \Delta_\varkappa(x-x') (-\overleftarrow{\partial}'_\mu) (-1)^{s-1} \overleftarrow{\Box}'^{s-1} \psi(x')\, d\sigma'_\mu = \psi(x). \tag{3.24}$$

Considering instead of (3.19) a function

$$f(x', x'') = \int_{\sigma} \Delta_\varkappa (x'-x) X_\mu \Delta_\varkappa (x''-x)\, d\sigma_\mu \tag{3.25}$$

which is, of course, also independent on the special choice of σ, and taking σ to

pass once through x' and for a second time through x'', we deduce easily the relation

$$\Delta_{\varkappa}(x) = -\Delta_{\varkappa}(-x). \tag{3.26}$$

Analogous equations as those deduced for ψ hold also for ψ^* and φ. In the last case $\Delta_{\varkappa}$ has to be replaced by Δ_{λ}, a solution of the third of equations (3.16).

Since the integral in (3.19) contains normal derivatives of ψ at the boundary σ' up to the order $2s-1$, we have to extend the commutation rules (2.26) to normal derivatives of the fields to this order. This may be done for the particular case of the Lagrange function considered in this section with help of relations (3.4). The corresponding calculations are carried out in appendix II. The results are

$$\begin{aligned} &[(\partial_n)^r \psi''(x),\ \psi(x')] = [(\partial_n)^r \varphi(x), \varphi(x')] = 0 \quad \text{for} \quad r = 0, 1, \ldots 2s-2 \\ &\varkappa_s [(\partial_n)^{2s-1} \psi^*(x), \psi(x')] = \lambda_s [(\partial_n)^{2s-1} \varphi(x), \varphi(x')] = \\ &\qquad = i\,\delta_{\sigma}(x - x') \quad \text{for} \quad (x_{\mu} - x'_{\mu})^2 > 0. \end{aligned} \tag{3.27}$$

Now everything is prepared to establish the general commutation rules. Given two arbitrary points x and x' we may by means of (3.19) express $\psi^*(x)$ by the boundary values at a surface σ'' passing through x'. Thus

$$[\psi^*(x), \psi(x')] = \left[\int_{\sigma''} \Delta_{\varkappa}(x - x'')\, X''_{\mu}\, \psi^*(x'')\, d\sigma''_{\mu}, \psi(x')\right]. \tag{3.28}$$

On account of (3.27) only the highest derivative of $\psi^*(x'')$ contributes to the commutator and we get

$$[\psi^*(x), \psi(x')] = -\varkappa_s \int_{\sigma''} \Delta_{\varkappa}(x - x'')\, [(\partial''_{\mu})^{2s-1} \psi^*(x), \psi(x')]. \tag{3.29}$$

On account of the last two equations (3.27), we thus obtain the following general commutation rules for the fields ψ, ψ^*, and analogously also for the field φ,

$$[\psi^*(x), \psi(x')] = -i\Delta_{\varkappa}(x - x'), \qquad [\varphi(x), \varphi(x')] = -i\Delta_{\lambda}(x - x'). \tag{3.30}$$

4. The Construction of the S-matrix and of the non-local Green's functions

To calculate the S-matrix we start from the Schrödinger equation (3.10) in the interaction representation

$$i\frac{d\Psi}{d\tau} = \int_{\sigma} d\sigma\, \mathcal{L}'(x) \cdot \Psi \tag{4.1}$$

and the definition

$$\Psi(\infty) = S\,\Psi(-\infty) \tag{4.2}$$

of the S-matrix. Equation (4.1) may be treated in exactly the same way as in Dyson's paper (1949) giving for S the well known expression

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{+\infty} dx_1 \ldots \int_{-\infty}^{+\infty} dx_n\, P[\mathcal{L}'(x_1) \ldots \mathcal{L}'(x_n)]. \tag{4.3}$$

The evaluation of the matrix elements demands certain assumptions concerning the initial and final states at $-\infty$ and $+\infty$ respectively, and the knowledge of the expectation values of $P[\psi(x)\psi^*(x')]$ and $P[\varphi(x)\varphi(x')]$ for the vacuum state.

The considerations of section 1 connected with equations (1.4), (1.5) and (1.12) indicate that the initial as well as the final state consist of a certain number of fields satisfying the first three of equations (1.5). In Fourier space these fields may be written as

$$\psi(x) = \psi(k)\, e^{ik_\mu x_\mu} \quad (k_\mu^2 = -\varkappa^2), \quad \varphi(x) = \varphi(k)\, e^{ik_\mu x_\mu} \quad (k_\mu^2 = -\lambda^2). \tag{4.4}$$

They correspond to real particles with mass $\varkappa$ and λ respectively. The energy of these fields as calculated by means of the energy-momentum four vector P_μ^0 [cf. (3.5)] is definite. With this assumption about the boundary values, only those real processes are possible in which some free particles of the type (4.4) comming in from $-\infty$ interact and go over into another set of free particles of the same type at $+\infty$. We have no right to associate with the other zeros of the polynomials (1.14) any real particles, since this would contradict our basic assumptions contained in equation (1.1) .These other zeros may be regarded formally as describing the functional form of the modified forces acting between the particles. Admission of other types of particles then (4.4) in the initial and final states connected with the other zeros of the polynomials (1.14) is very suggestive in a formalism based on a decomposition of the Lagrangian function (3.1) into a sum of Lagrangians containing only first order derivatives of the fields (cf. Pais and Uhlenbeck 1950) and leads to the well known difficulties connected with the non-determinacy of the energy.

The final step in the determination of the S-matrix consists in calculating the vacuum expectation values $\langle P[\psi(x)\psi^*(x')]\rangle_0$ and $\langle P[\varphi(x)\varphi(x')]\rangle_0$. These expectation values may be derived by a generalization of the conventional methods defining the vacuum state as the state of lowest energy.

We write the general solutions of equations (3.16) in the form

$$\psi(x) = \frac{1}{(2\pi)^4}\oint \frac{\psi(k)\, e^{ik_\mu x_\mu}}{\sum_{i=0}^{s} \varkappa_i (k_\mu^2)^i}\, dk, \qquad \varphi(x) = \frac{1}{(2\pi)^4}\oint \frac{\varphi(k)\, e^{ik_\mu x_\mu}}{\sum_{i=0}^{s} \lambda_i (k_\mu^2)^i}\, dk, \tag{4.5}$$

where $\oint$ indicate contour integrations in the complex k_0-plane taken in the positive sense along closed paths surrounding all zeros of the polynomials occuring in the denominators of the integrands. $\psi(k)$ are arbitrary coefficients.

Special solution of (3.16) satisfying the boundary conditions (3.23) may be written in the form

$$\Delta_\varkappa(x) = \frac{1}{(2\pi)^4}\oint \frac{e^{ik_\mu x_\mu}}{\sum_{i=}^{s} \varkappa_i (k_\mu^2)^i}\, dk, \qquad \Delta_\lambda(x) = \frac{1}{(2\pi)^4}\oint \frac{e^{ik_\mu x_\mu}}{\sum_{i=0}^{s} \lambda^i (k_\mu^2)^i} dk. \tag{4.6}$$

Equations (4.5) are obvious. The proof that (4.6) satisfy the prescribed boundary conditions is given in appendix III.

Now we shall decompose each of the quantities defined by (4.5) and (4.6) into two parts. If an operator or a function is defined by a contour integral of the form

$$F(x) = \frac{1}{(2\pi)^4} \oint \frac{F(k)\, e^{ik_\mu x_\mu}}{\sum_{i=0}^{s} c_i (k_\mu^2)^i}\, dk,$$

then we may always write it as a sum of $F^+(x)$ and $F^-(x)$, where

$$F^+(x) = \frac{1}{(2\pi)^4} \oint_{c+} \frac{F(k)\, e^{ik_\mu x_\mu}}{\sum_{i=0}^{s} c_i (k_\mu^2)^i}\, dk, \qquad F^-(x) = \frac{1}{(2\pi)^4} \oint_{c-} \frac{F(k)\, e^{ik_\mu x_\mu}}{\sum_{i=0}^{s} c_i (k_\mu^2)^i}\, dk, \tag{4.7}$$

and the contours c_+ and c_- are defined as follows: c_+ is a closed contour in the complex k_0-plane taken in the positive sense, surrounding all poles with positive $Re(k_0)$ and those of the poles for which $Re(k_0) = 0$ which are on the positive imaginary axis. c_- is a closed contour taken in the positive sense, surrounding the rest of the poles.

From the commutation rules (3.30) there follow (cf. appendix IV) the commutation rules for $\psi^+, \psi^-, \varphi^+, \varphi^-$

$$\begin{aligned} [\psi^{+*}(x), \psi^+(x')] &= i\Delta_\varkappa^+(x' - x), & [\psi^{-*}(x), \psi^-(x')] &= i\Delta_\varkappa^-(x' - x) \\ [\varphi^{+*}(x), \varphi^+(x')] &= i\Delta_\lambda^+(x' - x), & [\varphi^{-*}(x), \varphi^-(x')] &= i\Delta_\lambda^-(x' - x). \end{aligned} \tag{4.8}$$

To define the vacuum state we shall need Heisenberg's equations. In the interaction representation they take on account of (2.29) and (3.11) the form

$$[G(x), n_\mu P_\mu^0] = -in_\mu \partial_\mu G(x) \tag{4.9}$$

with P_μ^0 defined by (3.5). In a special frame of reference in which $n_i = 0$, $n_0 = 1$.

$$[G(x), P_0^0] = \frac{1}{i} \frac{\partial G(x)}{\partial x_0} \tag{4.10}$$

Taking for $G(x)$ one of its Fourier components

$$G(x) = G(k)\, e^{ik_\mu x_\mu} \tag{4.11}$$

and operating with (4.10) on an eigenstate $\Psi(P_0^{0\prime})$ of P_0^0 we obtain

$$P_0^0 G\Psi(P_0^{0\prime}) = (P_0^{0\prime} + k_0)\, G\Psi(P_0^{0\prime}). \tag{4.12}$$

The operator G operating on an eigenstate of the energy operator P_0^0 belonging to the eigenvalue $P_0^{0\prime}$ changes it to an eigenstate of the same operator belonging to the eigenvalue $P_0^{0\prime} + k_0$. Now if the Fourier component of G considered in (4.12) belongs to G^+, then k_0 has a positive real part and the energy is increased by a real factor apart from the change caused by the imaginary part of k_0. If we have to do with a Fourier component of G^- the energy is decreased by a real

factor. Now we may define the vacuum as that state Ψ_0 of the system in which the real part of the energy has its minimum. This definition leads to the following equations

$$\varphi^- \Psi_0 = \psi^- \Psi_0 = 0, \qquad \Psi_0^* \varphi^{-*} = \Psi_0^* \psi^{-*} = 0. \tag{4.13}$$

It is easily seen that the complex conjugate quantities $\varphi^{-*}, \varphi^{+*}, \psi^{-*}, \psi^{+*}$ have the opposite influence on $\Psi(P_0^{0\prime})$ then the original quantities and accordingly we must put

$$\varphi^{+*} \Psi_0 = \psi^{+*} \Psi_0 = 0, \qquad \Psi_0^* \varphi^+ = \Psi_0^* \psi^+ = 0. \tag{4.14}$$

With this definition of the vacuum state one can easily show that

$$\langle P[\psi(x)\psi^*(x')]\rangle_0 = i\Delta_\varkappa^F(x-x') = \begin{cases} -i\Delta_\varkappa^-(x-x') & \text{for } x_0 > x_0' \\ i\Delta_\varkappa^+(x-x') & \text{,, } x_0 < x_0' \end{cases}$$

$$\langle P[\varphi(x)\varphi(x')]\rangle_0 = i\Delta_\lambda^F(x-x') = \begin{cases} -i\Delta_\lambda^-(x-x') & \text{,, } x_0 > x_0' \\ i\Delta_\lambda^+(x-x') & \text{,, } x_0 < x_0'. \end{cases} \tag{4.15}$$

The explicit form of the generalized Feynman's functions $\Delta_\varkappa^F(x)$ and $\Delta_\lambda^F(x)$ may be derived from (4.6) and (4.7). It is

$$\Delta_\varkappa^F(x) = \frac{1}{(2\pi)^4}\int_c \frac{e^{ik_\mu x_\mu}}{\sum_{i=0}^{s} \varkappa_i (k_\mu^2)^i}\, dk, \qquad \Delta_\lambda^F(x) = \frac{1}{(2\pi)^4}\int_c \frac{e^{ik_\mu x_\mu}}{\sum_{i=0}^{s} \lambda_i (k_\mu^2)^i}\, dk \tag{4.16}$$

with the following definition of the contour c: The contour c in the complex k_0-plane is an open path beginning at $-\infty$ on the real k_0-axis and ending at $+\infty$ on the real k_0-axis. It passes above all poles with $\mathrm{Re}(k_0) < 0$, below all poles with $\mathrm{Re}(k_0) > 0$, leaves the poles with $\mathrm{Re}(k_0) = 0$, $Im(k_0) > 0$ on the left and the poles with $\mathrm{Re}(k_0) = 0$, $Im(k_0) < 0$ on the right looking in the direction of orientation of the path.

5. Discussion. From the results of section 4 it is seen that the only effect on the S-matrix of introducing non-localities of the special type considered in section 1 is to replace the original Feynman functions of the local theory by generalized functions defined by (4.15). This is a confirmation of the results obtained by direct (integral) methods of calculating the S-matrix (cf. Rzewuski 1951, Rayski 1951). However, the method of differential quantization used in this paper removes some ambiguities inherent in the integral methods. The result of the integral methods may be simply stated as follows. Each Feynman's function of the local theory

$$R_\varkappa(x) = \frac{1}{(2\pi)^4}\int_c \frac{e^{ik_\mu x_\mu}}{-k_\mu^2 - \varkappa^2}\, dk, \qquad R_\varphi(x) = \frac{1}{(2\pi)^4}\int_c \frac{e^{ik_\mu x_\mu}}{-k_\mu^2 - \lambda^2}\, dk \tag{5.1}$$

has to be replaced by the corresponding $K(x)$ or $L(x)$ function respectively (cf. (1.7) and (1.12)), the factors $|P(k^2)|^2$ and $[F(k^2)]^2$ being quite arbitrary. Now if these functions have poles, as in our case, the direct method gives no information concerning the choice of the integration path, whereas the differential method determines this path uniquely.

Another difference is that in the integral methods only those Δ-functions which satisfy the inhomogeneous equations (1.5) get divergence factors. Solutions of the homogeneous equations (1.5), as e.g. the $\Delta(x)$-function appearing in the general commutation rules, remain unchanged as compared with the local theory. In the differential method all Δ-functions are generalized and all of them may be written in the form (4.6) or (4.16) with an appropriate choice of the integration path.

The question arises, whether the results obtained in this paper may be generalized to the class of functions which may be expressed by infinite series

$$\sum_{i=o}^{\infty} a_i (k_\mu^2)^i. \tag{5.2}$$

This seems to be certainly possible for those infinite series which are expressible in form of convergent products of the form

$$\prod_{i=o}^{\infty} \left(1 - \frac{k_\mu^2}{c_i^2}\right). \tag{5.3}$$

The situation becomes more complicated if we wish to generalize our results to the whole class of entire functions as given by the Weierstrass factor representation in its general form. However from the results obtained by the integral methods which are valid for arbitrary form-functions $P(k_\mu^2)$ and $F(k_\mu^2)$ (cf. (1.12)) one may hope that it will be possible to extend also the differential treatment to a larger class of functions then those considered in this paper. This would certainly bring us new informations as to the nature of the form-factors, in a similar way as in our case it lead to the determination of the integration paths. It may be noted that the calculations of section 2 are certainly applicable to the general class of entire functions as given by (5.2). The dificulties arise later for the transition to interaction representation and the construction of general commutation rules and general Δ-functions.

Another question is the unitarity of the S-matrix. With the special choice of boundary values adopted in this paper one selects only a special type of processes, discarding other matrix elements corresponding to other boundary values as unphysical. This could have the effect that the S-matrix calculated in this way is not unitary. This question has not been investigated in this paper but it seems from the equivalence with the direct methods in which the S-matrix is unitary that also here unitarity may be obtained by appropriate normalization of the field operators.

The problem of the unitarity of the S-matrix is closely connected with the possibility of transition to equations of infinite order. Indeed, if such a transition were possible, one could obtain form-factors of exponential type having no poles at all and the question of unitarity would not arise.

APPENDIX I

Theorem:

$$\sum_{i=o}^{s} (-1)^i \varkappa_i \Box^i \psi = \sum_{i=o}^{s} (-1)^i \varkappa_i (\partial^t_{\nu\nu})^i \psi - \partial_n \pi^{\nu *}_n. \tag{I.1}$$

Proof (by induction): We first investigate the connection between the canonical momenta corresponding to problems of the order s and $s+1$. From (3.2) and (3.4), we easily get

$$\underset{\nu_1 \dots \nu_k}{p^{\nu *}}(s+1) = \underset{\nu_1 \dots \nu_k}{p^{\nu *}}(s) + (-1)^{s+1-k} \varkappa_{s+1} \Box^{s+1-k} \psi_{\nu_1 \dots \nu_k}, \quad (k = 1, \dots s),$$

$$\underset{\nu_1 \dots \nu_{s+1}}{p^{\nu *}}(s+1) = \varkappa_{s+1} \psi_{\nu_1 \dots \nu_{s+1}} \tag{I.2}$$

and

$$\underset{(r,n)}{\pi^{\nu *}}(s+1) = \underset{(r,n)}{\pi^{\nu *}}(s) + (-1)^{s+1} \varkappa_{s+1} (\partial_n)^r \sum_{i=o}^{s+1-r} \binom{i+r-1}{i} (\partial^t_{\nu\nu})^i \Box^{s+1-r-i} \psi,$$

$$(r = 1, \dots s)$$

$$\underset{(s+1,n)}{\pi^{\nu *}}(s+1) = (-1)^{s+1} \varkappa_{s+1} (\partial_n)^{s+1} \psi. \tag{I.3}$$

The important case is $r = 1$

$$\underset{n}{\pi^{\nu *}}(s+1) = \underset{n}{\pi^{\nu *}}(s) + (-1)^{s+1} \varkappa_{s+1} \partial_n \sum_{i=o}^{s} (\partial^t_{\nu\nu}) \Box^{s-i} \psi. \tag{I.4}$$

Now we assume that (I.1) is true for s. For $s+1$ we then have in virtue of (I.4) and (I.1)

$$\sum_{i=o}^{s+1} (-1)^i \varkappa_i \Box^i \psi - \sum_{i=o}^{s+1} (-1)^i \varkappa_i (\partial^t_{\nu\nu})^i \psi - \partial_n \underset{n}{\pi^{\nu *}}(s+1)$$

$$= (-1)^{s+1} \varkappa_{s+1} \left\{ \Box^{s+1} - (\partial^t_{\nu\nu})^{s+1} + \partial_{nn} \sum_{i=o}^{s} (\partial^t_{\nu\nu})^i \Box^{s-1} \right\} \psi = 0 \tag{I.5}$$

since the bracket $\{\ \}$ occuring in (I.5) vanishes identically. This again may be shown by induction

$$\Box^{s+1} - (\partial^t_{\nu\nu})^{s+1} + \partial_{nn} \sum_{i=o}^{s} (\partial^t_{\nu\nu})^i \Box^{s-i} = \Box^{s+1} - (\partial^t_{\nu\nu})^{s+1} + \partial_{nn} \sum_{i=o}^{s-1} (\partial^t_{\nu\nu})^i \Box^{s-i}$$

$$- \partial_{nn} (\partial^t_{\nu\nu})^s = \Box \left\{ \Box^s - (\partial^t_{\nu\nu})^s + \partial_{nn} \sum_{i=o}^{s-1} (\partial^t_{\nu\nu})^i \Box^{s-1-i} \right\} = 0 \tag{I.6}$$

if the bracket $\{\ \} = 0$ for s.

Both theorems are true for $s + 1$ if they are true for s. Since they are true for $s = 1$ as may be easily checked, they are true in general for arbitrary s. The same holds for the other identities in (3.6).

APPENDIX II

The commutation rules (3.27) are valid for $r = 0, 1, \ldots s - 1$ on account of (2.26). To prove their validity for $r = s, s + 1, \ldots 2s - 1$ we shall use formulae (3.4). We have for $r = s$ $(s > 1)$

$$\pi^{\nu}_{(s,n)} = (-1)^s \varkappa_s (\partial_n)^s \psi^* \quad \text{and} \quad [(\partial_n)^s \psi^*(x), \psi(x')] = 0 \tag{II.1}$$

on account of (2.26). For $r = s + 1$ we get the same result using (2.26), (3.27) and (I.1). Proceeding thus step by step, we get the result that all derivatives $(\partial)^s \psi^*(x)$ up to the order $2s - 2$ commute with $\psi(x')$. Thus the first part of the commutation rules (3.27) is proved. To prove the commutation rule for $r = 2s - 1$ we again use (3.4)

$$\pi^{\nu}_n = \sum_{i=o}^{s-1} \sum_{k=o}^{s-1-i} (-1)^{i+k+1} \varkappa_{i+k+1} (\partial^{\iota}_{\nu\iota})^i \partial_n \Box^k \psi^*. \tag{II.2}$$

On account of the commutation rules for $r = 0, 1, \ldots 2s - 2$ only the highest derivative in (II.2) will contribute to the commutator of this expression with $\psi(x')$. On account of (2.26) we have, therefore,

$$i\delta_{\sigma}(x - x') = -[\pi^{\nu}_n(x), \psi(x')] = \varkappa_s [(\partial_n)^{2s-1} \psi^*(x), \psi(x')] \tag{II.3}$$

which is exactly equation (3.27) for $r = 2s - 1$. The same proof holds for the field φ.

APPENDIX III

We separate in (4.6) the integration over the variable k_0,

$$\Delta_{\varkappa}(x) = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\vec{x}} \cdot \frac{1}{2\pi} \oint \frac{e^{-ik_0 x_0}}{\sum_{i=o}^{s} \varkappa_i (k_{\mu}^2)^i} dk_0 \tag{III.1}$$

and consider the integral

$$I(x_0) = \frac{1}{2\pi} \oint \frac{e^{-ik_0 x_0}}{\sum_{i=o}^{s} \varkappa^i (k_{\mu}^2)^i} dk_0 \tag{III.2}$$

n the complex k_0-plane. Since all expressions are invariant with respect to Lorentz transformations we may choose the surface σ to be the hyperplane $x_0 = \text{const}$. Normal derivatives become derivatives with respect to x_0, and we have

$$\frac{d^r I(x_0)}{dx_0^r} = \frac{(-i)^r}{2\pi} \oint \frac{k_0^r e^{-ik_0 x_0}}{\sum_{i=o}^{s} \varkappa_i (k_\mu^2)^i} dk_0 . \tag{III.3}$$

Since the boundary conditions (3.23) must be satisfied for points on σ, this means for our special choice of σ that we have to put in (III.3) $x_0 = 0$.

$$\left.\frac{d^r I(x_0)}{dx_0^r}\right|_{x_0=0} = \frac{(-i)^r}{2\pi} \oint \frac{k_0^r dk_0}{\sum_{i=o}^{s} \varkappa_i (k_\mu^2)^i} \tag{III.4}$$

This integral vanishes for $r < 2s - 1$. For eaven r it follows from the symmetry properties of the integrand. For odd r it may be shown by direct calculation. For $r = 2s - 1$ we have

$$\left.\frac{d^{2s-1} I(x_0)}{dx_0^{2s-1}}\right|_{x_0=0} = \frac{(-i)^{2s-1}}{2\pi} \oint \frac{k_0^{2s-1} dk_0}{\sum_{i=o}^{s} \varkappa_i (k_\mu^2)^i} . \tag{III.5}$$

The sum occuring in the integrand may be written

$$\sum_{i=o}^{s} \varkappa_i (k_\mu^2)^i = \varkappa_s \prod_{i=o}^{s} (k_\mu^2 + m_i^2) = \varkappa_s (-1)^s \prod_{i=o}^{s} (k_o^2 - c_i^2) , \quad c_i^2 = \vec{k}^2 + m_i^2 , \tag{III.6}$$

where some of the roots may be identical. On account of the vanishing of (III.4) for $r < 2s - 1$ we may add to (III.5) terms of the form (III.4) with coefficients chosen in such a way as to give together with k_0^{2s-1} all zeros of the polynomial in the denominator with respect of one. We may write, therefore,

$$\left.\frac{d^{2s-1} I(x_0)}{dx_0^{2s-1}}\right|_{x_0=0} = \frac{(-i)^{2s-1}(-1)^s}{2\pi\varkappa_s} \oint \frac{dk_0}{k_0 - a} = -\frac{1}{\varkappa_s} . \tag{III.7}$$

Consider now the integral occuring in (2.23) in our special frame of reference

$$\varkappa_s \int (\partial_n')^{2s-1} \Delta_\varkappa (x - x') d\sigma' = \varkappa_s \int (\partial_0')^{2s-1} \Delta_n (x - x') \Big|_{t_0 = t_0'} dx' dy' dz'$$

$$= \frac{1}{(2\pi)^3} \int dx' dy' dz' \int d\vec{k}\, e^{i\vec{k}(\vec{x} - \vec{x}')} = 1$$

The same demonstration applies also to the $\Delta_\lambda(x)$ function.

APPENDIX IV

Since $\psi(x)$ and $\Delta(x)$ as defined by (4.5) and (4.6) are a sum of contributions corresponding to the poles of the integrand, we may carry out calculations for one pair of poles which we assume to be complex. According to (III.6), we may write

$$\sum_{i=o}^{s} \varkappa_i (k_\mu^2)^i = (-1)^s \varkappa_s \prod_{i=o}^{s} (k_o^2 - c_i^2) =$$

$$= (-1)^s \varkappa_s (k_0 - a)(k_0 + a)(k_0 - a^*)(k_0 + a^*) \prod_{i=2}^{s} (k_o^2 - c_i^2) =$$

$$= (k_0 - a)(k_0 + a)(k_0 - a^*)(k_0 + a^*) f(k_0). \qquad \text{(IV.1)}$$

Carrying out the integrations in the complex k_0-plane we may write for $\psi(x)$, $\psi^*(x)$ and $\Delta_\varkappa(x - x')$

$$\psi(x) = \frac{i}{(2\pi)^3} \int \vec{dk}\, e^{i\vec{k}\vec{x}} \frac{1}{2} (a^2 - a^{*2})^{-1} \left\{ \frac{\psi(\vec{k}, a) e^{-iat}}{f(a)a} - \frac{\psi(\vec{k}, -a)^* e^{iat}}{f(-a)a} \right.$$

$$\left. - \frac{\psi(\vec{k}, a^*) e^{-ia^* t}}{f(a^*)a^*} + \frac{\psi(\vec{k}, -a^*) e^{ia^* t}}{f(-a^*)a^*} \right\},$$

$$\psi^*(x) = \frac{i}{(2\pi)^3} \int \vec{dk}\, e^{-i\vec{k}\vec{x}} \frac{1}{2} (a^2 - a^{*2})^{-1} \left\{ \frac{\psi^*(\vec{k}, a^*) e^{ia^* t}}{f(a^*)a^*} - \frac{\psi^*(\vec{k}, -a^*) e^{-ia^* t}}{f(-a^*)a^*} \right.$$

$$\left. - \frac{\psi^*(\vec{k}, -a) e^{ia^* t}}{f(a)a} + \frac{\psi^*(\vec{k}, -a) e^{-iat}}{f(-a)a} \right\}, \qquad \text{(IV.2)}$$

$$\Delta_\varkappa(x - x') = \frac{i}{(2\pi)^3} \int \vec{dk}\, e^{i\vec{k}(\vec{x} - \vec{x'})} \frac{1}{2} (a^* - a^{*2})^{-1} \left\{ \frac{e^{-ia(t-t')}}{f(a)a} - \frac{e^{ia(t-t')}}{f(-a)a} \right.$$

$$\left. - \frac{e^{-ia^*(t-t')}}{f(a^*)a^*} + \frac{e^{ia^*(t-t')}}{f(-a)a^*} \right\}.$$

To satisfy the identity (3.30):

$$[\psi(x), \psi^*(x')] = -i\Delta_\varkappa(x - x') \qquad \text{(IV. 3)}$$

we have first to equal to zero all commutators of the Fourier components of $\psi^*(x')$ and $\psi(x)$ belonging to different types of a: $a, -a, a^*, -a^*$. The only contribution to $[\psi(x), \psi^*(x')]$ is then

$$[\psi(x), \psi^*(x')] = \frac{1}{(2\pi)^6} \iint \vec{dk}\, \vec{dk'}\, e^{i(\vec{k}\vec{x} - \vec{k'}\vec{x'})} \frac{1}{4} (a^2 - a^{*2})^{-1} (a'^2 - a'^{*2})^{-1}$$

$$\left\{ \frac{[\psi(\vec{k}, a)\psi^*(\vec{k}', a')]}{f(a)f(a')aa'} e^{-i(at - a't')} + \frac{[\psi(\vec{k}, -a), \psi^*(\vec{k}', -a')]}{f(-a)f(-a')aa'} e^{i(at - a't')} \right. \qquad \text{(IV.4)}$$

$$\left. + \frac{[\psi(\vec{k}, a^*), \psi^*(\vec{k}', a'^*)]}{f(a^*)f(a'^*)a^*a'^*} e^{-i(a^* t - a'^* t')} + \frac{[\psi(\vec{k}, -a^*), \psi^*(\vec{k}', -a'^*)]}{f(-a^*)f(-a'^*)a^*a'^*} e^{i(a^* t - a'^* t')} \right.$$

To satisfy (IV.3) we must put

$$[\psi(\vec{k}, a), \psi^*(\vec{k}', a')] = (2\pi)^3 2(a^2 - a^{*2}) a f(a) \delta(\vec{k} - \vec{k}')$$

$$[\psi(\vec{k}, -a), \psi^*(k', -a')] = -(2\pi)^3 2(a^2 - a^{*2}) a f(-a)\, \delta(\vec{k} - \vec{k}') \quad \text{(IV.5)}$$

and the two complex. conjugate relations.

If we define

$$\psi^+(x) = \frac{i}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\vec{x}}\, \frac{1}{2}\,(a^2 - a^{*2})^{-1}\left\{\frac{\psi(\vec{k}, a)\, e^{-iat}}{f(a)\, a} - \frac{\psi(\vec{k}, a^*)\, e^{-ia^*t}}{f(a^*)\, a^*}\right\},$$

$$\psi^{+*}(x) = \frac{i}{(2\pi)^3}\int d\vec{k}\, e^{-i\vec{k}\vec{x}}\, \frac{1}{2}\,(a^2 - a^{*2})^{-1}\left\{\frac{\psi^*(\vec{k}, a^*)\, e^{ia^*t}}{f(a^*)\, a^*} - \frac{\psi\nu(\vec{k}, a)\, e^{iat}}{f(a)\, a,}\right\}. \quad \text{(IV.6)}$$

$$\Delta^+(x - x') = \frac{i}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}(\vec{x} - \vec{x}')}\, \frac{1}{2}\,(a^2 - ax^2)^{-1}\left\{\frac{e^{-ia(t-t')}}{f(a)\, a} - \frac{e^{-ia^*(t-t')}}{f(a^*)\, a^*}\right\},$$

and analogously ψ^-, ψ^{-*}, $\Delta_\varkappa^-$ by means of the remaining terms in expressions (IV.2), it follows from (IV.5) that the commutation relations between ψ^+, ψ^{+*} and ψ^-, ψ^{-*} are just (4.8). This is true of course also for the contributions to ψ and $\Delta_\varkappa$ from other poles. The calculation is somewhat more complicated for multiple poles, but offers no essential difficulties. (IV. 6) is that part of ψ and $\Delta_\varkappa$ which corresponds to poles with $Re(k_0) > 0$. Thus the definitions (IV.6) and (4.7) coincide.

КРАТКОЕ СОДЕРЖАНИЕ

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В настоящей работе проблема канонической квантизации теорий неместного поля сокращается, для известного класса форм — функций, к проблеме канонической квантизации теорий поля содержащих производные полей высшего (возможно даже бесконечного) порядка. Для производных конечного порядка вводится представление взаимодействия и создаются общие правила коммутации между величинами поля и двумя любыми точками пространства — времени. Матрица S высчитывается при помощи обобщённых функций Грина. Наконец, дискутируется связь с прямыми (интегральными) методами вычисления матрицы S для неместных теорий.

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ELECTRODYNAMICS WITHOUT POTENTIALS

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A general framework is given for a realy unitary theory of the electromagnetic field characterized by the skew-symmetric tensor $p_{\alpha\beta}$. The potentials appear only later as derived quantities. The Lagrangian in this theory is assumed to be an arbitrary function of $P = -{}^1/_4 p_{\alpha\beta}p^{\alpha\beta}$ and $|\varrho_0| = c^{-1}(j_\alpha j^\alpha)^{1/2}$, where j^α by definition $= c/4\pi p^{\alpha\beta}{}_{,\beta}$. Thus the Lagrangian is a function of the field and its derivatives. The field equations are obtained by variation with respect to the field. Generally this "potentialless" electrodynamics is equivalent to Mie's electrodynamics in which the potentials are fundamental. But this is not always so. Dirac's new electrodynamics can be formulated as a "potentialless" unitary theory (Lagrangian $= P + 4\pi k/c|\varrho_0|$) but remains outside the frame of Mie's electrodynamics.

1. Introduction. Potentials play a prominent rôle in all the current theories of classical and quantum electrodynamics, a rôle intimately connected with the variational principle lying at the base of the theory. The variation of the action integral is commonly performed with respect to the potentials and by so doing one considers implicitly these potentials as fundamental quantities, the field variables. This accounts for the essential importance of the potentials in the canonical formalism and explains among other things the fundamental significance in quantum electrodynamics of the commutation rules for the potentials. On the other hand, the theory has to be gauge invariant, in consequence of which the potentials have no immediate physical meaning.

In the present paper we intend to show that in the dualistic theories as well as in the unitary theory (in which there is no matter carrying electrical charges independently of the field) the variational principle may be so formulated that the compontens of the electromagnetic field are regarded as fundamental quantities: the potentials become then merely secondary quantities of distinctly secondary character.

In the case of dualistic theories the "potentialless" variational principle brings in nothing new from the physical point of view; the equations to which it leads are stricly equivalent to those given by the conventional theories. This mode of presentation casts, however, some additional light on the interesting problem of dualism between the quantities $\vec{B}$, $\vec{E}$ and $\vec{H}$, D, a dualism known before (Born and Infeld 1934).

In the case of the unitary theory which will be considered here, the fact of the fundamental rôle being played by the components of the electromagnetic field rather than the potentials may be of some importance on the quantum level.

2. The dualistic theories. In these theories the densities of current and charge are introduced as linked together with this form of matter that is independent of the electromagnetic field, e.g., as point singularities, extended sources (Mc Manus 1949), or currents connected with a complex field. The motion of matter carrying the charges is governed by equations of motion, the field complies with the field equations.

In the following discussion of the conventional variational principle with potentials, we shall introduce no restrictions whatever concerning the character of the sources and we shall assume general non-linear equations for the electromagnetic field (Born - Infeld 1934).

The equations characterizing the electromagnetic field $f_{\alpha\beta}$ ($f_{\alpha\beta} = -f_{\beta\alpha}$; $[f_{10}, f_{20}, f_{30}] = \vec{E}$, $[f_{23}, f_{31}, f_{12}] = \vec{B}$) have in general the form

$$f_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta} \tag{1}$$

or, what amounts to the same,

$$f_{[\alpha\beta,\gamma]} = f_{\alpha\beta,\gamma} + f_{\beta\gamma,\alpha} + f_{\gamma\alpha,\beta} = 0 \tag{1'}$$

and

$$p^{\alpha\beta}{}_{,\beta} = 4\pi/c j^{\alpha}, \tag{2}$$

where the quantities A_α are the potentials, and the components of the skew-symetrical tensor $p_{\alpha\beta}$ are bi-unique functions of the components of the $f_{\alpha\beta}$ tensor (physical meaning of $p_{\alpha\beta}$: $[p_{10}, p_{20}, p_{30}] = \vec{D}$, $[p_{23}, p_{31}, p_{12}] = \vec{H}$). The form of the current four-vector j^α depends on the chosen variety of the dualistic theory, e.g., in the case of point sources,

$$j^{\alpha} = c \sum_{(a)} \overset{a}{e} \int \delta_4(x^\nu - \overset{a}{\xi}{}^\nu) \frac{d\overset{a}{\xi}{}^\alpha}{d\lambda^a} d\lambda^a,$$

where $\overset{a}{\xi}{}^\alpha(\lambda^a)$ is the world-line of the a-th particle.

The equations characterizing the field and matter connected with the charges are usually deduced from the variational principle

$$\delta I = \delta(I_1 + I_2 + I_3) = 0, \tag{3}$$

where

$$I_1 = -\frac{1}{4\pi c} \int d_4 \mathrm{x} \mathbf{L}$$

is the action integral of the free field, L is a function of the invariants $F = {}^1/_4 f_{\alpha\beta} f^{\alpha\beta}$, $G^2 = ({}^1/_4 f_{\alpha\beta} f^{*\alpha\beta})^2$ (where $f^*{}_{\alpha\beta}$ is the dual tensor of $f_{\alpha\beta}$). Then

$$I_2 = 1/c^2 \int d_4 \mathrm{x}\, A_\alpha j^\alpha$$

describes the interaction of the field with matter carrying the charges, and I_3 is the action of free matter carrying the charges.

In spite of the fact that the potentials A_α appear explicitly in I_2, the theory is gauge-invariant when the structure of the expression for j_α implies the validity of the continuity equation $j^\alpha{}_{,\alpha} = 0$. Indeed, by substituting $A_\alpha + S,_\alpha$ for A_α one increases I_2 by

$$\Delta I_2 = 1/c^2 \int d_{(4)}\, \mathrm{x}\, S,_\alpha j^\alpha = 1/c^2 \oint d_{(3)} \mathrm{x} S n_\alpha j^\alpha - 1/c^2 \int d_{(4)}\, \mathrm{x}\, S j^\alpha{}_{,\alpha}.$$

When $j^\alpha{}_{,\alpha} = 0$, the variation ΔI_2, being a variation of a surface integral, vanishes.

Assuming (1), i.e., considering $f_{\alpha\beta}$ in **L** to be expressed in terms of A_α which is equivalent to assuming the validity of the first set of four field equations, (1'), and varying (3) with respect to the A_α's one gets the second set of four field equations, (2), where

$$p_{\alpha\beta} = 2 \frac{\partial \mathbf{L}}{\partial f^{\alpha\beta}} \tag{4}$$

By varying the parameters of matter connected with the charges, one gets equations of motion. Thus the potentials are in the usual formalism treated as fundamental quantities; this produces a marked effect upon the whole canonical formalism.

We propose below a formulation of the variational principle in which the components of the *electromagnetic field* and *not* the potentials appear as fundamental quantities describing the field. As equations (4) may be uniquely solved with respect to the $f_{\alpha\beta}$ we may choose the components $p_{\alpha\beta}$ as fundamental quantities. We shall define **H**, the dual function of **L**, depending on the $p_{\alpha\beta}$ by requiring that

$$f_{\alpha\beta} = -2 \frac{\partial \mathbf{H}}{\partial p_{\alpha\beta}} \tag{5}$$

has to be equivalent to (4). It is easily seen that **H** depends on the $p_{\alpha\beta}$ by means of the invariants $P = -{}^1/_4 p_{\alpha\beta} p^{\alpha\beta}$, $Q^2 = ({}^1/_4 p_{\alpha\beta} p^{*\alpha\beta})^2$. It follows also from (4) and (5) that

$$\mathbf{L} - \mathbf{H} = {}^1/_2 p_{\alpha\beta} f^{\alpha\beta}. \tag{6}$$

Knowing **L**, we can find the $f_{\alpha\beta}$ as functions of the $p_{\alpha\beta}$ by solving (4) with respect to the $f_{\alpha\beta}$, and then **H** from (6).

We express now the variational principle by requiring

$$\delta \left[-\frac{1}{4\pi c} \int d_4\, \mathrm{x}\, \mathbf{H} + I_3 \right] = 0 \tag{7}$$

with the restrictive condition

$$4\pi/c\, j^\alpha - p^{\alpha\beta}{}_{,\beta} = 0, \tag{8}$$

where the variation is produced by independent variations of the skew-symmetrical tensor $p_{\alpha\beta}$ and the parameters of matter connected with the charges.

In using the variational principle we assume equations (2) rather than equations (1), which were assumed previously; the result is the same as the previous one. Indeed, we can get rid of restriction (8) by the method of Lagrange's undetermined multipliers and write instead of (7) + (8)

$$\delta\left\{-\frac{1}{4\pi c}\int d_{(4)}\mathrm{x}\left[\mathbf{H}+(p^{\alpha\beta},_{\beta}-4\pi/cj^{\alpha})A_{\alpha}\right]+I_3\right\}=0, \tag{9}$$

where A_α are the Lagrange multipliers. Variation with respect to the $p_{\alpha\beta}$ yields, due to (5), $f_{\alpha\beta}=A_{\beta,\alpha}-A_{\alpha,\beta}$, and hence (1), equivalent to (1'), by identifying Lagrange's multipliers with the potentials. In the variation of (9) with respect to the parameters of matter carrying the charges, the essential terms are

$$1/c^2\int d_{(4)}\mathrm{x}j^{\alpha}A_{\alpha}+I_3.$$

They are identical in form with the corresponding ones in the variational principle (3) and lead, therefore, to the same equations of motion as before.

Our restrictive conditions (8) are equations describing the interaction of the field with matter carrying the charges. By formulating them, we assume, so to say, the "field current" $c/4\pi p^{\alpha\beta},_{\beta}$ to be equal to the "particle current" j^α; this allows us to dispense with the potentials as fundamental quantities. To say it once more, the $p_{\alpha\beta}$ are our fundamental quantities and the electromagnetic potentials A_α appear solely as Lagrange multipliers and play thus an auxiliary rôle only.

The variational principle (3) + (1) is equivalent to the variational principle (7) + (8). In the first instance the $f_{\alpha\beta}$ are clearly favoured, in the second one we must work with the quantities $p_{\alpha\beta}$ and the dual Lagrange function $\mathbf{H}$. Of course, in the case of a linear theory ($\mathbf{L}=F$), no distinction between the $f_{\alpha\beta}$ and the $p_{\alpha\beta}$ is needed.

3. The unitary theory. We shall now endeavour to construct a pure field theory, i.e., a unitary theory in which no matter connected with the charges is introduced independently of the field. Our fundamental concept will be the electromagnetic field, characterized by the skew-symmetrical tensor $p_{\alpha\beta}$ (consisting of the two vectors $\vec{D}$ and $\vec{H}$); all other physical quantities will be expressed in terms of the $p_{\alpha\beta}$. We shall attribute no physical significance to the potentials which will appear only afterwards as secondary quantities.

The simplest vector which can be formed from $p^{\alpha\beta}$ is

$$4\pi/c\,j^{\alpha}=p^{\alpha\beta},_{\beta}. \tag{10}$$

We shall adopt the above relations as *definition* of the current vector j^α. (It should be clearly understood that we do not consider these relations as field equations, but only as definitions of j^α in terms of the $p^{\alpha\beta}$). This current describes the motion of generally continuously distributed charges; the electromagnetic field $p_{\alpha\beta}$ is responsible for their existence. One has to connect with these charges the rest-density of charge ϱ_0 and the four-velocity v^α by means of the relations

$$c\,\varrho_0 v^{\alpha}=j^{\alpha}, \tag{11}$$

$$v_{\alpha}v^{\alpha}=1. \tag{12}$$

It follows from (10), (11), and (12) that

$$|\varrho_0|=c^{-1}(j_{\alpha}j^{\alpha})^{1/2}=1/4\pi(g_{\alpha\beta}p^{\alpha\nu},_{\nu}p^{\beta\mu},_{\mu})^{1/2}. \tag{13}$$

We consider the v^a field as possessing a well-defined physical meaning only in those regions in which $\varrho_0 \neq 0$; where the rest-density of charge ϱ_0 vanishes, the velocity field v^a becomes totally illusory[1].

We wish to base our theory on a variational principle in which the Lagrange function depends on the invariant $P = -{}^1/_4 p_{\alpha\beta}p^{\alpha\beta}$ and the quantity $|\varrho_0| = {}^1/_4\pi \times$ $\times (g_{\alpha\beta}p^{\alpha\nu}{}_{,\nu}p^{\beta\mu}{}_{,\mu})^{1/2}$. In general, we shall consider Lagrange functions depending on these quantities in a non-linear way. As the Lagrange functions must depend only on dimensionless quantities, we must look for dimensionless combinations of P and ϱ_0. The quantity

$$\varrho = 4\pi k|\varrho_0| = k(g_{\alpha\beta}p^{\alpha\nu}{}_{,\nu}p^{\beta\mu}{}_{,\mu})^{1/2}, \tag{14}$$

where $k = m_0c^2/|e|$, has the same dimensions as P (m_0 and e are respectively the rest-mass and charge). The quantities P/b^2 and ϱ/b^2, where $b = |e|/r_0{}^2$ and r_0 has the dimensions of length, are thus dimensionless. If we measure the field in natural units, we may put $b = 1$, and the Lagrangian function will depend on P and ϱ. Hence we assume that this function has the form

$$\mathbf{H} = \mathbf{H}(P, \varrho) \tag{15}$$

and depends, therefore, on the antisymmetrical $p_{\alpha\beta}$ and its derivatives. As

$$-4\pi c\delta I = \delta\int d_4\mathrm{x}\mathbf{H} = \int d_4\mathrm{x}\left\{-{}^1/_2\frac{\partial\mathbf{H}}{\partial P}p_{\alpha\beta} - \frac{4\pi k^2}{c}\left(\frac{\partial\mathbf{H}}{\partial\varrho}j_a/\varrho\right)_{,\beta}\right\}\delta p^{\alpha\beta}$$

$$+\frac{4\pi k^2}{c}\oint_J d_3\mathrm{x}\,\frac{\partial\mathbf{H}}{\partial\varrho}j_a n_\beta/\varrho\,\delta p^{\alpha\beta}, \tag{16}$$

hence, remembering that $\delta p^{\alpha\beta} = -\delta p^{\alpha\beta}$, we get from $\delta I = 0$ for the equations of motion

$$\frac{\partial\mathbf{H}}{\partial P}p_{\alpha\beta} = \frac{4\pi k^2}{c}\left[\left(\frac{\partial\mathbf{H}}{\partial\varrho}j_\beta/\varrho\right)_{,\alpha} - \left(\frac{\partial\mathbf{H}}{\partial\varrho}j_a/\varrho\right)_{,\beta}\right]. \tag{17}$$

These equations constitute a set of six differential equations of the second order for the six quantities $p_{\alpha\beta}$, where, of course, j_a stands for $c/4\pi p^{\alpha\beta}{}_{,\beta}$; together with proper initial conditions, they define uniquely the $p_{\alpha\beta}$, and through them also other physical quantities. We shall consider (17) as the fundamental equations of electrodynamics.

To understand better the meaning of these equations, we introduce the following notations

$$f_{\alpha\beta} = -2\frac{\partial\mathbf{H}}{\partial p^{\alpha\beta}} = \frac{\partial\mathbf{H}}{\partial P}p_{\alpha\beta} \tag{18}$$

$$A_a = \frac{c}{4\pi}\frac{\partial\mathbf{H}}{\partial j^a} = \frac{4\pi k^2}{c}\frac{\partial\mathbf{H}}{\partial\varrho}j_a/\varrho \tag{19}$$

by which equations (17) acquire the form

$$f_{\alpha\beta} = A_{\beta a} - A_{a,\beta}. \tag{20}$$

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[1] As $v^a = c^{-1}\varrho_0{}^{-1}j^a$ from (11), it may seen, from a purely formal point of view, that in some regions ϱ_0 may tend to zero in such a way as to make $v^a = 0/0$ tend to a sensible limit. We consider, however, the interpretation of such a v^a as velocity field conuected with the charges as not allowed.

The tensor $f_{\alpha\beta}$ is interpreted through the vectors $\vec{B}$ and $\vec{E}$: the quantities A_α which, for obvious reasons, we shall call potentials have a distinctly secondary character: we see that equations (18), (19) and (20) are equivalent to our fundamental equations (17). As a consequence of (20), we have the equations

$$f_{[\alpha\beta,\,\gamma]} = 0 \tag{21}$$

which say that magnetic poles do not exist.

Now we can easily find the energy-momentum tensor T^α_β of our unitary theory. Contracting equation (21) by means of the $p^{\alpha\beta}$ tensor and taking into consideration (10) and the equation of continuity of current, which follows from (10), we come after some easy transformations to the conclusion that the energy-momentum tensor

$$T^\alpha_\beta = {}^1/_{4\pi}\left[(\mathbf{H} + 1/2\, p^{\nu\mu} f_{\nu\mu})\,\delta^\alpha_\beta - p^{\alpha\nu} f_{\beta\nu}\right] + 1/c\left[A_\beta j^\alpha - A_\nu j^\nu\,\delta^\alpha_\beta\right] \tag{22}$$

satisfies the conservation equations

$$T^\alpha_{\beta,\alpha} = 0. \tag{23}$$

It is clear that we can consider all the quantities appearing in T^α_β as expressed in terms of the $p_{\alpha\beta}$ and its derivatives.

We shall now turn to the problem of reducing the integration of equations (17) to integration of four equations for the four A_α. If we put

$$F = 1/4 f_{\alpha\beta} f^{\alpha\beta}, \qquad \mu = k^{-1}(A_\alpha A^\alpha)^{1/2}$$

then, from (18) and (19), we get the relations

$$F = -\left(\frac{\partial \mathbf{H}}{\partial P}\right)^2 P, \tag{24}$$

$$\mu = \left|\frac{\partial \mathbf{H}}{\partial \varrho}\right|. \tag{25}$$

We shall investigate first the case when $D(F,\mu)/D(P,\varrho) \neq 0$; from relations (24) and (25) we obtain, after solving them for P and ϱ, both P and ϱ as functions of F and μ.

If we write now equations (18) and (19) in the form

$$p_{\alpha\beta} = \left(\frac{\partial \mathbf{H}}{\partial P}\right)^{-1} f_{\alpha\beta}, \tag{26}$$

$$j^\alpha = \frac{c\,k^{-2}}{4\pi}\,\varrho\left(\frac{\partial \mathbf{H}}{\partial \varrho}\right)^{-1} A^\alpha, \tag{27}$$

where $\mathbf{H}$ and ϱ are looked upon as functions of F and μ, we can consider the $p_{\alpha\beta}$ and j_α in (26) and (27) as functions of $f_{\alpha\beta}$ and A_α. Making use of (10), we get finally four equations for the A^α namely

$$k^{-2}\varrho\left(\frac{\partial H}{\partial \varrho}\right)^{-1} A^\alpha = \left\{\left(\frac{\partial H}{\partial P}\right)^{-1} f^{\alpha\beta}\right\}_{,\beta}, \tag{28}$$

if we consider the $f_{\alpha\beta}$ as expressed in terms of the A_α by means of (20). The knowledge of the A_α satisfying (28) gives us the solution of the fundamental equations (17), as we know then the $f_{\alpha\beta}$ by (20) and can calculate the $p_{\alpha\beta}$ from (26).

There follows then the interesting fact that in the case $D(F, \mu)/D(P, \varrho) \pm 0$ our unitary electrodynamics without potentials is equivalent to a general theory of the type of Mie's electrodynamics (1912). In fact, if we assume the validity of equations (20) and require, following Mie, the equations for the A_α to be a consequence of the variational principle

$$\delta\left\{-\frac{1}{4\pi c}\int d_{(4)}\ \mathrm{x}\ \mathbf{L}(F, \mu)\right\} = 0, \tag{29}$$

we get the equations

$$\left(\frac{\partial \mathbf{L}}{\partial F} f^{\alpha\beta}\right)_{,\beta} + k^{-2}\,\frac{\partial \mathbf{L}}{\partial \mu}\, A^\alpha/\mu = 0. \tag{30}$$

In order for equations (28) to have the same content as equations (30), it is necessary and sufficient that the relations (18) and (19):

$$\begin{aligned} f_{\alpha\beta} &= -2\,\frac{\partial \mathbf{H}}{\partial p^{\alpha\beta}} \\ A_\alpha &= c/4\pi\,\frac{\partial \mathbf{H}}{\partial j^\alpha} \end{aligned} \tag{31}$$

are consequences of the relations

$$\begin{aligned} j_\alpha &= -c/4\pi\,\frac{\partial \mathbf{L}}{\partial A^\alpha} \\ p_{\alpha\beta} &= 2\,\frac{\partial \mathbf{L}}{\partial f^{\alpha\beta}}, \end{aligned} \tag{32}$$

where H is treated as a function of P and $\varrho = 4\pi k/c\,(j_\alpha j^\alpha)^{1/4}$.

From (31) and (32) one sees that

$$\mathbf{L} - \mathbf{H} = 1/2 p_{\alpha\beta} f^{\alpha\beta} - 4\pi/c A_\alpha j^\alpha. \tag{33}$$

Therefore, if it is possible to solve equations (31) with respect to the $p_{\alpha\beta}$ and j_α, then thanks to (33), we can always find a function **L** which is dual to **H** and yields an electrodynamics of Mie, equivalent to our electrodynamics without potentials. The other way round: if it is possible to solve equations (32) with respect to the $f_{\alpha\beta}$ and A_α, then it may be easily shown that taking from (33) the function **H** dual to **L** we shall be able to find a potentialless formulation of every special case of Mie's electrodynamics.

To illustrate our arguments let us consider as an example the following expression for **L**, for which equations (32) can be explicitly solved[1],

[1] It is just a function of this form that has been investigated by Mie; for the theory to give significant results n has to be a large number; on computational grounds, in order to be able to solve certain equations, Mie had to assume that $n = 6$.

$$\mathbf{L} = F + a_n \mu^n \tag{34}$$

It is easily seen that the dual function $\mathbf{H}$ of $\mathbf{L}$ is given by

$$\mathbf{H} = P + b_n \varrho^{n/(n-1)}, \tag{35}$$

where b_n is dependent on the a_n and k. In particular let us take $n = 2$, then $\mathbf{L} = a_2\mu^2 + F$ is the Lagrangian of the vector-meson equations (of Proca) in vacuo. Thus we see that working with $\mathbf{H} = P + b_2\varrho^2$, we can formulate the theory of the Proca equations in a "potentialless" fashion; this is a rather unexpected result.

We notice further that if a_n are chosen in such a way that $\lim_{n \to \infty} b_n = b$ is a finite number and hence that there exists, so to say, a limit for $\mathbf{H} = P + b\varrho$, there will be no corresponding limit "in the $\mathbf{L}$ picture". The theory with $\mathbf{H} = P + b\varrho$ cannot be "translated" into the $\mathbf{L}$ picture and should be considered separately. Similarly, when $a_n = 0$, i.e., in the theory of Maxwell's equations in vacuo our theory cannot be formulated in a potentialless manner and cannot be therefore translated from the L picture into the H picture.

It may be also mentioned that as far as our "potentialless" theory is equivalent to a Mie-type theory, the energy-momentum tensor can be expressed by means of the quantities A_α, as follows

$$T^\alpha_\beta = 1/4\pi(\mathbf{L}\delta^\alpha_\beta - p^{\alpha\nu} f_{\beta\nu}) + 1/cA_\beta j^\alpha. \tag{36}$$

This equation follows from (22) and (33); in (36) the $p^{\alpha\nu}$ and j^α have to be understood as expressed in terms of the A_α by means of relations (32).

We shall now consider a special case in which equations (24) and (25) cannot be solved with respect to P and ϱ, i.e., when $D(F, \mu)/D(P, \varrho) = 0$: the vanishing of the Jacobian expresses generally the fact that there exists a functional relation between F and μ, a relation of the form

$$G(F, \mu) = 0. \tag{37}$$

In this case it is impossible to construct an electrodynamics of Mie's type equivalent to our theory, because of the existence of an additional condition connecting the four A_α; hence, it is then impossible to reduce equations (17) to four equations of second order for the four A_α.

Nevertheless it proves to be possible to replace equations (17) by five equivalent equations for the five quantities A_α and λ, whereby one of the equations acquires the character of an additional condition. Indeed, if we put

$$\lambda = k^{-2}\varrho \left(\frac{\partial \mathbf{H}}{\partial \varrho}\right)^{-1} \tag{38}$$

then, thanks to (25), we have $\varrho = k^2\mu\,|\lambda|$. Assuming $\dfrac{\partial \mathbf{H}}{\partial P} \neq 0$, we get from (18) $p_{\alpha\beta} = K(F, \varrho) f_{\alpha\beta}$, or

$$p_{\alpha\beta} = M(F, |\lambda|\mu) f_{\alpha\beta}. \tag{39}$$

Using now (18), (19), and (38), we can write

$$\lambda A^\alpha = \{M(F, |\lambda|\mu) f^{\alpha\beta}\}_{,\beta}, \tag{40}$$

which together with (37) forms five equations for the A_α and λ (it goes with saying that the A_α are connected with the $f_{\alpha\beta}$ by means of (20)).

To illustrate these considerations let us make the simplest possible assumptions as to **H**, namely

$$\mathbf{H} = P + \varrho. \tag{41}$$

In virtue of $\partial \mathbf{H}/\partial \varrho = 1$, we can easily find from (19) as condition for F and μ the relation

$$\mu = k^{-1}(A_\alpha A^\alpha)^{1/2} = 1. \tag{42}$$

Here equations (40) read

$$\lambda A^\alpha = f^{\alpha\beta}{}_{,\beta}. \tag{43}$$

Equations (42) and (43) together with (20) constitute the set of fundamental equations of Dirac's new electrodynamics as put forward by him in his first paper published on this subject (1951). Thus we see that a potentialless formulation of this theory is also possible.

It must be stated, however, that the potentialless theory with **H** defined by (41) is in a certain sense narrower than the theory of Dirac, i.e., the theory of equations (42), (43), and (20). We have namely, thanks to (38), $\lambda = k^{-2}\varrho \geqslant 0$ (as $\varrho = 4\pi k|\varrho_0|$), whereas in Dirac's theory the sign of λ remains indeterminate. It seems, however, that this property is rather encouraging, as it involves the fact that in the "potentialless" theory, as may be easily seen from (22),

$$T_0^0 = 1/8\pi(\vec{H}^2 + \vec{D}^2) + k|\varrho_0|v_0^2 \geqslant 0,$$
$$T_\alpha^\alpha = k|\varrho_0| \geqslant 0$$

and hence the energy-density and the trace of the energy-momentum tensor have correct signs, following unavoidably from the theory. Instead of this in Dirac's theory one gets

$$T_0^0 = 1/8\pi(\vec{H}^2 + \vec{D}^2) + \frac{\lambda}{4\pi} A_0^2; \quad T_\alpha^\alpha = 1/4\pi k^2 \lambda,$$

that is, quantities of indeterminate sign (as λ may be positive or negative).

Now, we shall discuss the question of the equations of motion of our unitary theory, without making any special assumptions as to the function $\mathbf{H}(P, \varrho)$. In accordance with the interpretation given in our preliminary considerations, we may, by (11) and (12), ascribe to the field $p_{\alpha\beta}$ at each point a rest-density of charge ϱ_0 and, if $\varrho_0 \neq 0$, also a velocity of the charge at that point.

One can expect that in all points in which v^α has a meaning the velocity field will be governed by some equations of motion, which are contained, in a certain sense, in the fundamental equations (17), and these equations of motion will contain terms which can be interpreted as density of the Lorentz force, charge density, and rate of change of momentum to be ascribed to the charges. In fact, it is easily shown that contracting (20) by means of $j^\alpha = c\varrho_0 v^\alpha$ and taking due account of (19), one gets the equations

$$\varrho_0 f_{\alpha\beta}\, v^\alpha = \varrho/4\pi \frac{d}{ds} \frac{\partial \mathbf{H}}{\partial \varrho} v_\beta - \varrho/4\pi \left(\frac{\partial \mathbf{H}}{\partial \varrho}\right)_{,\beta} \tag{44}$$

where d/ds is the substantial derivative defined by $da/ds = a_{,\alpha}\, v^\alpha$.

The term $\varrho_0 f_{\alpha\beta} v^\alpha$ must be obviously identified with the density of the Lorentz force F_β. The coefficient at dv_β/ds which we get after performing the substantial differentiation is associated with acceleration and has to be regarded as the density of rest-mass ϱ_{m_0}, so that

$$c^2 \varrho_{m_0} = 1/4\pi \frac{\partial \mathbf{H}}{\partial \varrho} \varrho. \tag{45}$$

Of course, we are here concerned with rest-mass in the sense of inertia possessed by the charges, which are produced by the field and are moving in it.

The remaining terms in (44) are related with the changes of $K = c^2 \varrho_{m_0}/\varrho_0$ from point to point, which are possible in the general framework of the theory in which $\dfrac{\partial \mathbf{H}}{\partial \varrho} \neq \text{const}$. In fact, due to (44) and (45), we can write

$$\varrho_0 f_{\alpha\beta} v^\alpha = c^2 \varrho_{m_0} dv_\beta/ds + c^2 \varrho_{m_0} (v^\alpha v_\beta - \delta^\alpha_\beta) \left(\ln \left|\frac{\varrho_{m_0}}{\varrho_0}\right|\right)_{,\alpha} \tag{46}$$

whenever $\varrho_0 \neq 0$ (only in such domains the motion of the charges has a meaning). In the special case

$$\varrho_{m_0}/\varrho_0 = \text{const}. \tag{47}$$

the equation of motion becomes

$$\varrho_0 v^\alpha f_{\alpha\beta} = c^2 \varrho_{m_0} dv_\beta/ds \tag{48}$$

which is only possible when $\dfrac{\partial \mathbf{H}}{\delta \varrho} = \text{const}$. We have to deal with equations of this type when $\mathbf{H}$ has the form

$$\mathbf{H} = \Omega(P) + a\varrho, \qquad a = \text{const}. \tag{49}$$

in particular in the case of Dirac's theory. For $\mathbf{H}$ given by (49), $c^2 \varrho_{m_0} = a/4\pi\varrho$, so that here $a/4\pi c^2 \varrho$ plays the rôle of rest-mass density. In such a theory $\varrho_{m_0} = ka/c^2|\varrho_0|$ and hence the density of rest-mass is proportional to the rest-density of charge.

In his second paper on the new unitary electrodynamics Dirac (1952) gives up the first variant of this theory because of the fact that the equations of motion following from the theory, i.e. (48), do not describe in the conventional way the motion of the charges in a magnetic field.[1] He generalizes therefore his theory by introducing two new variables ξ and η in such a way as to obtain the possibility of a uniform motion of the charges in the direction of the magnetic field.

Leaving aside the question if just so a modification of the theory as was given by Dirac is really necessary, we shall show that a potentialless formulation of the variational principle of Dirac's theory given in his second paper is also possible.

[1] This problem was considered more fully by one of us (Infeld 1952).

Let us add to the variables on which **H** depends two new variables ξ, η and introduce the invariant

$$I = 2\pi k/c(\eta_{,a}\,\xi + \xi_{,a}\,\eta)j^a. \tag{50}$$

Let us also put

$$\mathbf{H} = P + \varrho + I. \tag{51}$$

Then, a straightforward calculation gives us for the Euler-Lagrange equations of the variational principle $\delta \int d_4 x H = 0$ in which $p_{a\beta}$, ξ, and η are considered as variables the following equations

$$p_{a\beta} = \frac{4\pi k^2}{c}\,[(j_\beta/\varrho)_{,a} - (j_a/\varrho)_{,\beta}] + k(\xi_{,a}\,\eta_{,\beta} - \xi_{,\beta}\,\eta_{,a}) \tag{52}$$

$$j^a\,\xi_{,a} = 0 \quad (53), \qquad\qquad j^a\,\eta_{,a} = 0, \tag{54}$$

which are easily seen to be equivalent to the corresponding equations of Dirac's second paper. They involve also the following equations of motion

$$k\,\frac{dv_a}{ds} = v^\beta f_{\beta a} \tag{55}$$

which are formally identical with (48).

It may be interesting to add that the variational principle with **H** given by (51) makes the variables ξ and η a little more intuitive. In **H** the invariant P is responsible for the field, the invariant ϱ for its corpuscular aspect and I, which has an analogous structure to the conventional interaction term and appears in consequence of the introduction of ξ and η, accounts for the influence of the motion of the charges in the field, and vice versa. Of course, also in the case of a general $\mathbf{H}(P, \varrho)$, if in the variational principle we add to it I from (50), we get a modified theory analogous to the above mentioned modification of Dirac's theory. Formally, the equations of motion undergo hereby no change and retain their form (44).

4. Concluding remarks. Our unitary theory in its potentialless formulation forms without doubt to broad a framework and additional physical postulates are needed to single out the proper Lagrange function. We have deliberately retained, however, the general non-linear structure of the theory. Unitary theories analogous in essence to Dirac's or Mie's theories yield equations of motion in a satisfactory way but, as it seems, only through non-linear equations can we hope to describe properly particles and obtain a fully correct theory.

The terms $c^2\varrho_{m_0}(v^a\,v_\beta - \delta^a_\beta)\left(\ln\left|\frac{\varrho_{m_0}}{\varrho_0}\right|\right)_{,a}$ in the equations of motion of our theory may seem at first a little odd. It may be noticed, however, that these equations have a quite analogous structure to the conformly invariant equations of motion following from other considerations (Infeld and Schild 1945, 1946). It is possible that a physically well founded introduction of the postulate of conform invariance could help us to find the proper framework of a unitary theory, forming the right classical bases for quantization.

КРАТКОЕ СОДЕРЖАНИЕ

Л. Инфельд и Й. Плебанский, Электродинамика без потенциалов. Получена общая схема расчета унитарной теории электромагнитного поля, характеризующегося антисимметричным тензором $p_{\alpha\beta}$. Потенциалы являются величинами, которые выводятся из теории. Лагранжева функция является произвольной функцией от $P = -\frac{1}{4}\, p_{\alpha\beta}\, p^{\alpha\beta}$ и $|\varrho_0| = C^{-1}(j_\alpha j^\alpha)^{\frac{1}{4}}$ где j^α, по определению, равно $c/4\pi\, p^{\alpha\beta}{}_{,\beta}$. Следовательно, лагранжева функция есть функция поля и его производных. Уравнения поля (17) получены варьированием относительно поля. Вообще, эта „электродинамика без потенциалов" эквивалентна электродинамике Ми, в которой потенциалы авляются основными величинами. Но это не всегда так. Новая электродинамика Дирака может быть сформулирована „без потенциалов" (лагранжян $= P + 4\pi k/c \cdot |\varrho_0|$), в то же время она находится вне пределов электродинамики Ми.

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ON THE ELECTROSTATIC NEUTRON-ELECTRON INTERACTION

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The neutron-electron static interaction is investigated by means of the Dyson technique with the use of formalistic regularization. It is shown that besides the convection current also the polarization current yields a non-vanishing contribution to the static interaction, the latter contribution being about three times as large as the former.

Introduction. The electrostatic neutron-electron interaction constitutes only a small correction to the main effects responsible for the scattering of neutrons by matter (nucleon-nucleon interaction and the interaction between the magnetic moments of nuclei and electrons with the magnetic moment of the neutron). Nevertheless, this correction could be measured, although the experimental errors are rather large (Fermi and Marshall 1947; Havens et al. 1947, 1949, 1951; Hughes 1952, Hammermesh et al. 1952). Also several papers on this subject have been published by theoreticians (Luttinger 1948; Slotnick and Heitler 1949; Case 1949; Dancoff and Drell 1949; Borowitz and Kohn 1949; Foldy 1951, 1952), but the results of these authors show some discrepancies. The comparison between theory and experiment meets with difficulties in consequence of large experimental errors as well as of lack of exact information concerning the type of interaction and the value of the nucleon-meson coupling constant.

We assume in this paper a pseudoscalar charged meson field with a pseudoscalar coupling to the nucleon field, and we use the well known technique of Dyson (1949) together with the method of formalistic regularization of Pauli and Villars (1949). The field produced by charged particles (electrons) is treated occasionally as a non-quantized, external electromagnetic potentials A_μ, specialized to a pure Coulomb field.

§1. The effective potential. The effect of the electrostatic interaction shall be described by the effective potential operator $H^F(x_0)$ which, after Dyson, reads

$$H^F(x_0) = \sum_{n=0}^{\infty} (-i)^n/n! \int_{-\infty}^{+\infty} d^4x_1 \dots d^4x_n \, P[(H^e(x_0) H^i(x_1) \dots H^i(x_n)], \tag{1}$$

where H^i is the Hamiltonian of interaction between the nucleonic and mesonic fields

$$H^i = ig(\bar{\psi}\gamma_5\chi\varphi + \bar{\chi}\gamma_5\psi\varphi^*) \tag{2}$$

(φ, φ^* denotes the mesonic field variables, ψ, $\bar{\psi}$, χ, $\bar{\chi}$, denote the proton and neutron field variables, respectively). $H^e(x_0)$ is the Hamiltonian of interaction between protons and charged mesons with the external electromagnetic field A_μ. It may be divided into the meson part

$$H_1^e = -ie(\varphi\partial_\mu\varphi^* - \varphi^*\partial_\mu\varphi)A_\mu + \text{terms of order } e^2 \tag{3'}$$

and the proton part

$$H_2^e = -ie\bar{\psi}\gamma_\mu\psi A_\mu\,. \tag{3''}$$

Consequently, the operator H^F may be divided into two parts: H_1^F and H_2^F. Up to the order eg^2 we have

$$H_1^{F(2)}(x_0) = (-i)^2/2\int_{-\infty}^{+\infty} d^4x_1 d^4x_2 P[H_1^e(x_0)H^i(x_1)H^i(x_2)], \tag{4'}$$

$$H_2^{F(2)}(x_0) = (-i)^2/2\int_{-\infty}^{+\infty} d^4x_1 d^4x_2 P[H_2^e(x_0)H^i(x_1)H^i(x_2)] \tag{4''}$$

to be computed in detail.

The computation of the matrix elements of the operators (4') and (4'') is facilitated by the well known rules of Feynman and Dyson. In the approximation eg^2 we have two graphs describing the scattering of the neutron by an electromagnetic field (see Fig. 1), corresponding to the intuitive picture of a partial

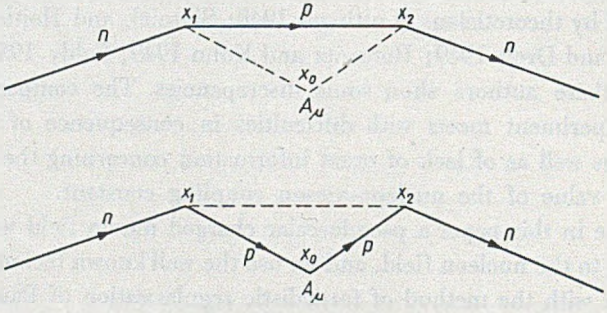

Fig. 1. Scattering of a neutron on an electromagnetic field.
(Nucleon line continuous, meson line dashed)

dissociation of the neutron into a pair proton-meson, a scattering of the proton (or meson) by the electromagnetic field A_μ, and a final recombination of the pair into the scattered neutron.

Each graph corresponds to a matrix element of H^F by means of the following rules:

1) the external neutron lines correspond to spinor quantities $\bar{\chi}(k^2)$, $\chi(k^1)$, where k^1 (k^2) denotes the energy-momentum fourvector of the neutron before (after) the collision,

2) every internal spinor (proton) line corresponds to a factor

$$(\gamma_\mu p_\mu + iM)/(p_\mu^2 + M^2),$$

where M is the nucleon mass, p_μ is the energy-momentum four-vector of the virtual proton,

3) every internal boson (meson) line corresponds to a factor

$$-i/(p_\mu'^2 + m^2),$$

where m is the meson mass, p_μ' is the energy-momentum four vector of the virtual meson,

4) every vertex where nucleonic lines meet mesonic lines corresponds to a factor

$$g\gamma_5$$

in the case of pseudoscalar mesons with pseudoscalar coupling,

5) every vertex x_0 where the spinor line is broken by the external field A_μ corresponds to a factor

$$-ie\gamma_\mu A_\mu(x_0),$$

6) every vertex x_0 where the boson line is broken by the external field A_μ corresponds to a factor

$$e(p_\mu^1 + p_\mu^2) A_\mu(x_0),$$

where p_μ^1, p_μ^2 are the momenta of the meson before and after the scattering by A_μ.

7) The whole expression is to be multiplied by $(2\pi)^{-4(F-n+1)}$ and integrated over the $(F - n + 1)$ independent momenta, where F is the number of internal lines in the graph, while n is the number of vertices x_i (except the vertex x_0).

8) The conservation law of energy and momentum is satisfied for every vertex where the lines meet, which enables the elimination of $n - 1$ momenta. There remain integrals over the $F - n + 1$ independent momenta. In our case $n - 1 = 1$, $F - n + 1 = 1$.

Making use of the above stated rules the matrix elements of H_1^F and H_2^F assume the form

$$\langle H_1^{F(2)}(x_0)\rangle \tag{5'}$$

$$= -\frac{eg^2}{(2\pi)^4} A_\mu(x_0) \int d^4p \frac{\langle \chi(x_0)\gamma_5(\gamma_\varrho p_\varrho + iM)\gamma_5\chi(x_0)\rangle [2p_\mu - (k^1 + k^2)_\mu]}{[p^2 + M^2][(p - k^1)^2 + m^2][(p - k^2)^2 + m^2]},$$

$$\langle H_2^{F(2)}(x_0)\rangle \tag{5''}$$

$$= -\frac{eg^2}{(2\pi)^4} A_\mu(x_0) \int d^4p \frac{\langle \bar{\chi}(x_0)\gamma_5[\gamma_\varrho(p_\varrho + k_\varrho^2) + iM]\gamma_\mu[\gamma_\sigma(p_\sigma + k_\sigma^1) + iM]\gamma_5\chi(x_0)\rangle}{[p^2 + m^2][(p + k^1)^2 + M^2][(p + k^2)^2 + M^2]}.$$

With the aid of the Dirac equations and the well-known properties of γ_5, the matrix elements become

$$\langle H_1^{F(2)}(x_0)\rangle \tag{6'}$$

$$= \frac{eg^2}{(2\pi)^4} A_\mu(x_0)\langle\bar{\chi}(x_0)\gamma_\varrho\chi(x_0)\rangle \int d^4p \frac{2\left(p_\varrho - \frac{k_\varrho^1 + k_\varrho^2}{2}\right)\left(p_\mu - \frac{k_\mu^1 + k_\mu^2}{2}\right)}{[p^2 + M^2]\,[(p-k^1)^2 + m^2]\,[(p-k^2)^2 + m^2]},$$

$$\langle H_2^{F(2)}(x_0)\rangle \tag{6''}$$

$$= \frac{eg^2}{(2\pi)^4} A_\mu(x_0)\langle\bar{\chi}(x_0)\gamma_\varrho\gamma_\mu\gamma_\sigma\chi(x_0)\rangle \int d\,p \frac{p_\varrho p_\sigma}{[p^2+m^2][(p-k^1)^2+M^2]\,[(p-k^2)^2+M^2]}$$

§ 2. Evaluation of the integrals. The integrals in (6′) and (6″) diverge logarithmically but their divergences compensate. Thus, we should consider rather the integral over the sum of the two fractions than the sum of the two separate integrals. However, the integration over the sum of the two fractions being very cumbersome, we may avoid the ambiguities inherent in the non-well defined integrals (6′) and (6″) by applying the method of formalistic regularization (Pauli and Villars 1949). Thus, we consider the differences between the fractions in (6′) and (6″) and similar fractions with an additional meson mass R. These differences are convergent after integration. The results of the regularized (6′) and (6″) are added together and the transition $R \to \infty$ performed. The result of this manipulation is the same as if we computed the integral of the sum of the fractions in (6), but the integrations are easier since the common denominators are simpler.

After simple manipulations the regularized expressions are

$$\langle H_1^{F(2)}(x_0)\rangle_R = \langle c\rangle \int d^4p \times \tag{7'}$$

$$\frac{2\gamma_\varrho(p_\varrho - K_\varrho)\,(p_\mu - K_\mu)\,A_\mu(x_0)R^4}{[p^2+M^2][(p-k^1)^2+m^2][(p-k^1)^2+m^2+R^2][(p-k^2)^2+m^2][(p-k^2)^2+m^2+R^2]},$$

$$\langle H_2^{F(2)}(x_0)\rangle_R$$

$$= \langle c\rangle \int d^4p \frac{\gamma_\varrho\gamma_\mu\gamma_\sigma p_\varrho p_\sigma A_\mu(x_0)\,R^2}{[p^2+m^2]\,[p^2+m^2+R^2]\,[(p-k^1)^2+M^2]\,\,[(p-k^2)^2+M^2]} \tag{7''}$$

where $\langle c\rangle$ is short for

$$\langle c\rangle = eg^2(2\pi)^{-4}\langle\bar{\chi}\ldots\chi\rangle, \tag{8'}$$

while:

$$K_\mu = \frac{(k^1+k^2)_\mu}{2}. \tag{8''}$$

The integrals (7′) and (7″) are computed with the aid of the well-known formula due to Feynman (1949)

$$\frac{1}{A_0A_1\ldots A_n} \equiv n!\int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-1}} dx_n\{(A_n - A_{n-1})x_n + \ldots + (A_1 - A_0)x_1 + A_0\}^{-(n+1)} \tag{9}$$

Introducing new variables:

$$p_\mu \to \sqrt{P - Q_\mu^2}\, p_\mu - Q_\mu, \tag{10}$$

where

$$Q_\mu^{(1)} = (k^1 - k^2)_\mu z - k_\mu^1 x, \tag{11'}$$

$$P^{(1)} = R^2(u - z + y) - (2M^2 - m^2)x + M^2, \tag{11''}$$

for H_1^F, and

$$Q_\mu^{(2)} = k_\mu^2 (y - x) - k_\mu^1 (1 - x), \tag{12'}$$

$$P^{(2)} = R^2 z + m^2 y \tag{12''}$$

for H_2^F, the integrals are symmetrized. We shall not write them down. Their computation is particularly easy in this frame of reference where $\vec{A} = 0$, $A_4 = i\varphi$. Some of the integrals vanish on reasons of symmetry, which enables the substitutions

$$\gamma_\varrho p_\varrho p_\mu A_\mu(x_0) \to \gamma_4 A_4(x_0) p_4^2, \tag{13'}$$

$$\gamma_\varrho \gamma_\mu \gamma_\sigma p_\varrho p_\sigma A_\mu(x_0) \to \gamma_4 A_4(x_0)\,(2p_4^2 - p_\varrho p_\varrho), \tag{13''}$$

$$\gamma_\varrho \gamma_\mu \gamma_\sigma Q_\varrho Q_\sigma A_\mu(x_0) \to \gamma_\sigma Q_\sigma Q_4\, A_4(x_0) - \gamma_4 A_4(x_0) Q_\varrho^2, \tag{13'''}$$

from where we have

$$\langle H_1^{F(2)}(x_0)\rangle_R = \langle c\rangle \left\{ 48 \int d^4p \frac{\gamma_4 p_4^2 A_4(x_0)}{(p_\mu^2 + 1)^5} \int_0^1 dx \int_0^x dy \int_0^y dz \int_0^z du \frac{R^4}{(P^{(1)} - Q_\mu^{(1)2})^2} + \right.$$

$$\left. + 48 \int \frac{d^4p}{(p_\mu^2 + 1)^5} \int_0^1 dx \int_0^x dy \int_0^y dz \int_0^z du \frac{R^4 \gamma_\varrho (K_\varrho + Q_\varrho^{(1)})\,(K_4 + Q_4^{(1)}) A_4(x_0)}{(P^{(1)} - Q_\mu^{(1)2})^3} \right\}, \tag{14'}$$

$$\langle H_2^{F(2)}(x_0)\rangle_R = \langle c\rangle \left\{ 6 \int d^4p \frac{\gamma_4 A_4(x_0)\,(2p_4^2 - p_\varrho p_\varrho)}{(p_\mu^2 + 1)^4} \int_0^1 dx \int_0^x dy \int_0^y dz \frac{R^2}{P^{(2)} - Q_\mu^{(2)2}} + \right.$$

$$+ 6 \int \frac{d^4p}{(p_\mu^2 + 1)^4} \int_0^1 dx \int_0^x dy \int_0^y dz \tag{14''}$$

$$\left. \times R^2 (\gamma_\sigma Q_\sigma^{(2)} Q_4^{(2)} A_4(x_0) - \gamma_4 A_4(x_0) Q_\sigma^{(2)2}) / (P^{(2)} - Q_\mu^{(2)2})^2 \right\}.$$

Making use of the formulae derived in the appendix and using the well known decomposition

$$(k^1 + k^2)_\mu = i(2M\gamma_\mu - \sigma_{\mu\nu} \Delta k_\nu), \qquad \sigma_{\mu\nu} = -\frac{1}{2} i[\gamma_\mu, \gamma_\nu] \tag{15}$$

into the "convection current" and the "polarization current", we find

$$\langle H_1^{F(2)}(x_0)\rangle_R = A_1 + B_1 + C_1, \tag{16'}$$

$$\langle H_2^{F(2)}(x_0)\rangle_R = A_2 + B_2 + C_2, \tag{16''}$$

$$A_1 = \langle c' \rangle \int_0^1 dx \int_0^x dy \int_0^y dz \int_0^z du \frac{R^4}{[R^2(u - z + y) + \Phi(x) + z(x - z)(\Delta k)_\mu^2]^2}, \quad (17')$$

$$A_2 = \langle c' \rangle \int_0^1 dx \int_0^x dy \int_0^y dz \frac{-R^2}{R^2 z + \Phi(y) + (x - y)(1 - x)(\Delta k)_\mu^2}, \quad (17'')$$

$$B_1 = \langle c' \rangle \int_0^1 dx \int_0^x dy \int_0^y dz \int_0^z du \frac{-4M^2(1-x)^2 R^4}{[R^2(u - z + y) + \Phi(x) + z(x - z)(\Delta k)_\mu^2]^3}, \quad (18')$$

$$B_2 = \langle c' \rangle \int_0^1 dx \int_0^x dy \int_0^y dz \frac{[(x - y)(1 - x)(\Delta k)_\mu^2 - M^2(1 - y)^2] R^2}{[R^2 z + \Phi(y) + (x - y)(1 - x)(\Delta k)_\mu^2]^2}, \quad (18'')$$

$$C_1 = \langle c \rangle i\pi^2 A_\mu(x_0) \sigma_{\mu\nu} \Delta k_\nu \int_0^1 dx \int_0^x dy \int_0^y dz \int_0^z du$$

$$\times \frac{2M(1-x)^2 R^4}{[R^2(u - z + y) + \Phi(x) + z(x - z)(\Delta k)_\mu^2]^3}, \quad (19')$$

$$C_2 = \langle c \rangle i\pi^2 A_\mu(x_0) \sigma_{\mu\nu} \Delta k_\nu \int_0^1 dx \int_0^x dy \int_0^y dz \frac{M(1-y)^2 R^2}{[R^2 z + \Phi(y) + (x - y)(1 - x)(\Delta k)_\mu^2]^2}, \quad (19'')$$

where

$$\Phi(x) \equiv M^2(1 - x)^2 + m^2 y = M^2 x^2 - (2M^2 - m^2)x + M^2, \quad (20)$$

while

$$(\Delta k)_\mu = (k^1 - k^2)_\mu \quad (21)$$

is the change of the energy-momentum of the neutron due to scattering, while $\langle c' \rangle$ is short for

$$\langle c'(x_0) \rangle = -eg^2(2\pi)^{-4} \langle \bar{\chi} \gamma_4 \chi \rangle \varphi(x_0). \quad (22)$$

The integrals (17)—(19) are rather complicated, but we are interested only in the volume of the "potential well" and not in the matrix element $\langle H^F(x_0) \rangle$ itself. This volume or rather its depth is a quantity to be compared with experiment. This quantity may be computed comparatively easily. Let us introduce the power series expansion

$$H^F(x_0) = a_0 + a_2(\Delta k)_\mu^2 + a_4(\Delta k)_\mu^4 + \ldots \quad (23)$$

As shown by Case, only the coefficient a_2 contributes to the value of the volume and depth of the "potential well". The argument of Case will be reproduced later. Meanwhile, we compute the integrals (A) and (B) up to the order $(\Delta k)^2$ which is quite easily done and yields the result

$$A_1 + A_2 + B_1 + B_2 = \langle c' \rangle (\Delta k)_\mu^2 / 6 \int_0^1 dx \left\{ \frac{2(1-x)^3 - x^3}{\Phi(x)} + \frac{M^2[2x^3(1-x)^2 + (1-x)^5]}{[\Phi(x)]^2} \right\}. \quad (24)$$

The transition to the limit $R \to \infty$ is to be performed after, but not before, summing together the divergent integrals A_1 and A_2, and after integrating over the parameters z and u.

Formula (24) gives just the effect caused by the "convection part" of the neutron current. The other effect caused by the "polarization part" has been omitted in the above-mentioned papers of Case and Borowitz & Kohn, who argued that this effect (caused by the electric moment of the neutron) vanishes in the static limit, i.e. for infinitely slow neutrons.

In fact it may be easily shown that this part of the effect yields nothing to the coefficient a_0 of the series expansion (23) but it yields a non-vanishing a_2, and we shall see later that it is just a_2 that is responsible for the electrostatic interaction.

The computation of the part of a_2 due to the "polarization current" is complicated by the fact that the result cannot be interpreted simply in form of a single "potential" with a radial symmetry. On the other hand, we may define the static potential as that quantity which is derivable from those elements of the matrix $\langle H^F \rangle$ which describe *central collisions*.

On account of (15) the polarization current is

$$\langle \bar{\chi} A_\mu(x_0) \sigma_{\mu\mu} \Delta k_\mu \chi \rangle = i(k^1 + k^2)_\mu A_\mu(x_0) \langle \bar{\chi} 1 \chi \rangle + 2M A_\mu(x_0) \langle \bar{\chi} \rangle \gamma_\mu \chi \rangle. \quad (25)$$

With the aid of the Dirac equations we have also for $\vec{A} = 0$

$$\langle \bar{\chi} A_4(x_0) \sigma_{4i} \Delta k_i \chi \rangle = \frac{(k^1 + k^2)_4^2 + 4M^2}{2M} A_4(x_0) \langle \bar{\chi} \gamma_4 \chi \rangle + \frac{(k^1 + k^2)_4 (k^1 + k^2)_i}{2M} A_4(x_0) \langle \bar{\chi} \gamma_i \chi \rangle, \ (i = 1, 2, 3). \quad (26)$$

Assuming that the electromagnetic potentials are produced by a charged particle (electron with *spin neglected*) whose momenta are p_μ^1 and p_μ^2, before and after the collision, respectively, the potentials (as well as the current four-vector) are proportional to $(p^1 + p^2)_\mu$ so that the frame of reference in which

$$\vec{p^1} + \vec{p^2} = 0, \qquad \vec{p^1} = -\vec{p^2}, \qquad (\vec{p^1})^2 = (\vec{p^2})^2 \quad (27)$$

is the desired electrostatic frame where $\vec{A} = 0$. From the energy and momentum conservation during the collision we find in this frame of reference

$$(\vec{k^1})^2 = (\vec{k^2})^2. \quad (28)$$

Assuming the collision to be central, we see that

$$\vec{k^1} = -\vec{k^2} \quad \text{or} \quad \vec{k^1} + \vec{k^2} = 0, \quad (29)$$

whence the second term in (26) disappears. On account of

$$(k^1 + k^2)_4^2 = (k^1 + k^2)_\mu^2 = -4M^2 - (\Delta k)_\mu^2 \tag{30}$$

we find a reasonable expression[1]

$$\langle \bar{\chi} A_4(x_0) \sigma_{4i} \Delta k_i \chi \rangle = -(\Delta k)_\mu^2 / 2M \langle \bar{\chi} \gamma_4 \chi \rangle A_4(x_0). \tag{31}$$

Substituting (31) into (19), we get

$$C_1 + C_2 = -\langle c' \rangle (\Delta k)_\mu^2 / 2 \int_0^1 dx \frac{(1-x)^2}{\Phi(x)} \tag{32}$$

and we see that the polarization current does not contribute to a_0 but contributes to a_2.

Summing up the contributions to a_0 in A_1, A_2, B_1 and B_2 and performing the remaining integrations, we get

$$a_0 = \langle \varrho \varphi(x_0) \rangle \tag{33}$$

where

$$\varrho = -(eg^2/32\pi^2)\, \chi^* \chi \tag{33'}$$

is obviously a (infinite, see below) charge density of the neutron and must be renormalized to the value zero. (Instead of performing a charge renormalization we may get rid of the self-charge by regularizing also with respect to the nucleon mass).

After the removal of the unphysical neutron self-charge we are left with the contribution of the order $(\Delta k)_\mu^2$. After integration over x we have

$$a_2 = -(eg^2/16\pi^2)\, (\Lambda/6M^2) \langle \bar{\chi}(x_0) \gamma_4 \chi(x_0) \rangle \varphi(x_0), \tag{34}$$

where

$$\Lambda = (14 - 5\lambda)(4 - \lambda)^{-1} + (3 - 5\lambda) \ln \lambda^{1/2} \tag{34'}$$
$$+ \lambda^{1/2} (-46 + 33\lambda - 5\lambda^2)(4 - \lambda)^{-3/2} \arccos \lambda^{1/2}/2,$$

while $\lambda = (m/M)^2$ is the square of the ratio of the meson and nucleon masses

§ 3. The volume and depth of the potential well.

From the matrix element $\langle H^F(x_0) \rangle$ we should go over to the volume and depth of the potential well which is directly connected with the experimental results on static interaction. The potential of static interaction $V(r)$ is defined by

$$H^F = \chi^* \chi\, V(r). \tag{35}$$

[1] A partition into an "electrostatic" (i.e. involving A_4 only) and a "non-static" (involving $\vec{A}$) part is a conventional procedure depending on the frame of reference. The above considerations show, however, that central collisions enable a most natural definition of an "electrostatic" interaction independent of any frame of reference. It is immediately seen that the same "electrostatic" part of the interaction is obtained also in the case of non-central collisions by means of a partition into terms depending on $\vec{A}$ and A_4 in that particular frame of reference where $\vec{k}^1 = -\vec{k}^2$, i.e. in the system of the "centre of mass" of the neutron before and after the collision.

With the aid of the Fourier transform $V(p)$, we have

$$\int V(r)\,d\tau = (2\pi)^{-3/2}\int d^3p \int d^3x\ exp(i\vec{p}\cdot\vec{r})V(p) = (2\pi)^{3/2}\int d^3p V(p)\delta^{(3)}(p)$$
$$= (2\pi)^{3/2}\ V\,(p=0), \tag{36}$$

from where it is seen that the volume and depth depend merely on the Fourier coefficient with $p = 0$.

In order to find $V(p)$ we write (34), on account of (23), in the form

$$H^F = \chi^*\chi f([(\Delta k)_\mu^2]\varphi(\vec{r}), \tag{37}$$

where $f[(\Delta k)_\mu^2]$ is

$$f[(\Delta k)_\mu^2] = -(eg^2/16\pi^2)\,(\Lambda/6M^2)\,(\Delta k)_\mu^2$$
$$+ \text{terms of the order } (\Delta k)_\mu^0,\ (\Delta k)_\mu^4,\ \ldots \text{ from (23).} \tag{38}$$

Representing $\varphi(\vec{r})$ in (37) by means of a Fourier integral and comparing (37) with (35) we get

$$V(p) = \varphi(p) f[(\Delta k)_\mu^2], \tag{39}$$

but the only non-vanishing terms of $\langle H(x_0)\rangle$ are those with $p_\mu = (\Delta k)_\mu$, so that

$$V(p) = \varphi(p) f(p^2) \quad \text{with} \quad p_\mu = (\Delta k)_\mu. \tag{40}$$

Now, the Fourier transform of a Coulomb potential is

$$\varphi(p) = -e(2\pi)^{-3/2}\,p^{-2}, \tag{41}$$

whence (40) and (36) yields

$$\int V(r)d\tau = -e(\Delta k)_\mu^{-2} f\,[(\Delta k)_\mu^2]\Big|_{(\Delta k)_\mu \to 0} \tag{42}$$

showing that only the terms of the order $(\Delta k)_\mu^2$ contribute to the volume and depth of the potential well, as assumed previously.

Taking account of the terms of the order $(\Delta k)_\mu^2$, we have finally

$$\int V(r)\,d\tau = (e^2g^2/16\pi^2)\,(\Lambda/6M^2) \tag{43}$$

with Λ given by (34).

The quantity that is directly given in experiment is the depth V_0 of the rectangular potential well extending over the classical volume of the electron:

$$\int V(r)\,d\tau = 4\pi/3\,(e^2/4\pi m_e c^2)^3\,V_0. \tag{44}$$

In *CGS* — units we get

$$V_0 = -(g^2/8\pi\hbar c)\,\alpha^{-2}(m_e/M)^2 m_e c^2 (3/2\pi)\,(-\Lambda/6) \tag{45}$$

or, with the aid of (34),

$$V_0 = -(g^2/8\pi\hbar c)\,\alpha^{-2}(m_e/M)^2 m_e c^2 (3/2\pi)\,\{(-7/3 + 5\lambda/6)\,(4-\lambda)^{-1}$$
$$+ (1/2 - 5\lambda/6)\,/\ln\lambda^{-1/2} + \lambda^{1/2}\,(4-\lambda)^{-3/2}\,(23/3 - 11\lambda/2$$
$$+ 5\lambda^2/6)\ \text{arc cos}\ \lambda^{1/2}/2\} \tag{45'}$$

or

$$V_0 \cong -1,36\,(g^2/8\pi\hbar c)\,\{N\}\,kev \tag{45''}$$

where the bracket $\{N\}$ is the same as in (45'). The numerical value of V_0 depends upon the values of the coupling constant g and the mass ratio λ.

§ 4. Discussion of the results. The result of this investigation differs from those of Case (1949) and Borowitz and Kohn (1949), who dropped the polarization current responsible for a part of the effect. To illustrate the difference we have plotted V_0 as function of λ (Fig. 2).

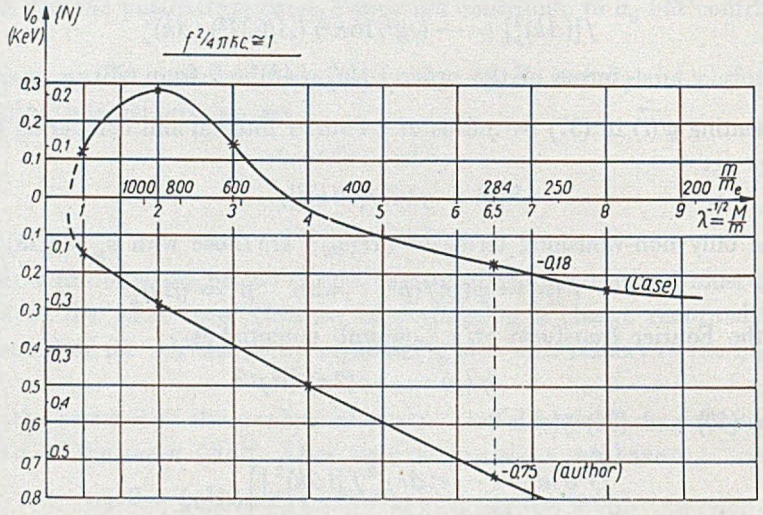

Fig. 2. Depth V_0 of the rectangular potential well extending over the classical volume of the electron as function of $\lambda^{-1/2} = M/m$ or m/m_e (M = nucleon mass, m = meson mass, m_e = electron mass, f = coupling constant)

In particular, for the value $\lambda^{-1/2} \approx 6{,}5$ corresponding to the π-meson mass $m \approx 284\,m_e$ the results are as follows[1]

$$\text{Case:}\quad V_0 \cong -180\,(f^2/4\pi\hbar c)\,eV \tag{46'}$$

$$\text{author:}\quad V_0 \cong -750\,(g^2/8\pi\hbar c)\,eV \tag{46''}$$

showing that our result is about 4 times larger, than that of previous authorst Thus, an agreement with experiment may be achieved with a smaller value of the coupling constant g.

The method of regularization, applied by Case, is different from the method of Pauli's regulators. It is based on the generalized Feynman identities

[1] On account of a different definition our coupling constant g is related to the Case's coustan f by:

$$g^2 = 2f^2$$

$$\frac{1}{(A+m^2)(B+m^2)C} \equiv 4! \int_0^1 dx \int_0^x dy \int_{m^2}^{\infty} da \int_{m^2}^{\infty} db \frac{(x-y)(1-x)}{\{(A+a)(1-x)+(B+b)(x-y)+Cy\}^5}, \tag{47'}$$

$$\frac{1}{(A+m^2)BC} \equiv 4! \int_0^1 dx \int_0^x dy \int_{m^2}^{\infty} da \int_{m^2}^{\infty} db \frac{y(x-y)}{\{(A+m^2)(1-x)+(B+a-m^2)(x-y)+(C+b-m^2)y\}^5}. \tag{47''}$$

Both methods lead to identical results.

A quite different result would be obtained, however, without use of regularization. By applying (11) directly to (6) and performing similar calculations, as before, one would obtain

$$A_1' + A_2' = -\langle c' \rangle \frac{1}{i\pi^2} \int d^4p \frac{\vec{p}^2 + 3p_0^2}{(p_\mu^2+1)^3} = \frac{1}{2}\langle c' \rangle, \tag{48'}$$

while the regularized expression (7) gives

$$A_1 + A_2 = \langle c' \rangle \left\{ 1 + \int_0^1 dx \frac{x(x-1)\Phi'(x)}{\Phi(x)} + \frac{(\Delta k)_\mu^2}{6} \int_0^1 dx \frac{(1-x)^3 - x^3}{\Phi(x)} \right\}. \tag{48''}$$

For the proof of the value of the integral in (48'), see Appendix.

I would like to express my thanks to Professor Rayski for his extensive help and continuous encouragement and to Komisja Popierania Twórczości Naukowej i Artystycznej for a grant-in-aid.

APPENDIX

The momentum space integrals appearing in H^F were derived from the Fourier representation of the known Feynman (or "causal") functions Δ^F, S^F

$$\Delta^F(p) = (-2i)/(p_\mu^2 + m^2), \tag{A.1'}$$

$$S^F(p) = 2(\gamma_\mu p_\mu + iM)/(p_\mu^2 + M^2). \tag{A.1''}$$

The integration over p_0 in these integrals must be performed over a known contour (omitting the poles of the integrands below and above the real axis).

We shall evaluate the symmetrical integrals over a momentum space dealt with in our calculations.

a) The integral $\int \frac{d^4p}{(p_\mu^2+1)^n}$, $n =$ natural number.

By decomposing this integral into a (3-dimensional) space integral and the integral over p_0, denoting the abscissas of the poles p_1 and p_2 by $\mp A$, where:

$$A = \sqrt{\vec{p}^2 + 1} \tag{A.2}$$

and introducing the new variable

$$q_0 = p_0 - A, \tag{A.3}$$

we get

$$\int_c \frac{d^4p}{(p_\mu^2 + 1)^n} = (-1)^n \int d^3p \oint \frac{dq_0}{q_0^n (q_0 + 2A)^n} \,. \tag{A.4}$$

By expanding the denominator in powers of $(q_0/2A)$ and taking the residue $2\pi i$ at the pole P_2 we get (A.4) in the form

$$\int_c \frac{d^4p}{(p_\mu^2 + 1)^n} = \frac{(-1)^{n+1} \binom{-n}{n+1} 2\pi i}{2^{2n-1}} \int \frac{d^3p}{\left(\sqrt{\vec{p}^2 + 1}\right)^{2n-1}} \tag{A.5}$$

which, after transforming to spherical coordinates and integrating over the angles, leads to the ordinary integral

$$\int_c \frac{d^4p}{(p_\mu^2 + 1)^n} = \frac{(-1)^{n+1} \binom{-n}{n+1} 2\pi i \cdot 4\pi}{2^{2n-1}} \int_0^\infty dp \, \frac{p^2}{\left(\sqrt{p^2 + 1}\right)^{2n-1}} \tag{A.6}$$

which, after elementary integration, assumes the form

$$\int_c \frac{d^4p}{(p_\mu^2 + 1)^n} = \frac{(-1)^{n+1} \binom{-n}{n+1}}{2^{2n-4}} \cdot \frac{1}{2n-3} \cdot \frac{2n-6}{2n-5} \cdot \ldots \cdot \frac{2}{3} \cdot 1 \, i\pi^2. \tag{A.7}$$

After expansion of the binomial coefficient and some elementary algebraic transformations, we get finally

$$\int_c \frac{d^4p}{(p_\mu^2 + 1)^n} = \frac{(n-3)!}{(n-1)!} \, i\pi^2 \quad \text{for} \quad n \geqslant 3. \tag{A.8}$$

For $n < 3$ the integrals are divergent.

b) The integral $\int d^4p \, \frac{p_\mu^2}{(p_\mu^2 + 1)^n}$.

Writing

$$p_\mu^2 = (p_\mu^2 + 1) - 1,$$

we get integrals of the form (A.8) and hence

$$\int d^4p \, \frac{p_\mu^2}{(p_\mu^2 + 1)^n} = \frac{2(n-4)!}{(n-1)!} \, i\pi^2 \quad \text{for} \quad n \geqslant 4 \tag{A.9}$$

c) The integrals $\int d^4p \, \frac{\vec{p}^2}{(p_\mu^2 + 1)^n}$ and $\int d^4p \, \frac{p_4^2}{(p_\mu^2 + 1)^n}$.

To compute them it suffices to multiply (A.6) by p^2 which gives an integral of the type

$$\int_0^\infty dp \frac{p^4}{\left(\sqrt{p^2+1}\right)^{2n-1}} \tag{A.10}$$

which can be evaluated by elementary means and gives, after some algebraic transformations

$$\int_c d^4p \frac{\vec{p}^2}{(p_\mu^2+1)^n} = \frac{3\ (n-4)!}{2\ (n-1)!}\, i\pi^2 \quad \text{for} \quad n \geqslant 4 \tag{A.11}$$

By the decomposition

$$p_\mu^2 = \vec{p}^2 - p_o^2 = \vec{p}^2 + p_4{}^2 \tag{A.12}$$

the other integral can be computed by means of (A.11) and (A.9). We get

$$\int_c d^4p \frac{p_4^2}{(p_\mu^2+1)^n} = -\int_c d^4p \frac{p_0^2}{(p_\mu^2+1)^n} = \frac{1\ (n-4)!}{2\ (n-1)!}\, i\pi^2 \text{ for } n \geqslant 4 \tag{A.13}$$

d) The integral $\int d^4p \dfrac{\vec{p}^2 + 3p_0^2}{(p_\mu^2+1)^3}$

The consituents of this integral are divergent, but their sum is finite. By using (A.12) we can write it in the form

$$\int_c d^4p \frac{\vec{p}^2 + 3p_0^2}{(p_\mu^2+1)^3} = 4\int d^4p \frac{\vec{p}^2}{(p_\mu^2+1)^3} - 3\int \frac{d^4p}{(p_\mu^2+1)^2} + 3\int \frac{d^4p}{(p_\mu^2+1)^3}. \tag{A.14}$$

The last integral can be calculated by means of (A.8), the first one by multiplying (A.6) by p^2, which yields

$$\int_c d^4p \frac{\vec{p}^2}{(p_\mu^2+1)^3} = \frac{3}{2}\, i\pi^2 \int_0^\infty dp \frac{p^4}{(\sqrt{p^2+1})^5}, \tag{A.15}$$

and the second one by applying directly (A.6) with $n = 2$.

After summing up we get finally

$$\int_c d^4p \frac{\vec{p}^2 + 3p_0^2}{(p_\mu^2+1)^3} = -\frac{1}{2}\, i\pi^2. \tag{A.16}$$

КРАТКОЕ СОДЕРЖАНИЕ

Я. Ольшевский, О электростатическом взаимодействии между нейтроном и электроном.

Нейтронно-электронное взаимодействие исследовалось техникой Дайсона применяя формалистическую регуляризацию. Оказалось, что кроме конвекционного тока также поляризационный ток даёт неисчезающий вклад в статическое взаимодействие, при нём вклад поляризационного тока приблизительно втрое больше конвекционного.

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LABORATORY EQUIPMENT AND TECHNIQUES

TWO SIMPLE METHODS OF MEASUREMENT OF THE RATE OF POLARIZATION OF LIGHT

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(received January 22, 1953)

Two experimental arrangements are described by means of which measurements of the rate of polarization of light can be made without such special devices as double image prisms, Savart plates etc. Only simple means are needed which can be found in every laboratory.

The first arrangement described is very similar to that used many years ago for the intensity measurements of *D*-fluorescence by Jabłoński and Pringsheim (1931). The only new elements are a nicol (or a polaroid) and a compensating set of glass plates (which is a very useful but not a necessary element of the arrangement). This arrangement, as used for measurement of the rate of polarization of fluorescence, is illustrated in Fig. 1.

The light beam from the source S is directed by means of the lens L_1 towards the fluorescent body (e.g., a fluorescent solution) F and polarized by the polaroid (or nicol) N_1. The fluorescent light goes through the lens L_3, the set of glass plates A, the polaroid N_2 and enters the slit Sl. A thin glass filament G illuminated by the source of the exciting light S serves as a self-glowing slit of the "monochromator" consisting

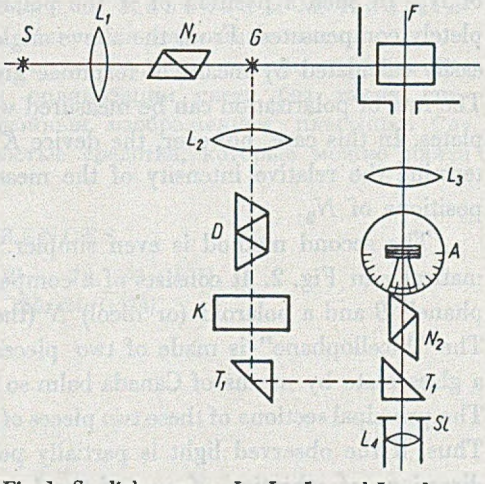
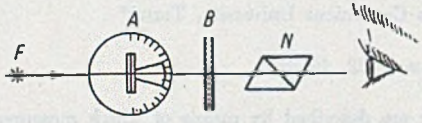


Fig. 1. S – light source. L_1, L_2, L_3, and L_4 – lenses. G – glass filament. F – fluorescent body. D – prism *a vision directe*. K – a device for adjusting the intensity of light scattered by the glass filament G (e.g. a nicol or polaroid). T_1 – totally reflecting prisms. A – a set of glass plates (compensator of Arago). N_1 and N_2 – nicols or polaroids. Sl – slit

* The experiments concerning the first of the described methods were carried out by A. Kawski, those concerning the second method by M. Kryszewski with the aid of P. Drzewiecki.

of the filament G, the lens L_2, the prism *à vision directe* D and the slit Sl which lets through a certain region of the spectrum. By shifting G along the direction of the exciting beam, the spectral compositions of the two beams entering the slit Sl and observed through L_4 can by made similar. The totally reflecting prisms are so adjusted as to give on the slit Sl two light spots (one above the other) with a sharp boundary. By means of the device K, the illuminations of the adjacent light spots observed through L_4 could be made equal. In our case this device consisted simply of a polaroid. Since the light scattered by G is partially polarized, the intensity of the comparison beam can be altered by turning the polaroid. Instead of a polaroid a gray wedge can be also used. In order to determine the rate of polarization of fluorescence one has to find such a position (angle of turn) of the compensating set of glass plates A at which the



ig.F 2. A — compensating set of glass plates. B – "bicellophane." N – nicol (or polaroid)

Fig. 3. „Bicellophane." The principal sections of the two adjacent pieces of cellophane make an angle of 45°

brightness of fluorescence observed through L_4 is independent of the azimuth of N_2. In such a position of A the polarization of the fluorescent light is completely compensated. From the above angle of turn the rate of polarization can eb easily calculated by means of formulae given by Gaviola and Pringsheim (1924t) The rate of polarization can be measured without using a compensating set of glass plates. In this case, however, the device K must be arranged so as to allow to determine the relative intensity of the measured light of fluorescence at different positions of N_2.

The second method is even simpler. The arrangement is shown diagrammatically in Fig. 2. It consists of a compensating set of glass plates A a "bicellophane" B and a polaroid (or nicol) N (the lenses are not shown in the figure). The "bicellophane" is made of two pieces of birefringent cellophane stuck on a glass plate by means of Canada balm so as to exhibit a sharp line of demarcation. The principal sections of these two pieces of cellophane make an angle of 45° (Fig. 3). Thus, if the observed light is partially polarized parallel to one of the principal directions of vibration of one of the halves of the bicellophane B (Fig. 2), the turning of the polaroid N will cause changes of brightness of this half of the bicellophane, but no changes whatever of the brightness of the other half. Thus, unless the observed light is completely depolarized, there results in general a difference of brightness of the two halves of the bicellophane B, observed through N_2. The difference is maximum when the direction of vibration of light transmited by N is parallel or perpendicular to that of polarization of light emitted by the source F.

The difference can be made to disappear by means of the compensating set of glass plates A, thus enabling the determination of the degree of polarization in the same manner as in the first method.

This method could be used also without the compensating set of glass plates (in this case N should be used to compensate the difference of brightness in B), but the accuracy of this modification would be much lower.

Both methods were tried with the use of the compensating set of glass plates only. Their accuracy proved to be not much lower than that of the standard methods. It depends of course on several conditions (as the intensity of the investigated light, its colour, etc.) and also on the rate of polarization to be measured, but even such small degrees of polarization as 1—3 per cent can still be measured by the above methods. We hope that the methods may prove useful for laboratories lacking in standard equipment for measuring the rate of polarization.

Note added in proof.

We are indebted to M. Frąckowiak for calling our attention to the fact that the cellophane itself might slightly polarize the transmitted light. Thus, if the bicellophane-method is used, one has to make sure that such unwanted effect on light transmitted by the bicellophane be compensated by a suitable setting of the nicol N (Fig. 2).

КРАТКОЕ СОДЕРЖАНИЕ

П. Држевецкий, А. Яблонский, А. Кавский и М. Кришевский, Два простых метода измерения степени поляризации света.

Описаны два экспериментальных устройства, при помощи которых могут производится измерения степени поляризации света без таких специальных приборов как призмы с двойным изображением, пластинка Саварта и т. д. Необходимы только простые средства, которые можно найти во всякой лаборатории.

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QUENCHING UNIT FOR CO_2 FILLED G. M. COUNTERS

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(received February 11, 1953)

A new quenching circuit organically coupled with a high frequency HV generator has been constructed and tested. The discharge pulse of the G.M. counter is fed to a univibrator generating a square pulse of adjustable length. The square pulse goes to a modulator which lowers the amplitude of the radiofrequency oscillations and thus the output HV on the G.M. counter.

It is well known that $CO_2 + CS_2$ filled G.M. counters need external quenching. For the purpose of supplying and quenching conventional circuits may be used; these circuits present, however, some disadvantages. With a CO_2 filling of 0,5 atm. the plateau voltage is c. 4000 volts and rises with pressure. Much trouble is experienced with conventional transmitting tubes working beyond their rating in such circuits. In order to get rid of these troubles a new quenching circuit combined with a high-voltage radio-frequency supply and a stabiliser on low voltage level was designed. High voltage in this circuit is derived from a radio-frequency generator coupled to a Teslacoil (Elmore and Sands 1949).

A radio-frequency high-voltage generator needs only a stabilized voltage of the order of some hundred volts. With a high radio-frequency one can use small time constants in the HV filter so that the HV can respond to fast amplitude modulation. If one makes the pulses from the G.M. counter actuate a modulator lowering the output voltage below the threshold value of the counter for a sufficient lapse of time, the counter discharge will be quenched. This new quenching circuit is organically coupled with the high frequency HV generator and this plays an important rôle in the quenching process. Three possibilities of quenching were considered. They are given in Fig. 1 a, b, c.

(a) The voltage drop across R_1 is fed by $C1$ to the univibrator A which generates a square pulse of adjustable length. The square pulse goes to the modulator B which lowers the amplitude of the radio-frequency oscillations and thus the output HV on the GM counter.

(b) This circuit is based on the fact that a radio-frequency HV supply can be easily overloaded. A positive square wave is fed to the overloading element, which for simplicity is assumed to be a biassed-off triode. This tube is rendered conducting for the duration of the square pulse and overloads thus the radio-frequency supply.

(c) Another possibility of overloading is based on "loss" or absorption modulation. In such a circuit the biassed-off tube is connected across a part of the HV transformer. When the tube is cut off, the secondary of the RF circuit is not damped. Positive pulse from the univibrator opens the cut-off tube. The plate

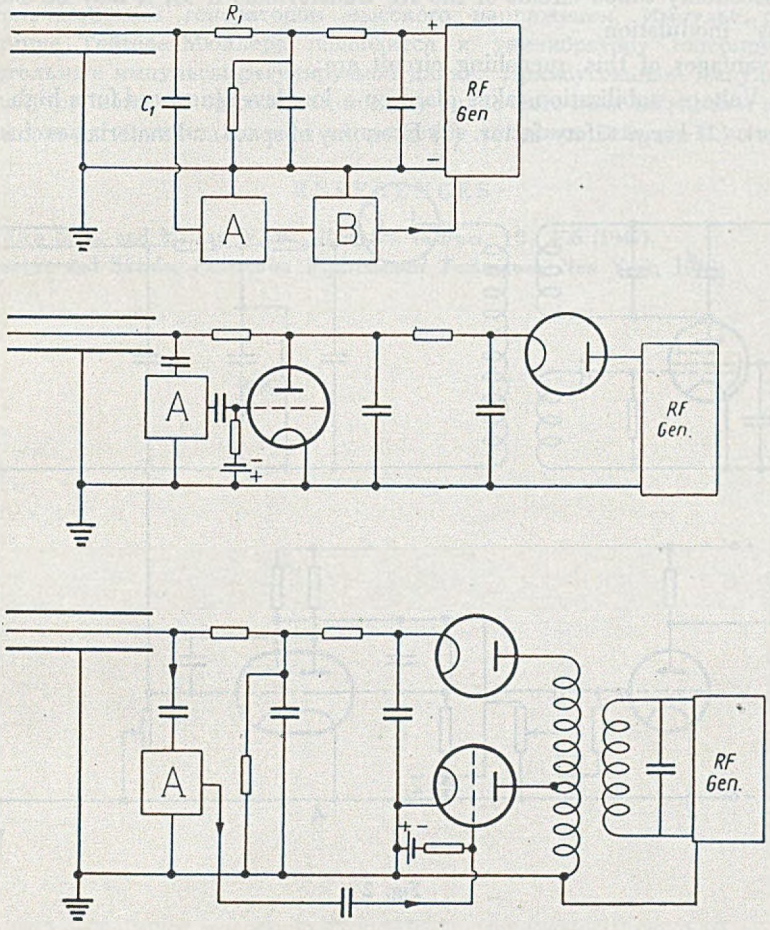

Fig. 1 a, b, c

current damps the secondary of the RF transformer, so that the HV drops and the discharge of the G.M. counter is extinguished.

Circuit (a) was built and tested. The RF was about 800 kc/S. The supply gave a maximum voltage of 5500 volts at about 1 mA. The detailed circuit diagram is represented on Fig. 2. The discharge pulse of the G.M. counter is fed to the single-shot multivibrator which generates a square pulse of adjustable length. The positive square wave opens the modulator tube. The additional current flowing through the plate resistor gives a voltage drop on the oscillator's screen-grid and this lowers the RF amplitude. This amplitude drop can be regulated by means of the potentiometer *P*. The leak resistor of 5 Mohms discharges the filter and the

G.M. capacities to the value of the new established radio-frequency high-voltage. For a quenching-voltage drop of 200 volts at 4000 working voltage, the 90% drop takes 0,7 msek., the rise time is less than 0,1 msek. (for a 2 msek. pulse duration). It is believed that the slow time of drop is caused mainly by the small damping of the secondary tuned circuit of the RF transformer; if so, it may be shortened by "loss" modulation.

Advantages of this quenching circuit are:

(1) Voltage stabilization takes place on a low level (no need for a high voltage stabilizer). (2) Large safety factor. (3) Economy of space and material, exclusive use

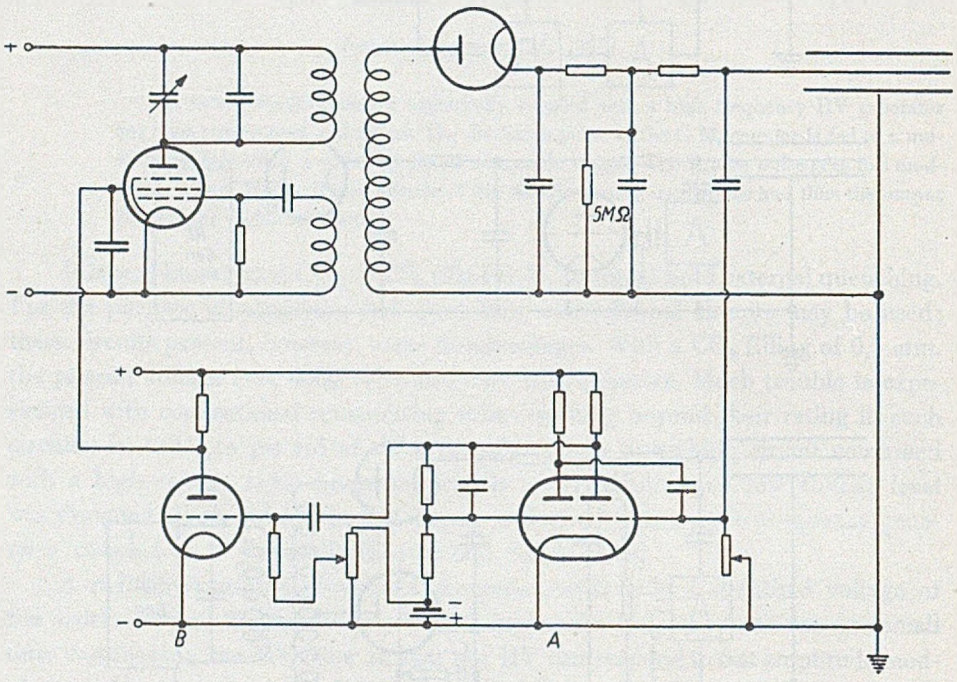

Fig. 2

of conventional radio tubes (no part of quenching circuit on HV). (4) Short rise time of counter voltage. (5) Possibility of regulation of resolving time and of quenching-voltage drop. (6) Positive or negative pulses from the univibrator with constant amplitude. (7) The cylinder of the counter may be grounded.

The chief disadvantage of this circuit is the slow voltage-drop on the counter (0,7 msek.) limiting its resolving time.

Details of circuits as well as working characteristics will be published later.

I wish to express my sincere gratitude to Professor S. Szczeniowski under whose guidance this work has been completed, as well as my great indebtness to Professor W. Mościcki, Chief of the Laboratory of Cosmic Ray Physics of the University of Poznań, who by his friendly advice and criticism has helped me throughout my work.

КРАТКОЕ СОДЕРЖАНИЕ

Ст. Горголевский, Тушащий элемент для счётчиков Гейгера-Мюллера наполненных CO_2.

Был построен и испытан новый тушащий контур органически связанный с высокочастотным генератором высокого напряжения. Импульс разряда в счётчике Гейгера-Мюллера подводится к унивибратору генерирующему прямоугольные импульсы регулируемой длины. Прямоугольный импульс идёт к модулятору, который снижает амплитуду радиочастотных колебаний и таким образом также снижает высокое напряжение счётчика Гейгера-Мюллера.

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LETTERS TO THE EDITOR

LÖSUNG DER G-GLEICHUNGEN VON JÁNOSSY FÜR DIE KOSMISCHEN SCHAUER

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(Eingegangen am 28. Mai 1953)

Nach Jánossy (1950) lauten die Grundgleichungen für die Theorie der Nukleonen-und Elektronen-Photonen-Schauer bzw.

$$\frac{\partial G}{\partial x}(\varepsilon,\underline{u},x) + \alpha G(\varepsilon,\underline{u},x) = \int_0^\infty\int_0^\infty G\left(\frac{\varepsilon}{\varepsilon'},\underline{u},x\right) G\left(\frac{\varepsilon}{\varepsilon''},\underline{u},x\right) w(\varepsilon',\varepsilon'')\, d\varepsilon'\, d\varepsilon'', \tag{1}$$

$$\frac{\partial G^{(i)}}{\partial x}(\varepsilon,\underline{u},x) + a_i G^{(i)}(\varepsilon,\underline{u},x) = \int_0^1 G^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'},\underline{u},x\right) G^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'},\underline{u},x\right) \tag{2}$$

$$\times w^{(i)}(\varepsilon')\, d\varepsilon', \qquad (i=1,2)$$

$$\underline{u} = (u_1,u_2) \quad \text{bzw.} \quad \underline{u} = (u_1,u_2,u_3,u_4),$$

$$\alpha = \int_0^\infty\int_0^\infty w(\varepsilon',\varepsilon'')\, d\varepsilon'\, d\varepsilon'' \quad \text{bzw.} \quad a_i = \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon'. \ (i=1,2)$$

Die Ionisierungsverluste sind vernachlässigt (Approximation A). Hier wurden dieselben Bezeichnungen wie in der Arbeit von Jánossy verwendet. Es treten noch gewisse Randbedingungen hinzu.

Diese Gleichungen wurden von mehreren Autoren (Jánossy, Messel, Bhabha u. a.) vielfach behandelt. Die Arbeiten befassten sich fast ausschliesslich mit der Berechnung der Momente der Verteilung der kosmischen Teilchen, nicht aber mit den erzeugenden Funktionen G bzw. $G^{(i)}$ selbst. Messel, Potts und Gardner gaben eine indirekte Methode zur Lösung der G-Gleichungen an.

Wir wollen nun im Folgenden eine direkte Lösungsmethode anführen, die sowohl auf die Nukleonenkomponente, wie auch auf die Elektronen-Photonen-Komponente mit und ohne Ionisierungsverluste anwendbar ist. Für $\varepsilon > 1$ lässt sich die Lösung

der obigen Gleichungen ohne grössere Schwierigkeiten ermitteln. Diese Lösung hängt nur von $\underline{u}$ und x nicht aber von ε ab. Ist die Lösung für $\varepsilon > 1$ bekannt, so lassen sich die Integrodifferentialgleichungen (1) bzw. (2) schrittweise lösen, und zwar gibt der erste Schritt die strenge Lösung des Problems für $\varepsilon > 1/2$, der zweite Schritt — für $\varepsilon > 1/3$, der $(n-1)$ Schritt — für $\varepsilon > 1/n$. Die Lösung für $\varepsilon > 1$ wird also sukzessiv auf die Streifen der Breite $\frac{1}{n-1} > \varepsilon > \frac{1}{n}$ in der (ε, x)-Ebene erweitert.

Wir wollen den Gedankengang der Lösungsmethode an einem möglichst einfachen Modell erläutern und betrachten dazu den Fall der Elektronen-Photonen-Kompenente mit $u_2 = u_4 = 1$ und $u_1 = u_3 = 0$. Führt man statt $G^{(i)}$ die Grösse $F^{(i)}(\varepsilon, x) = 1 - G^{(i)}$ ein, so lassen sich die Gleichungen schreiben

$$\frac{\partial F^{(i)}(\varepsilon,x)}{\partial x} + a_i F(\varepsilon,x) = \int_0^1 w^{(i)} \left\{ F^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'},x\right) + F^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'},x\right)\right\} d\varepsilon'$$

$$- \int_1^1 w^{(i)} F^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'},x\right) F^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'},x\right) d\varepsilon'. \qquad (3)$$

Es treten gewisse Randbedingungen hinzu. Dieses Problem lässt für $\varepsilon > 1$ eine einfache Lösung zu:

$$F^{(i)}(\varepsilon > 1, x) = 0.$$

Wir suchen zuerst die Lösung für $\varepsilon > 1/2$. Berücksichtigt man dann in den Gleichungen (3), dass

$$F^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'},x\right) = 0 \qquad \text{für } \varepsilon' < \varepsilon$$

$$F^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'},x\right) = 0 \qquad \text{für } \varepsilon' > 1-\varepsilon,$$

so lassen sich die Gleichungen auf die Form

$$\frac{\partial F^{(i)}(\varepsilon,x)}{\partial x} + a_i F^{(i)}(\varepsilon,x) = \int_0^1 w^{(i)} \left\{ F^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'},x\right) + F^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'},x\right)\right\} d\varepsilon' \qquad (4)$$

bringen. Diese Gleichungen sind linear und mit den Gleichungen für die Erwartungswerte der Besetzungszahlen in der Jánossyschen Theorie der kosmischen Strahlung identisch (Jánossy 1950, Jánossy und Messel 1950, Messel 1951). Sie lassen sich, wie üblich, mittels der Mellin Transformation lösen[1].

Wir erweitern nun das ε-Intervall zu $\varepsilon > 1/3$. Bezeichnet man die Lösung der linearen Gleichung (4) und zugleich die Lösung unseres Problems für $\varepsilon > 1/2$

[1] In unserem Fall verliert die inverse Mellin Transformation ihres Sinn für $\varepsilon = 0$, die Gleichungen (4) bzw. (5) lassen sich aber ohne Verwendung der Mellin Transformation leicht lösen.

mit $u_2^{(i)}$, während $u_1^{(i)} = O$ die Lösung für $\varepsilon > 1$ ist, so gilt für das Intervall $\varepsilon > 1/3$

$$\begin{aligned} F^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) &= 0 && \text{für} && 0 \leqslant \varepsilon' \leqslant \varepsilon \\ &= u^{(3-i)}_2 && \text{„} && \varepsilon \leqslant \varepsilon' \leqslant 2\varepsilon \\ F^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) &= u^{(1)}_2 && \text{„} && 1-2\varepsilon \leqslant \varepsilon' \leqslant 1-\varepsilon \\ &= 0 && \text{„} && 1-\varepsilon \leqslant \varepsilon' \leqslant 1. \end{aligned}$$

Dies in die Gleichung (3) gesetzt, führt zu den Gleichungen

$$\frac{\partial F^{(i)}(\varepsilon,x)}{\partial x} + a_i F^{(i)}(\varepsilon,x) - \int_0^1 \left\{ F^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) + F^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right)\right\} w^{(i)}\, d\varepsilon'$$

$$= -\int_0^1 u_2^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) u_2^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) w^{(i)}(\varepsilon')\, d\varepsilon' \qquad (5)$$

Diese Gleichungen sind wiederum linear und den Gleichungen für die zweiten Momente in der Jánossyschen Theorie ähnlich (Jánossy und Messel 1950).

Allgemein, im n-ten Schritt, bekommen wir statt (3) die Gleichungen

$$\frac{\partial F^{(i)}(\varepsilon,x)}{\partial x} + a_i F^{(i)}(\varepsilon,x - \int_0^1 \left\{ F^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) + F^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right)\right\} w^{(i)}\, d\varepsilon$$

$$= -\int_0^1 u_{n-1}^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) u_{n-1}^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) w^{(i)}(\varepsilon')\, d\varepsilon'. \qquad (6)$$

Auch diese Gleichungen lassen sich, wie üblich, mittels der Mellin Transformation lösen. Dann lässt sich die Rekurenzformel für die Lösung der transformierten Gleichung im n-ten Schritt wie folgt schreiben

$$P_n^{(i)}(s,x) = P_2^{(i)}(s,x) - \int_0^x \sum_{m=1}^{2} Q^{(im)}(s, x-\xi) \frac{1}{2\pi i} \int_{\varrho - i\infty}^{\varrho_0 + i\infty} dr\, s\, P_{n-1}^{(3-m)}(r,\xi)$$

$$\times P_{n-1}^{(1)}(s-r,\xi)\, A^{(m)}(r, s-r); \qquad P_n^{(i)} = \int_0^1 \varepsilon^{s-1} u_n(\varepsilon, x)\, d\varepsilon. \qquad (7)$$

Diese Methode lässt sich unmittelbar auf den Fall der Approximation B (Ionisierungsverluste berücksichtigt) erweitern. Sie lässt sich auf Integralgleichungen in denen das Glied

$$\int_0^u w\left(\frac{u'}{u}\right) du'\, H^{(1)}(u')\, H^{(2)}(u-u'),$$

d. h. eine Faltung mit Gewicht, auftritt und die Werte der Funktionen $H^{(i)}(u)$ für $o < u < 1$ bekannt sind, anwenden. Die Arbeit wird fortgesetzt, eine ausführliche Mitteilung folgt.

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