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SYNTHESIS AND SIMULATION OF RANDOM PROCESSES

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Contents

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Spis treści

Chapter 1

Introduction

This monograph presents an approach to the synthesis and simulation of wide-sense stationary scalar and multivariate, one- and multi- dimensional random processes given by diagrams of their power spectral densities. The approach is based on multisine random time-series and the finite discrete Fourier transform. In the sequel:

- *synthesis* means the determination of the spectrum of a multivariate multisine random time-series (1-D random process) or a multidimensional (M-D) multisine random process based on the corresponding power spectral density of the random process to be simulated,
- *simulation* means the generation of the corresponding multisine random process approximation by performing the inverse finite discrete Fourier transform of the synthesised spectrum.

1.1 STATE OF THE ART

Multisine time-series are known for a long time. They are sums of discrete-time sines with amplitudes and phase shifts determined by a variety of methods, depending upon the purpose for which the multisine time-series will serve. Traditionally, they are generated by solving their difference equations [52]. Recently, their popularity has increased due to the possibility of generating them by a numerically efficient implementation of the finite discrete Fourier transform and opportunities offered by digital computers equipped with new hardware for boosting numerical calculations like coprocessors or signal processors.

The most popular application of a multisine time-series is spectral analysis [46]. It determines the distribution of power or energy in the frequency-domain. The recent approach to spectral analysis has its roots in Fourier series representation of periodic functions. Amplitudes and phase shifts of Fourier series are chosen so as to fit this series to a given function in the mean square sense. These ideas are also a basic tool for analysis of random processes by using the Wiener generalised harmonic analysis.

Multisine time-series may be used also as basic building blocks for synthesising and simulating various deterministic signals and random processes with predetermined spectral or correlation properties. A theoretical foundation for such synthesis is given by the famous Gauss sum [90]. Its individual complex terms with the period length equal to any prime number exhibit interesting property of whiteness. They have strictly zero correlation function for nonzero shifts. For a long time, this idea has received no attention [6] and the interesting potential of multisine time-series seemed to be largely unexplored. More attention was given to the synthesis of binary signals for the purpose of excitation in system identification [2], [8], [17], [18], [43], [57] and for generation of cryptographically secure sequences [44], [86]. Some discussion of multisine time-series may be found in the books of Kay [50] and Marple [58]. However, in the last decade the significance of multisine time-series has increased. They have been successfully applied to synthesise excitations for process identification [42] and to synthesise different random processes: white noises of scalar [32], bivariate [66] and multivariate [34] type as well as random processes given by their power spectral densities [24], [25], [26], [28], [85].

If the multisine time-series is synthesised for the purpose of using it as an excitation signal during an identification experiment, it is worthwhile to choose its amplitudes so as to fit to the desired shape of spectrum [35], [23], [27] and its phase shifts so as to deliver to identified system as much power as possible in a limited amplitude range [42]. The choice of phase shifts which meets the above constraints is given by a number of suboptimum algorithms [7], [9], [10], [41], [76], [89]. The solution of this problem has contributed to the attractiveness of multisine excitations in system identification [9], [10], [87], [88], [91]. The properties of multisine excitations are basic to new methods of single- and multi-input system identification [23], [27], [29], [30], [31], [33], [35], [36].

This monograph concentrates on the application of multisine time-series to synthesis and simulation of random processes given by their power spectral densities.

The problem of synthesising and simulating random processes defined by their power spectral densities given in an analytical form has so far been solved satisfactorily only for rational 1-D power spectral densities [1], [49]. This approach is applied as an approximation for nonrational cases [78]. In the case of a multidimensional (M-D) random process given by its power spectral density, the problem of factorising the power spectral density is more complicated. It is well known that the M-D ($M > 1$) power spectral density of rational form almost never has a rational factorisation [11]. When the spectral factorisation problem is solved, the resulting random process is both synthesised and simulated as the output of a discrete-time linear filter excited by white noise. Spectral and correlation properties of the obtained random process realisations depend highly on:

- the quality of white noise used as a driving input. A considerable research effort has been made to develop various pseudo-random number generators and to compare their properties [53]. A recent comparison of several Gaussian white noise generators may be found in [12]. The generators currently used for simulation purposes belong as a rule to the class of linear congruential recursive generators. For a given initial state x_0 the future states of such generators evolve according to a linear recursion with modular arithmetic as:

$$x_n = (Ax_{n-1} + C) \bmod m.$$

  This basic scheme has been generalised to non-linear generators [69] and generators by inversion [70].

  The literature concerning multivariate orthogonal white noise generation seems to be scarce. A few ad hoc attempts to generate multivariate white noise series can be found but they are limited to bivariate white noise series. As a rule they tried to decorrelate binary random or binary pseudo-random series by various devices, the main one being time shifts [14], [37]. There exist congruential linear and non-linear generators producing sequences of multivariate white noise [71] but they have no mechanism providing, important in the multivariate case, orthogonality of its elements. However no systematic approach to deal with this problem is known so far.

  Congruential generators produce realisations of independent and uniformly distributed random variables. These random variables are sufficient to construct a random number generator for any desired continuous random variable distribution by means of inverting the distribution function [62]. In the case of Gaussian random variables the rivalling tool is provided by the Central Limit Theorem. An approximation to a Gaussian random variable may be obtained by summing many uniformly distributed random variables [83];

- the filter parameters accuracy obtainable for any given rational 1-D power spectral density by using spectral factorisation. In the 1-D nonrational case, the corresponding rational approximation can be calculated using minimax or least-squares error criteria [19] applied to the power spectral density. There exist extensions of the classical 1-D spectral factorisation concept to the 2-D case based on nonrational factors [15], [59] for which rational approximations are obtained using least squares, Padé or minimax approximation theory; however there is no general solution of the M-D ($M > 2$) spectral factorisation problem;

- the filter structure form implementation [38], [39], [54], [61], [64] and the rounding errors accumulating in recursive calculations.

Besides, analytical representations of power spectral densities are hardly ever available. Very often the power spectral density of the random process to be simulated is given only by a nonparametric representation, e.g. as a diagram or table.

1.2 THE CONTRIBUTION

This monograph presents a new approach to the problem of numerically synthesising and simulating wide-sense stationary time-series and multidimensional random processes for which only the nonparametric power spectral density representation as diagram or table is given.

The essence of the presented approach is to approximate the power spectral density by the periodogram of a multisine time-series with deterministic amplitudes chosen so that for a given number of equally spaced frequencies from the range $[0, 2\pi)$, the periodogram is equal to the original power spectral density [24], [25], [28]. The periodogram may be used in turn to construct the corresponding spectrum provided the phase shifts for each sine component are chosen. It is well known, that any periodogram corresponds to infinitely many different time-series with different phase shifts. It is demonstrated in the monograph, that to get ergodic random processes, the phase shifts should be chosen with some well-defined random properties. This concludes the synthesis part of the procedure. To simulate the synthesised time-series, the spectrum with the chosen phase shifts is transformed into the time-domain using the inverse finite discrete Fourier transform. Using this approach a broad range of scalar and multivariate random processes may be synthesised and simulated provided, their power spectral densities are available.

Multisine approximations of wide-sense stationary scalar and multivariate random processes obtained by this approach have discrete spectra. However, the original processes have continuous power spectral densities. It turns out that by fulfilling certain conditions on sampling in the frequency domain, the approximation of continuous power spectral densities by discrete spectra is not resulting in loss of information.

Additionally, the original random processes have autocorrelation functions converging to zero for large lags. This property holds for multisine time-series provided the number of sines is sufficiently large. For any real random process simulation, it is usually possible to choose the necessary number of sine components.

Multidimensional random processes given also by power spectral densities may be synthesised and simulated in the same way as 1-D random processes. The main building block used is an M-D multisine random process consisting of a sum of M-D sine components with deterministic amplitudes and random phase shifts.

A powerful theoretical justification of the approach is given by the Doob's Spectral Representations Theorem [16], [80] which states that any wide-sense stationary random process can be approximated arbitrarily close by a sum of sines and cosines with amplitudes being zero mean independent random variables and with deterministic phase shifts equal to zero.

1.3 ADVANTAGES OF THE APPROACH

The following factors are at the root of the attractiveness of the proposed approach to synthesis and simulation of wide-sense stationary random processes:

- there is no need to solve the spectral factorisation problem for a given power spectral density in order to calculate the corresponding parametric approximation needed for simulation;
- time-series or multidimensional random processes may be precisely defined in the frequency-domain, which is of importance for a number of applications (e.g. design of optimum excitations for identification [92], data encryption [67] and computer simulation of plants to be controlled [40], [84]);
- the frequency-domain definitions are directly used to generate, by means of the inverse finite discrete Fourier transform, the simulated random process which satisfy the ergodic hypothesis and are asymptotically Gaussian;
- particular realisations of the simulated random processes may be obtained by inversely Fast-Fourier-Transforming realisations of the synthesised spectrum;
- the approach may be used for nonparametrically defined wide-sense stationary random, rational and nonrational, scalar and multivariate time-series and multidimensional random processes, for which only the diagram or table of the power spectral density is available [24], [25], [28];
- the approach may be used to synthesise and simulate various types of scalar and multivariate white noises [32], [34], [66], which turn out to have interesting properties while compared with standard approaches, e. g. congruential generators;
- it gives an opportunity to reduce radically the simulation effort by a simulation time-scale contraction, which forms a new technique for the simulation of Gaussian random processes;
- there is a direct extension of the proposed method to the generation of wide-sense stationary continuous-time band-limited random signals, defined also by their power spectral densities [26].

1.4 ORGANISATION OF THE MONOGRAPH

In Chapter 2, the time- and frequency- domain definitions of scalar as well as different multivariate multisine random time-series are introduced. Their statistical properties, resulting from ensemble and time-domain averaging, are discussed. The weak ergodicity of multisine random time-series is examined. It is shown that periodograms of weakly ergodic multisine random time-series as well as expected values of periodograms for nonergodic multisine random time-series are uniquely defined by amplitudes of their sine components. This chapter is recapitulated with the idea of multisine random time-series synthesis and simulation based on the inverse finite discrete Fourier transform.

Chapter 3 is devoted to the synthesis and simulation of multisine random time-series defined by power spectral densities. Statistical properties of synthesised multisine random process approximations are determined. Asymptotic Gaussianess and ergodicity of synthesised time-series are discussed. An extension of the proposed random process synthesis and simulation method to the generation of wide-sense stationary continuous-time band-limited random signals, given also by their power spectral densities, is included.

Multisine white noise approximations obtained by using the proposed random process synthesis and simulation method are presented in Chapter 4. The following cases are discussed: weakly ergodic scalar and bivariate white and pseudo-white multisine random time-series which are asymptotically Gaussian, weakly ergodic multivariate orthogonal asymptotically Gaussian and white multisine random time-series, and nonergodic multivariate orthogonal white and pseudo-white multisine random time-series which are asymptotically ergodic and Gaussian.

Simulation of Gaussian random processes is the subject of Chapter 5. Simulation schemes based on the proposed approach are established, including a proposition of simulation time-scale contraction. The proposed schemes are illustrated by simulation examples.

In Chapter 6, an extension of multisine random time-series ideas given in Chapter 2 to a multidimensional (M-D) case is presented. Scalar and multivariate M-D multisine random processes are formally defined and their time- and frequency- domain properties are established. It is shown that multidimensional multisine random processes inherit properties of the 1-D multisine random time-series. The defined M-D multisine random processes are used to synthesise and simulate wide-sense stationary M-D random process given by their power spectral densities. Asymptotic properties of synthesised M-D multisine random process approximations are discussed.

The problem of synthesising and simulating various types of scalar, bivariate and multivariate ergodic and nonergodic multidimensional white multisine random processes is summarised in Chapter 7.

The proposed synthesis and simulation method of wide-sense stationary random processes given by their power spectral densities is recapitulated in Chapter 7.

All simulation experiments presented in the monograph have been done using the EFPI (*Expert for Process Identification* [63], [65]) and Multi-EDIP (*Multivariate System and Signal Analyser* [68]) software packages.

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1.5 NOTATIONS

Throughout this monograph:

- any multisine random time-series - one dimensional (1-D) multisine random process - is denoted by the stem MRS,
- any multidimensional (M-D) multisine random process is denoted by the stem MRS$^{M\text{-}D}$.

These stems can be preceded by additional letters with the meaning:

- B - bivariate,
- G - Gaussian,
- M - multivariate,
- N - nonergodic,
- O - orthogonal,
- PW - pseudo-white,
- S - scalar,

- W - white,

and followed by figures 1 or 2 denoting type of pseudo-whiteness or ergodocity.

Lower- and upper- case letters denote scalar quantities. Vectors and matrices are denoted by lower- and upper- case letters with bold type faces. Additionally, the following shorthand notation is used:

-
$$[a_r]_{r=1,2,\ldots,p} = [a_1, a_2, \ldots, a_p]^T, \tag{1.1}$$

-
$$\operatorname{diag}[b_{u_r u_r}]_{r=1,2,\ldots,p} = \begin{bmatrix} b_{u_1u_1} & 0 & \cdots & 0 \\ 0 & b_{u_2u_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{u_pu_p} \end{bmatrix}, \tag{1.2}$$

-
$$[c_{u_r u_s}]_{r,s=1,2,\ldots,p} = \begin{bmatrix} c_{u_1u_1} & c_{u_1u_2} & \cdots & c_{u_1u_p} \\ c_{u_2u_1} & c_{u_2u_2} & \cdots & c_{u_2u_p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{u_pu_1} & c_{u_pu_2} & \cdots & c_{u_pu_p} \end{bmatrix}. \tag{1.3}$$

Superscripts ^{T} and * represent the transpose and complex conjugate transpose operations, respectively.

The shorthand notation (x_ν) denotes M-tuples with M consecutive elements $x_1, x_2, \ldots, x_M$:

$$(x_\nu) = (x_1, x_2, \ldots, x_M). \tag{1.4}$$

The expected value operator $\mathcal{E}\{\cdot\}$ for a function $\theta(x)$ of the random variable x is defined as:

$$\mathcal{E}\{\theta(x)\} = \int_{-\infty}^{\infty} \theta(x)p(x)dx, \tag{1.5}$$

where $p(\cdot)$ represents the probability density function of the random variable x.

The mean value operator $\mathcal{M}\{\cdot\}$ is defined for:

- the 1-D case as:

$$\mathcal{M}\{x(i)\} = \lim_{q\to\infty} \frac{1}{qN} \sum_{i=0}^{qN-1} x(i), \tag{1.6}$$

where $x(i)$ for $i = 0, 1, \ldots, \infty$ denotes a time-series which is periodic modulo N, i.e.: $x(i) = x(i + qN)$ for $q = 0, 1, \ldots, \infty$;

- the M-D case as:

$$\mathcal{M}\{x(i_\nu)\} = \lim_{q_1\to\infty} \cdots \lim_{q_M\to\infty} \frac{1}{\prod_{\nu=1}^{M} q_\nu N_\nu} \sum_{i_1=0}^{q_1N_1-1} \cdots \sum_{i_M=0}^{q_MN_M-1} x(i_\nu), \tag{1.7}$$

where $x(i_\nu)$ for $(i_\nu) \in \{0, 1, \ldots, \infty\} \times \{0, 1, \ldots, \infty\} \times \cdots \times \{0, 1, \ldots, \infty\}$ denotes a periodic multidimensional (M-D) series with the period M-tuple (N_ν).

The following detailed notation is used:

$\square$	end of definition, lemma or proof
A	amplitude of a sine component
$\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi+\alpha\}\right)$	Bernoulli distribution with the probability $\frac{1}{2}$ on a set of events $\{\alpha, \pi+\alpha\}$
$\mathcal{C}$	complex numbers
$\delta(\cdot), \delta_e(\cdot), \delta_o(\cdot)$	the Kronecker's delta, even delta and odd delta function
DFT	the one- or multi- dimensional discrete Fourier transform
FFT	the Fast Fourier Transform algorithm
$g(\cdot)$	Gaussian multisine random process
$i, (i_\nu)$	discrete time instants and M-tuple of independent variables
$Im\{\cdot\}$	imaginary part of a complex number
$\mathbf{I}$	the unit matrix
j	a complex unit $j^2 = -1$
λ^2	the value of white noise power spectral density
N	period length of any multisine random time-series
(N_ν)	period M-tuple of any M-D multisine random process
$\mathcal{N}^1_{r,p}, \mathcal{N}^{c,1}_{r,p}$	set of relative frequencies of the rth element of $\mathbf{u}(i)$
$\mathcal{N}^M_{r,p}, \mathcal{N}^{c,M}_{r,p}$	set of frequency M-tuples of the rth element of $\mathbf{u}(i_\nu)$
$\mathcal{N}(0, \sigma^2)$	Gaussian distribution with mean 0 and variance σ^2
$\mathbf{o}$	the zero matrix
Ω	fundamental relative frequency
Ω_ν	fundamental relative frequency for the νth frequency axis
ωT	relative frequency from the range $[0, 2\pi)$
p	number of elements of any multivariate random process
ϕ, φ	phase shifts of sine components
$\Phi^B_{\mathbf{uu}}(\cdot)$	periodogram of $\mathbf{u}^B(\cdot)$
$\Phi_{vv}(\cdot)$	power spectral density of a random process $v(\cdot)$
q	number of repeated sequences of any basic multisine random process
$\mathcal{R}$	real numbers
$Re\{\cdot\}$	real part of a complex number
$R_{uu}(\cdot)$	autocorrelation function of $u(\cdot)$
σ^2_{uu}	variance of $u(\cdot)$
T	sampling interval
T_ν	sampling interval of the νth independent variable
$\tau, (\tau_\nu)$	lag and M-tuple lag in autocorrelation function
$u^B(\cdot)$ and $u(\cdot)$	basic and extended multisine random processes
$U^B(\cdot)$	finite discrete Fourier transform of $u^B(\cdot)$
z^{-1}	unit delay operator

Chapter 2

Multisine Random Time-Series

The purpose of this chapter is to introduce time- and frequency- domain definitions of basic and extended multisine random time-series. Scalar and different multivariate multisine random time-series are discussed. Their statistical properties resulting from ensemble and time-domain averaging are presented. The weak ergodicity of multisine random time-series is examined. It is shown that periodogram for weakly ergodic multisine random time-series and expected value of periodogram for nonergodic multisine random time-series are uniquely defined by amplitudes of their sine components. This chapter is recapitulated with the idea of multisine random time-series synthesis and simulation based on the inverse finite discrete Fourier transform.

2.1 SCALAR MULTISINE RANDOM TIME-SERIES

Definitions

The basic N-sample real-valued scalar multisine random time-series (SMRS) is defined in the time-domain as:

Definition 2.1 *The basic N-sample SMRS $u^B(i)$ is defined in the time-domain by a sum of $\frac{N}{2}+1$ discrete-time harmonic sines, including a constant component:*

$$u^B(i) = \sum_{n=0}^{\frac{N}{2}} A_n \sin(\Omega n i + \phi_n), \tag{2.1}$$

where $\Omega = \frac{2\pi}{N}$ denotes the fundamental relative frequency, $n = 0, 1, \ldots, \frac{N}{2}$ denotes consecutive harmonics of this frequency in the range $[0, \pi]$, $i = 0, 1, \ldots, N-1$ denotes consecutive discrete time instants, A_n are deterministic amplitudes of the sine components ($A_n \in \mathcal{R}$), ϕ_n are phase shifts, of which ϕ_0 is deterministic and the remaining phase shifts are random, independent and:

- *uniformly distributed on $[0, 2\pi)$ for $n = 1, 2, \ldots, \frac{N}{2} - 1$,*
- *Bernoulli distributed $B\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$ for $n = \frac{N}{2}$, i.e.:*

$$P\left\{\phi_{\frac{N}{2}} = \alpha\right\} = P\left\{\phi_{\frac{N}{2}} = \pi + \alpha\right\} = \frac{1}{2}, \tag{2.2}$$

where $P\{X\}$ denotes the probability of an event X.

□

The basic N-sample SMRS can be defined in the frequency-domain by its finite discrete Fourier transform [22]. This spectrum is determined as follows:

$$\begin{aligned} U^B(j\Omega m) &= \sum_{i=0}^{N-1} u^B(i) e^{-j\Omega m i} = \sum_{i=0}^{N-1} \sum_{n=0}^{\frac{N}{2}} A_n \sin(\Omega n i + \phi_n) e^{-j\Omega m i} \\ &= \sum_{n=0}^{\frac{N}{2}} \frac{A_n}{2j} \left[e^{j\phi_n} \sum_{i=0}^{N-1} e^{j(\Omega n - \Omega m)i} - e^{-j\phi_n} \sum_{i=0}^{N-1} e^{-j(\Omega n + \Omega m - 2\pi)i} \right] \\ &= \frac{N}{2j} \sum_{n=0}^{\frac{N}{2}} A_n \left[e^{j\phi_n} \delta(m-n) - e^{-j\phi_n} \delta(m - (N-n)) \right], \end{aligned} \tag{2.3}$$

where $j^2 = -1$, $\delta(\cdot)$ is the Kronecker's delta function:

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{2.4}$$

and use has been made of:

$$\sum_{i=0}^{N-1} e^{-j\Omega k i} = \begin{cases} N & \text{if } k = 0, N, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

The spectrum $U^B(j\Omega m)$ constitutes the frequency-domain definition of the basic N-sample SMRS:

Definition 2.2 *The basic N-sample SMRS $u^B(i)$ is defined in the frequency-domain for the (relative) frequency range $[0, 2\pi)$ by its finite discrete Fourier transform:*

$$\begin{aligned} U^B(j\Omega m) = \frac{N}{2} \Bigg\{ & (2A_0 \sin\phi_0 + j0)\delta(m) + \sum_{n=1}^{\frac{N}{2}-1} A_n \left[(\sin\phi_n - j\cos\phi_n)\, \delta(m-n) \right. \\ & \left. + (\sin\phi_n + j\cos\phi_n)\, \delta(m - (N-n)) \right] + (2A_{\frac{N}{2}} \sin\phi_{\frac{N}{2}} + j0)\delta(m - \frac{N}{2}) \Bigg\}, \end{aligned} \tag{2.6}$$

where $\Omega = \frac{2\pi}{N}$ denotes the fundamental relative frequency, $m = 0, 1, \ldots, N-1$ denotes consecutive harmonics of this frequency in the range $[0, 2\pi)$, A_n are amplitudes of the sine components ($A_n \in \mathcal{R}$), ϕ_n are phase shifts, of which ϕ_0 is deterministic and the remaining phase shifts are random, independent and:

- *uniformly distributed on $[0, 2\pi)$ for $n = 1, 2, \ldots, \frac{N}{2} - 1$,*
- *Bernoulli distributed $B\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$ for $n = \frac{N}{2}$.*

□

These two definitions of the basic SMRS are equivalent by means of the finite discrete Fourier transform. Definition 2.1 can be determined from Definition 2.2 by using the inverse finite discrete Fourier transform:

$$u^B(i) = \frac{1}{N} \sum_{m=0}^{N-1} U^B(j\Omega m) e^{j\Omega m i} = \frac{1}{N} \sum_{m=0}^{N-1} \frac{N}{2} \sum_{n=0}^{\frac{N}{2}} A_n \left[\frac{e^{j\phi_n}}{j} \delta(m-n) - \frac{e^{-j\phi_n}}{j} \delta(m-(N-n)) \right] e^{j\Omega m i}$$

$$= \frac{1}{2}\sum_{n=0}^{\frac{N}{2}} A_n \left[\frac{e^{j\phi_n}}{j}e^{j\Omega ni} - \frac{e^{-j\phi_n}}{j}e^{j\Omega(N-n)i}\right] = \sum_{n=0}^{\frac{N}{2}} A_n \sin(\Omega ni + \phi_n). \quad (2.7)$$

The range of relative frequencies in (2.1) is constrained from below by the constant component ($\Omega m = 0$) and from above by the Nyquist frequency ($\Omega m = \pi$). It can be shown that higher relative frequency sine components are represented by sine components from this range. The number of relative frequencies included in the spectrum $U^B(j\Omega m)$ is equal to N. All sine components (including the constant and Nyquist frequency components!) of the SMRS are represented in the relative frequency range $[0, 2\pi)$ by two lines each. These components are free from leakage [45] because their frequencies are harmonics of the frequency bin Ω. It implies periodicity of the basic SMRS in the time-domain window of the length N. Additionally, the spectrum $U^B(j\Omega m)$ of the real-valued SMRS satisfies, for the harmonic frequencies from the range $(\pi, 2\pi)$, the following condition:

$$U^B(j(2\pi - \Omega m)) = U^B(-j\Omega m). \quad (2.8)$$

Each frequency Ωm ($m = 0, 1, \ldots, N-1$) is related to the absolute frequency ω_m by

$$\Omega m = \omega_m T, \quad (2.9)$$

where T is the sampling interval of the corresponding (hypothetical) continuous-time sine.

Expanding the time range up to $i = 0, 1, \ldots, \infty$, an extended SMRS is obtained.

Definition 2.3 *The extended SMRS $u(i)$ is defined in the time-domain by a sum of $\frac{N}{2}+1$ discrete-time harmonic sines including a constant component:*

$$u(i) = \sum_{n=0}^{\frac{N}{2}} A_n \sin(\Omega ni + \phi_n), \quad (2.10)$$

where $\Omega = \frac{2\pi}{N}$ denotes the fundamental relative frequency, $n = 0, 1, \ldots, \frac{N}{2}$ denotes consecutive harmonics of this frequency in the range $[0, \pi]$, $i = 0, 1, \ldots, \infty$ denotes consecutive discrete time instants, A_n are amplitudes of the sine components ($A_n \in \mathcal{R}$), ϕ_n are phase shifts, of which ϕ_0 is deterministic and the remaining phase shifts are random, independent and:

- *uniformly distributed on $[0, 2\pi)$ for $n = 1, 2, \ldots, \frac{N}{2} - 1$,*
- *Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$ for $n = \frac{N}{2}$.*

□

In the sequel, it is assumed that the definitions of extended multisine time-series are obtained from the corresponding definitions of the basic multisine random time-series by changing the time-range from $i = 0, 1, \ldots, N-1$ up to $i = 0, 1, \ldots, \infty$.

The extended SMRS is periodic modulo N because $u(i) = u(i + qN)$ for $q = 0, 1, \ldots, \infty$. Besides:

$$u(i) = u^B(i) \quad (2.11)$$

for $0 \leq i \leq N-1$. It implies that the extended SMRS belongs to the space of periodic signals [48] with the period N.

The spectrum $U'(j\Omega' m')$ of the first qN samples of the extended SMRS $u(i)$ is related to the $U^B(j\Omega m)$ by:

$$U'(j\Omega' m') = \begin{cases} qU^B(j\Omega' m') & \text{if } \Omega' m' \in \{0, \Omega, \ldots, \Omega(N-1)\} \\ 0 + j0 & \text{if } \Omega' m' \notin \{0, \Omega, \ldots, \Omega(N-1)\} \end{cases}, \quad (2.12)$$

where $\Omega' = \frac{2\pi}{qN} = \frac{\Omega}{q}$ denotes the relative fundamental frequency for the qN-sample time-series and $m' = 0, 1, \ldots, qN-1$ denotes consecutive harmonics of this fundamental frequency in the range $[0, 2\pi)$.

Properties

By finite Fourier transform techniques [4], spectral properties of the basic N-sample SMRS can be stated as:

Lemma 2.1 *Consider the basic N-sample SMRS. Its periodogram is given by:*

$$\Phi^B_{uu}(\Omega m) = \frac{NT}{4}\left\{4A_0^2\sin^2\phi_0\delta(m) + \sum_{n=1}^{\frac{N}{2}-1} A_n^2[\delta(m-n) + \delta(m-(N-n))] + 4A_{\frac{N}{2}}^2\sin^2\alpha\,\delta\left(m - \frac{N}{2}\right)\right\}, \quad (2.13)$$

where $m = 0, 1, \ldots, N-1$.

□

Proof: *It follows from the periodogram definition [4] that:*

$$\Phi^B_{uu}(\Omega m) = \mathcal{E}\left\{\frac{T}{N}\left[\sum_{i=0}^{N-1} u^B(i)e^{-j\Omega mi}\right]\left[\sum_{i=0}^{N-1} u^B(i)e^{j\Omega mi}\right]\right\}$$

$$= \frac{T}{N}\left[\sum_{i=0}^{N-1} u^B(i)e^{-j\Omega mi}\right]\left[\sum_{i=0}^{N-1} u^B(i)e^{j\Omega mi}\right] = \frac{T}{N}U^B(j\Omega m)U^B(-j\Omega m). \quad (2.14)$$

This ends the proof when Definition 2.2 is taken into account.

□

The statistical properties of the extended SMRS, which result from the ensemble averaging, are given by:

Lemma 2.2 *Consider the extended SMRS. For each time instant $i = 0, 1, \ldots, \infty$:*

1. *its expected value is given by $\mathcal{E}\{u(i)\} = A_0\sin\phi_0$.*
2. *its autocorrelation function is:*

$$\mathcal{E}\{u(i)u(i-\tau)\} = A_0^2\sin^2\phi_0 + \frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1} A_n^2\cos(\Omega n\tau) + (-1)^\tau A_{\frac{N}{2}}^2\sin^2\alpha, \quad (2.15)$$

where $\tau = 0, 1, \ldots, \infty$.

3. *its variance is:*

$$\mathcal{E}\left\{(u(i) - \mathcal{E}\{u(i)\})^2\right\} = \frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1} A_n^2 + A_{\frac{N}{2}}^2\sin^2\alpha. \quad (2.16)$$

□

Proof: *The uniform distribution of the random phase shifts ϕ_n on $[0, 2\pi)$ for each frequency Ωn ($n = 1, 2, \ldots, \frac{N}{2} - 1$) implies that for any time instant i the random variable (the nth extended SMRS sine component):*

$$u_n(i) = A_n\sin(\Omega ni + \phi_n) \quad (2.17)$$

is characterised [77] by:

$$\mathcal{E}\{u_n(i)\} = 0 \quad (2.18)$$

and

$$\mathcal{E}\{u_n(i)u_n(i-\tau)\} = \frac{A_n^2}{2}\cos(\Omega n\tau). \tag{2.19}$$

Bernoulli distribution $B\left(\frac{1}{2},\{\alpha,\pi+\alpha\}\right)$ *of the random phase shift* $\phi_{\frac{N}{2}}$ *for the frequency* $\Omega\frac{N}{2}=\pi$ *and the properties of sine function imply that for any time instant* i *the random variable*

$$u_{\frac{N}{2}}(i) = A_{\frac{N}{2}}\sin(\pi n i+\phi_{\frac{N}{2}}) \tag{2.20}$$

is also characterised by Bernoulli distribution $B\left(\frac{1}{2},\left\{A_{\frac{N}{2}}\sin\alpha,-A_{\frac{N}{2}}\sin\alpha\right\}\right)$ *with:*

$$\mathcal{E}\{u_{\frac{N}{2}}(i)\} = 0 \tag{2.21}$$

and

$$\mathcal{E}\{u_{\frac{N}{2}}(i)u_{\frac{N}{2}}(i-\tau)\} = (-1)^\tau A_{\frac{N}{2}}^2\sin^2\alpha. \tag{2.22}$$

It follows from the above remarks and Definition 2.1 that:

$$\mathcal{E}\{u(i)\} = A_0\sin\phi_0. \tag{2.23}$$

The independence of random phase shifts under the Disjoint Blocks Theorem [51] implies that autocorrelation function of $u(i)$ *is:*

$$\mathcal{E}\{u(i)u(i-\tau)\} = A_0^2\sin^2\phi_0 + \sum_{n=1}^{\frac{N}{2}-1}\frac{A_n^2}{2}\cos(\Omega n\tau) + (-1)^\tau A_{\frac{N}{2}}^2\sin^2\alpha. \tag{2.24}$$

□

It follows from this lemma that the extended SMRS is a wide-sense stationary random process. Any change of the assumption about distributions of the random phase shifts ϕ_n for $n = 1,2,\ldots,\frac{N}{2}$ in the extended SMRS definition would result in extended SMRS's for which the expected value and autocorrelation function will depend on the time instant i. For instance, when the Nyquist-frequency phase shift $\phi_{\frac{N}{2}}$ is assumed to be deterministic and the remaining phase shifts are defined as in Definition 2.1, the resulting extended SMRS has time-dependent expected value:

$$\mathcal{E}\{u(i)\} = A_0\sin\phi_0 + (-1)^i A_{\frac{N}{2}}\sin\phi_{\frac{N}{2}}. \tag{2.25}$$

The choice of all random phase shifts as Bernoulli distributed $B\left(\frac{1}{2},\{\alpha,\pi+\alpha\}\right)$ leads to the extended SMRS's which exhibit interesting symmetries:

- for $\alpha = 0$ and additionally $A_0 = 0$ or $\phi_0 = 0$, the resulting extended SMRS's are odd sequences:
$$u(i+qN) = -u(qN-i), \tag{2.26}$$
- for $\alpha = \frac{\pi}{2}$, the resulting extended SMRS's are even sequences:
$$u(i+qN) = u(qN-i), \tag{2.27}$$

where $i = 1,2,\ldots,N-1$ and $q = 1,2,\ldots,\infty$.

The following lemma presents properties of the extended SMRS obtained for the time-domain averaging on any particular time-series:

Lemma 2.3 *Consider the extended SMRS.*

1. *Its mean value is* $\mathcal{M}\{u(i)\} = A_0\sin\phi_0$.

2. *Its autocorrelation function is given by:*
$$R_{uu}(\tau) = A_0^2\sin^2\phi_0 + \frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1}A_n^2\cos(\Omega n\tau) + (-1)^\tau A_{\frac{N}{2}}^2\sin^2\alpha, \tag{2.28}$$
where $\tau = 0,1,\ldots,\infty$.
3. *Its variance is:*
$$\sigma_{uu}^2 = \frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1}A_n^2 + A_{\frac{N}{2}}^2\sin^2\alpha. \tag{2.29}$$
4. *Its sine components are mutually orthogonal:*
$$\sum_{i=0}^{qN-1}A_s\sin(\Omega si+\phi_s)A_t\sin(\Omega ti+\phi_t) = 0 \tag{2.30}$$
for all $s \neq t$, $s,t = 0,1,\ldots,\frac{N}{2}$ *and* $q = 1,2,\ldots,\infty$.

□

Proof:

1. *It follows from Definition 2.1 that:*
$$\mathcal{M}\{u(i)\} = \lim_{q\to\infty}\frac{1}{qN}\sum_{i=0}^{qN-1}u(i) = \frac{1}{N}\sum_{i=0}^{N-1}u(i) = \frac{1}{N}U^B(j0) = A_0\sin\phi_0. \tag{2.31}$$
The limit disappeared because of the periodicity modulo N *of* $u(i)$.
2. *It follows from the definition of the time-domain averaged autocorrelation function that:*
$$R_{uu}(\tau) = \lim_{q\to\infty}\frac{1}{qN}\sum_{i=0}^{qN-1}u(i)u(i-\tau) = \frac{1}{T}\lim_{q\to\infty}\mathcal{DFT}^{-1}\left\{\Phi'_{uu}(\Omega'm')\right\}, \tag{2.32}$$
where $\Omega' = \frac{2\pi}{qN}$, $m' = 0,1,\ldots,qN-1$, $\mathcal{DFT}^{-1}$ *denotes the inverse discrete Fourier transform and* $\Phi'_{uu}(\Omega'm')$ *is the periodogram of* qN *samples of* $u(i)$. *The above definition can be presented as:*
$$\begin{aligned}R_{uu}(\tau) &= \lim_{q\to\infty}\frac{1}{qNT}\sum_{m'=0}^{qN-1}\frac{T}{qN}\left[\sum_{i=0}^{qN-1}u^B(i)e^{-j\Omega'm'i}\right]\left[\sum_{i=0}^{qN-1}u^B(i)e^{j\Omega'm'i}\right]e^{j\Omega'm'\tau}\\ &= \lim_{q\to\infty}\frac{1}{qNT}\sum_{m'=0}^{qN-1}\frac{qT}{N}\left[\sum_{i=0}^{N-1}u^B(i)e^{-j\Omega'm'i}\right]\left[\sum_{i=0}^{N-1}u^B(i)e^{j\Omega'm'i}\right]e^{j\Omega'm'\tau}\\ &= \frac{1}{NT}\sum_{m=0}^{N-1}\frac{T}{N}U^B(j\Omega m)U^B(-j\Omega m)e^{j\Omega m\tau} = \frac{1}{NT}\sum_{m=0}^{N-1}\Phi_{uu}^B(\Omega m)e^{j\Omega m\tau},\end{aligned} \tag{2.33}$$
where the limit disappeared because of the periodicity modulo N *of* $u(i)$. *Additionally, the autocorrelation function* $R_{uu}(\tau)$ *is a deterministic function. It follows from Lemma 2.1 that:*
$$R_{uu}(\tau) = \frac{1}{4}\sum_{m=0}^{N-1}\left\{4A_0^2\sin^2\phi_0\delta(m) + \sum_{n=1}^{\frac{N}{2}-1}A_n^2\left[\delta(m-n)+\delta(m-(N-n))\right]\right.$$

$$\left. + 4A_{\frac{N}{2}}^2 \sin^2 \alpha \delta(m - \frac{N}{2}) \right\} e^{j\Omega m \tau}. \tag{2.34}$$

It results in (2.28).

3. *It follows from autocorrelation function (2.28) that:*

$$\sigma^2 = R_{uu}(0) - A_0^2 \sin^2 \phi_0. \tag{2.35}$$

4. *It follows immediately from the Parseval Theorem and from (2.6).*

□

The same results of the time-domain averaging as presented in the above lemma may be obtained for any distributions of the random phase shifts ϕ_n $(n = 1, 2, \ldots, \frac{N}{2} - 1)$ assuming only that ϕ_0 is deterministic, and $\phi_{\frac{N}{2}}$ is deterministic or Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$.

It follows from Lemma 2.2 and Lemma 2.3 that the extended SMRS is a wide-sense stationary random process for which the time-domain averaged results from any time-series realisation are equal to the corresponding ensemble averaged results over collection of the time-series. It implies weak ergodicity of the extended SMRS. When the random phase shifts distributions are different from these presented in the SMRS definition, the results of the time-domain averaging are different from the results of ensemble averaging and obtained time-series are nonergodic.

It should be noticed that, in spite of random phase shifts, the autocorrelation function and periodogram of the SMRS are deterministic, real-valued functions. Additionally, the autocorrelation function is periodic modulo N.

The orthogonality of sine components of the SMRS is independent of the choice of these sine components phase shifts and amplitudes.

2.2 MULTIVARIATE ORTHOGONAL MULTISINE RANDOM TIME-SERIES

2.2.1 Ergodic Case

Definitions

Consider any scalar multisine random time-series with a sufficiently large number $\frac{N}{2} + 1$ of sine components. Each element $u_r(i)$ $(r = 1, 2, \ldots, p)$, of a multivariate orthogonal multisine random time-series (MOMRS) $\mathbf{u}(i)$ is a sum of some of the SMRS sine components with the constraint that the same frequency may not appear in more than one MOMRS element and each SMRS sine component belongs to one and only one MOMRS element. It is formalised by the following time-domain definition:

Definition 2.4 *The basic N-sample MOMRS is defined in the time-domain by the p-dimensional multivariate time-series* $\mathbf{u}^B(i) = \left[u_r^B(i)\right]_{r=1,2,\ldots,p}$ *with the rth MOMRS element given by:*

$$u_r^B(i) = \sum_{\Omega n \in \mathcal{N}_{r,p}^1} A_n \sin(\Omega n i + \phi_n). \tag{2.36}$$

$\mathcal{N}_{r,p}^1$ *is the set of all frequencies* Ωn *present in the rth MOMRS element* $u_r(i)$ *and:*

$$\mathcal{N}_{1,p}^1 \cup \mathcal{N}_{2,p}^1 \cup \cdots \cup \mathcal{N}_{p,p}^1 = \{0, \Omega, \ldots, \pi\}. \tag{2.37}$$

These sets are pairwise disjoint:

$$\mathcal{N}_{s,p}^1 \cap \mathcal{N}_{t,p}^1 = \emptyset \tag{2.38}$$

for $s \neq t$, *and* $s, t = 1, 2, \ldots, p$. $\Omega = \frac{2\pi}{N}$ *denotes the fundamental relative frequency,* $n = 0, 1, \ldots, \frac{N}{2}$ *denotes consecutive harmonics of this frequency in the range* $[0, \pi]$, $i = 0, 1, \ldots, N - 1$ *denotes consecutive discrete time instants,* A_n *are deterministic amplitudes of the sine components* $(A_n \in \mathcal{R})$, ϕ_n *are phase shifts, of which* ϕ_0 *is deterministic and the remaining phase shifts are random, independent and:*

- *uniformly distributed on* $[0, 2\pi)$ *for* $n = 1, 2, \ldots, \frac{N}{2} - 1$,
- *Bernoulli distributed* $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$ *for* $n = \frac{N}{2}$.

□

The basic N-sample MOMRS is represented in the frequency-domain for the (relative) frequency range $[0, 2\pi)$ by the p-dimensional vector $\mathbf{U}^B(j\Omega m) = \left[U_r^B(j\Omega m)\right]_{r=1,2,\ldots,p}$ of finite discrete Fourier transforms with the rth element given by:

$$U_r^B(j\Omega m) = \frac{N}{2j} \sum_{\Omega n \in \mathcal{N}_{r,p}^1} A_n \left[e^{j\phi_n} \delta(m - n) - e^{-j\phi_n} \delta(m - (N - n))\right], \tag{2.39}$$

where $m = 0, 1, \ldots, N - 1$ denotes consecutive harmonics of the fundamental relative frequency Ω in the range $[0, 2\pi)$.

Elements of the basic MOMRS can be regarded as real-valued SMRS. They inherit properties of the scalar multisine random time-series. The spectrum vector $\mathbf{U}^B(j\Omega m)$ of the real-valued MOMRS satisfies, for the harmonic frequencies from the range $(\pi, 2\pi)$, the condition:

$$\mathbf{U}^B(j(2\pi - \Omega m)) = \mathbf{U}^B(-j\Omega m). \tag{2.40}$$

Similarly as for the scalar case, the extended MOMRS is periodic modulo N, i.e. $\mathbf{u}(i) = \mathbf{u}(i + qN)$ for $q = 0, 1, \ldots, \infty$. The extended time-series is related to the basic N-sample MOMRS by $\mathbf{u}(i) = \mathbf{u}^B(i)$ for $0 \leq i \leq N - 1$. The spectrum vector $\mathbf{U}'(j\Omega' m')$ of the first qN samples of the extended MOMRS can be expressed using the $\mathbf{U}^B(j\Omega m)$ as:

$$\mathbf{U}'(j\Omega' m') = \begin{cases} q\mathbf{U}^B(j\Omega m) & \text{if } \Omega' m' \in \{0, \Omega, \ldots, \Omega(N-1)\} \\ 0 + j0 & \text{if } \Omega' m' \notin \{0, \Omega, \ldots, \Omega(N-1)\} \end{cases}, \tag{2.41}$$

where $\Omega' = \frac{2\pi}{qN} = \frac{\Omega}{q}$ denotes the relative fundamental frequency for the qN-sample time-series and $m' = 0, 1, \ldots, qN - 1$ denotes consecutive harmonics of this frequency in the range $[0, 2\pi)$.

The fact that elements of the MOMRS have no common frequencies under the Parseval theorem implies orthogonality of its elements for the ensemble averaging:

$$\mathcal{E}\{u_r(i) u_s(i)\} = 0 \tag{2.42}$$

as well as for the time-domain averaging:

$$\frac{1}{qN} \sum_{i=0}^{qN-1} u_r(i) u_s(i) = 0, \tag{2.43}$$

where $r \neq s$, $r, s = 1, 2, \ldots, p$ and $q = 1, 2, \ldots, \infty$.

Properties

The periodogram matrix of the basic N-sample MOMRS is given by the lemma:

Lemma 2.4 *Consider the basic N-sample MOMRS. Its periodogram matrix is $\Phi^B_{\mathrm{uu}}(j\Omega m) = \left[\Phi^B_{u_r u_s}(j\Omega m)\right]_{r,s=1,2,\ldots,p}$, where for $m = 0,1,\ldots,N-1$:*

$$\Phi^B_{u_r u_s}(j\Omega m) = \begin{cases} \Phi^B_{u_r u_r}(\Omega m) + j0 & \text{if } r = s \\ 0 + j0 & \text{if } r \neq s \end{cases} \tag{2.44}$$

$\Phi^B_{u_r u_r}(\Omega m)$ is the periodogram of the rth MOMRS element:

$$\Phi^B_{u_r u_r}(\Omega m) = \frac{NT}{4} \sum_{\Omega n \in \mathcal{N}^1_{r,p}\setminus\{0,\pi\}} A_n^2\left[\delta(m-n) + \delta(m-(N-n))\right] + \Phi^B_{0,\pi}(\Omega m), \tag{2.45}$$

where:

$$\Phi^B_{0,\pi}(\Omega m) = \begin{cases} NT\left(A_0^2 \sin^2\phi_0 \delta(m) + A_{\frac{N}{2}}^2 \sin^2\alpha\,\delta(m-\frac{N}{2})\right) & \text{if } (0 \in \mathcal{N}^1_{r,p}) \wedge (\pi \in \mathcal{N}^1_{r,p}) \\ NTA_0^2 \sin^2\phi_0 \delta(m) & \text{if } (0 \in \mathcal{N}^1_{r,p}) \wedge (\pi \notin \mathcal{N}^1_{r,p}) \\ NTA_{\frac{N}{2}}^2 \sin^2\phi_{\frac{N}{2}} \delta(m-\frac{N}{2}) & \text{if } (0 \notin \mathcal{N}^1_{r,p}) \wedge (\pi \in \mathcal{N}^1_{r,p}) \\ 0 & \text{if } (0 \notin \mathcal{N}^1_{r,p}) \wedge (\pi \notin \mathcal{N}^1_{r,p}) \end{cases} \tag{2.46}$$

□

Proof: *The proof of the above lemma proceeds similarly as for Lemma 2.1, when it is noticed that the periodogram matrix of MOMRS is:*

$$\Phi^N_{\mathrm{uu}}(\Omega m) = \mathcal{E}\left\{\frac{T}{N}\left[\sum_{i=0}^{N-1} \mathrm{u}^B(i)e^{-j\Omega m i}\right]\left[\sum_{i=0}^{N-1} \mathrm{u}^{T,B}(i)e^{j\Omega m i}\right]\right\} = \frac{T}{N}\mathrm{U}^B(j\Omega m)\mathrm{U}^{T,B}(-j\Omega m). \tag{2.47}$$

□

It follows from this lemma that for all frequencies Ωm ($m = 0,1,\ldots,N-1$) the MOMRS periodogram matrix is a singular matrix.

When the ensemble averaging is taken into account, properties of the MOMRS are given by the lemma:

Lemma 2.5 *Consider the extended MOMRS. For each time instant $i = 0,1,\ldots,\infty$:*

1. *its expected value vector is $\mathcal{E}\{\mathrm{u}(i)\} = [\mathcal{E}\{u_r(i)\}]_{r=1,2,\ldots,p}$, where:*

$$\mathcal{E}\{u_r(i)\} = \begin{cases} A_0 \sin\phi_0 & \text{if } 0 \in \mathcal{N}^1_{r,p} \\ 0 & \text{if } 0 \notin \mathcal{N}^1_{r,p} \end{cases} \tag{2.48}$$

2. *its correlation function matrix is $\mathcal{E}\{\mathrm{u}(i)\mathrm{u}^T(i-\tau)\} = [\mathcal{E}\{u_r(i)u_s(i-\tau)\}]_{r,s=1,2,\ldots,p}$, where for $\tau = 0,1,\ldots,\infty$:*

$$\mathcal{E}\{u_r(i)u_s(i-\tau)\} = \begin{cases} \mathcal{E}\{u_r(i)u_r(i-\tau)\} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \tag{2.49}$$

$\mathcal{E}\{u_r(i)u_r(i-\tau)\}$ is the autocorrelation function of the rth MOMRS element:

$$\mathcal{E}\{u_r(i)u_r(i-\tau)\} = \sum_{\Omega n \in \mathcal{N}^1_{r,p}\setminus\{0,\pi\}} \frac{A_n^2}{2}\cos(\Omega n\tau) + \mathcal{E}\{u_{0,\pi}(i)u_{0,\pi}(i-\tau)\}, \tag{2.50}$$

where:

$$\mathcal{E}\{u_{0,\pi}(i)u_{0,\pi}(i-\tau)\} = \begin{cases} A_0^2 \sin^2\phi_0 + (-1)^\tau A_{\frac{N}{2}}^2 \sin^2\alpha & \text{if } (0 \in \mathcal{N}^1_{r,p}) \wedge (\pi \in \mathcal{N}^1_{r,p}) \\ A_0^2 \sin^2\phi_0 & \text{if } (0 \in \mathcal{N}^1_{r,p}) \wedge (\pi \notin \mathcal{N}^1_{r,p}) \\ (-1)^\tau A_{\frac{N}{2}}^2 \sin^2\phi_{\frac{N}{2}} & \text{if } (0 \notin \mathcal{N}^1_{r,p}) \wedge (\pi \in \mathcal{N}^1_{r,p}) \\ 0 & \text{if } (0 \notin \mathcal{N}^1_{r,p}) \wedge (\pi \notin \mathcal{N}^1_{r,p}) \end{cases} \tag{2.51}$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.2.

It follows from this lemma that the extended MOMRS is a wide-sense stationary multivariate random process. Similarly, as for the scalar case of multisine random time-series, any change of the assumption about distributions of the random phase shifts ϕ_n in the MOMRS definitions results in an extended MOMRS for which elements of the expected value vector and autocorrelation function matrix are time-dependent. For instance, when the Nyquist-frequency phase shift $\phi_{\frac{N}{2}}$ is assumed to be deterministic and the remaining phase shifts are as in the MOMRS definition, the resulting extended MOMRS expected value vector is time-dependent, i.e. elements of the $\mathcal{E}\{\mathrm{u}(i)\} = [\mathcal{E}\{u_r(i)\}]_{r=1,2,\ldots,p}$ are:

$$\mathcal{E}\{u_r(i)\} = \begin{cases} A_0 \sin\phi_0 + (-1)^i A_{\frac{N}{2}}^2 \sin\phi_{\frac{N}{2}} & \text{if } 0 \in \mathcal{N}^1_{r,p} \\ 0 & \text{if } 0 \notin \mathcal{N}^1_{r,p} \end{cases} \tag{2.52}$$

The choice of all random phase shifts ϕ_n as Bernoulli distributed $\mathcal{B}\left(\frac{1}{2},\{\alpha,\pi+\alpha\}\right)$ leads to the extended MOMRS's which exhibit the following symmetries:

- for $\alpha = 0$ and additionally $A_0 = 0$ or $\phi_0 = 0$, the time-series are odd sequences, i.e. $\mathrm{u}(i+qN) = -\mathrm{u}(qN-i)$,
- for $\alpha = \frac{\pi}{2}$, the time-series are even sequences, i.e. $\mathrm{u}(i+qN) = \mathrm{u}(qN-i)$, where $i = 1,2,\ldots,N-1$ and $q = 1,2,\ldots,\infty$.

When the time-domain averaging on any particular extended MOMRS is analysed, the following lemma can be formulated:

Lemma 2.6 *Consider the extended MOMRS.*

1. *Its mean value vector is $\mathcal{M}\{\mathrm{u}(i)\} = [\mathcal{M}\{u_r(i)\}]_{r=1,2,\ldots,p}$, where:*

$$\mathcal{M}\{u_r(i)\} = \begin{cases} A_0 \sin\phi_0 & \text{if } 0 \in \mathcal{N}^1_{r,p} \\ 0 & \text{if } 0 \notin \mathcal{N}^1_{r,p} \end{cases} \tag{2.53}$$

2. *Its correlation function matrix is $\mathrm{R}_{\mathrm{uu}}(\tau) = [R_{u_r u_s}(\tau)]_{r,s=1,2,\ldots,p}$, where for $\tau = 0,1,\ldots,\infty$,:*

$$R_{u_r u_s}(\tau) = \begin{cases} R_{u_r u_r}(\tau) & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \tag{2.54}$$

$R_{u_r u_r}(\tau)$ is the autocorrelation function of the rth MOMRS element:

$$R_{u_r u_r}(\tau) = \frac{1}{2} \sum_{\Omega n \in \mathcal{N}^1_{r,p} \setminus \{0,\pi\}} A_n^2 \cos(\Omega n \tau) + R_{0,\pi}(\tau), \tag{2.55}$$

where:

$$R_{0,\pi}(\tau) = \begin{cases} A_0^2 \sin^2 \phi_0 + (-1)^\tau A^2_{\frac{N}{2}} \sin^2 \alpha & \text{if } (0 \in \mathcal{N}^1_{r,p}) \wedge (\pi \in \mathcal{N}^1_{r,p}) \\ A_0^2 \sin^2 \phi_0 & \text{if } (0 \in \mathcal{N}^1_{r,p}) \wedge (\pi \notin \mathcal{N}^1_{r,p}) \\ (-1)^\tau A^2_{\frac{N}{2}} \sin^2 \phi_{\frac{N}{2}} & \text{if } (0 \notin \mathcal{N}^1_{r,p}) \wedge (\pi \in \mathcal{N}^1_{r,p}) \\ 0 & \text{if } (0 \notin \mathcal{N}^1_{r,p}) \wedge (\pi \notin \mathcal{N}^1_{r,p}) \end{cases}. \tag{2.56}$$

3. Its variance matrix is $\sigma^2_{\mathbf{uu}} = diag\left[\sigma^2_{u_r u_r}\right]_{r=1,2,\ldots,p}$, where:

$$\sigma^2_{u_r u_r} = \sum_{\Omega n \in \mathcal{N}^1_{r,p} \setminus \{0,\pi\}} \frac{A_n^2}{2} + \sigma^2_\pi, \tag{2.57}$$

and:

$$\sigma^2_\pi = \begin{cases} A^2_{\frac{N}{2}} \sin^2 \alpha & \text{if } \pi \in \mathcal{N}^1_{r,p} \\ 0 & \text{if } \pi \notin \mathcal{N}^1_{r,p} \end{cases}. \tag{2.58}$$

Proof of the above lemma proceeds similarly as for Lemma 2.3 □

The same results of the time-domain averaging as presented in the above lemma may be obtained for any distributions of the random phase shifts ϕ_n assuming only that ϕ_0 is deterministic, and $\phi_{\frac{N}{2}}$ is deterministic or Bernoulli distributed $B\left(\frac{1}{2}, \{\alpha, \pi+\alpha\}\right)$.

Lemma 2.5 and Lemma 2.6 allow us to say that the extended MOMRS is a weakly ergodic multivariate orthogonal random process.

Frequencies distribution

In the sequel, it is assumed that all elements of the MOMRS have similar frequency contents. It is achieved by ordering consecutive frequencies circularly to consecutive elements of the MOMRS, i.e. the frequency Ωn is a member of the $\mathcal{N}^{c,1}_{r,p}$ when:

$$r = n \bmod p + 1. \tag{2.59}$$

Such ordering will be called consecutively circular ordering and denoted by the upper index c in symbols $\mathcal{N}^{c,1}_{r,p}$ $(r = 1, 2, \ldots, p)$ describing sets of frequencies.

If $\frac{N}{2p}$ is an integer number then the zero- and Nyquist- frequencies are elements of the set $\mathcal{N}^{c,1}_{1,p}$. This set consists of $n_1 = \frac{N}{2p} + 1$ elements:

$$\mathcal{N}^{c,1}_{1,p} = \left\{0, \Omega p, \ldots, \Omega\frac{N}{2}\right\}. \tag{2.60}$$

The remaining sets $\mathcal{N}^{c,1}_{r,p}$ $(r = 2, 3, \ldots, p)$ have $n_r = \frac{N}{2p}$ elements and the rth set is defined as:

$$\mathcal{N}^{c,1}_{r,p} = \left\{\Omega(r-1), \Omega(r-1+p), \ldots, \Omega(\frac{N}{2} - p + r - 1)\right\}. \tag{2.61}$$

For $p = 1$ the MOMRS with consecutive circularly ordered frequencies reduces to the SMRS. The set of its sine components frequencies is given by:

$$\mathcal{N}^{c,1}_{1,1} = \left\{0, \Omega, \ldots, \Omega\frac{N}{2}\right\}. \tag{2.62}$$

It should be noticed that constant bin spacing equal to Ω is kept throughout the relative frequency range $[0, 2\pi)$. This property allows us to synthesise a scalar multisine white noise for which whiteness holds for finite N-sample random process representations [32].

When $p = 2$, a bivariate orthogonal multisine random time-series (BOMRS) [66] is obtained. Elements of the basic BOMRS $\mathbf{u}^B(i) = \left[u^B_r(i)\right]_{r=1,2}$ have no common frequencies:

- the $u^B_1(i)$ time-series contains the constant component and sine components with frequencies from the set of even harmonics of Ω:

$$\mathcal{N}^{c,1}_{1,2} = \{0, 2\Omega, 4\Omega, \ldots, \pi\}. \tag{2.63}$$

  Its frequency bin is equal to 2Ω;

- the $u^B_2(i)$ time-series contains only sine components with frequencies from the set of odd harmonics of Ω:

$$\mathcal{N}^{c,1}_{2,2} = \left\{\Omega, 3\Omega, \ldots, (\frac{N}{2} - 1)\Omega\right\}. \tag{2.64}$$

  Its frequency bin is also equal to 2Ω.

The frequency-domain representation of each BOMRS element have the same frequency bin 2Ω throughout the range $[0, 2\pi)$. It implies that each element of the BOMRS is represented in the frequency range $[0, 2\pi)$ by $\frac{N}{2}$ relative frequencies. This property offers the possibility to synthesise a finite-sample bivariate orthogonal white multisine random time-series [66].

For the SMRS $(p = 1)$ and BOMRS $(p = 2)$ constant bin spacings were kept for adjacent frequencies below and above the Nyquist frequency $\Omega\frac{N}{2} = \pi$ and 2π frequency. It follows from definition (2.60) that for $p > 2$ a constant bin spacing equal to $p\Omega$ can be kept only for the first MOMRS element $u_1(i)$. For the remaining elements $u_r(i)$ $(r = 2, 3, \ldots, p)$ of the MOMRS sets of frequencies of its sine components in the range $[0, 2\pi)$ are given by:

$$\left\{\Omega(r-1), \Omega(r-1+p), \ldots, \Omega(\frac{N}{2} - p + r - 1), \Omega(\frac{N}{2} + p - r + 1), \ldots, \Omega(N - r + 1)\right\} \tag{2.65}$$

The distance between the first-above and last-below the Nyquist frequency is equal to:

$$\Delta^c_\pi(r, p) = \Omega(\frac{N}{2} + p - r + 1) - \Omega(\frac{N}{2} - p + r - 1) = [2(p - r) + 2]\,\Omega. \tag{2.66}$$

Values of the distance $\Delta^c_\pi(r, p)$ for different numbers p $(p \geq 2)$ of the MOMRS elements and elements $r = 2, 3, \ldots, p$ are presented in Tab. 2.1.

The corresponding distance between the first-above and last-below the frequency 2π is given by:

$$\Delta^c_{2\pi}(r, p) = 2(r - 1)\Omega. \tag{2.67}$$

It should be noticed that the distance $\Delta^c_{2\pi}(r, p)$ is invariant to the number p of MOMRS elements. Values of $\Delta^c_{2\pi}(r, p)$ calculated for different $p \geq 2$ and elements $r = 2, 3, \ldots, p$ are presented in Tab. 2.2.

It can be noticed from Tabs 2.1 and 2.2 that, for any even N, a constant frequency bin spacing equal to $p\Omega$ throughout the entire frequency range $[0, 2\pi)$ is kept for all MOMRS elements only for the case of $p = 1, 2$.

Let us assume that N is any odd number such that $\frac{N-1}{2p}$ is an integer number. The consecutive circular ordering of frequencies gives:

Table 2.1

	$\Delta^{e}_{\pi}(r,p)$				
p	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
2	2Ω	–	–	–	–
3	4Ω	2Ω	–	–	–
4	6Ω	4Ω	2Ω	–	–
5	8Ω	6Ω	4Ω	2Ω	–
6	10Ω	8Ω	6Ω	4Ω	2Ω

Values of the distance $\Delta^{e}_{\pi}(r,p)$ between the first-above and last-below Nyquist frequency for different numbers p of MOMRS elements and elements $r = 2,3,\ldots,p$ (N even)

Table 2.2

	$\Delta^{e}_{2\pi}(r,p)$				
p	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
2	2Ω	–	–	–	–
3	4Ω	4Ω	–	–	–
4	6Ω	6Ω	6Ω	–	–
5	8Ω	8Ω	8Ω	8Ω	–
6	10Ω	10Ω	10Ω	10Ω	10Ω

Values of the distance $\Delta^{e}_{2\pi}(r)$ between the first-above and last-below frequency 2π for different numbers p of MOMRS elements and elements $r = 2,3,\ldots,p$

- the set $\mathcal{N}^{c,1}_{1,p}$ as:

$$\mathcal{N}^{c,1}_{1,p} = \left\{0, p\Omega, \ldots, \frac{N-1}{2}\Omega\right\} \tag{2.68}$$

- and sets $\mathcal{N}^{c,1}_{r,p}$ for $r = 2,3,\ldots,p$ as:

$$\mathcal{N}^{c,1}_{r,p} = \left\{\Omega(r-1), \Omega(r+p-1), \ldots, \Omega(\frac{N-1}{2}-p+r-1)\right\}. \tag{2.69}$$

It follows from the definitions of the sets $\mathcal{N}^{c,1}_{r,p}$ that:

$$\Delta^{o}_{\pi}(1,p) = \Omega \tag{2.70}$$

and

$$\Delta^{o}_{\pi}(r,p) = [2(p-r)+3]\Omega. \tag{2.71}$$

for $r = 2,3,\ldots,p$. Values of the distance $\Delta^{o}_{\pi}(r,p)$ are presented in Tab. 2.3.

The distance $\Delta^{o}_{2\pi}(r,p)$ between the first-above and last-below the frequency 2π in the case of N odd can be calculated from the corresponding expression on $\Delta^{e}_{2\pi}(r,p)$ for N even.

It follows from Tab. 2.3 that for any N odd there exists a possibility to keep constant bin spacing equal to $p\Omega$ only for the SMRS ($p=1$). From the theoretical point of view, the case of N even is more interesting because it offers possibilities to synthesise scalar and bivariate white or pseudo-white multisine random time-series. In the sequel, we return to the assumption that N is any even number.

When $\frac{N}{2p}$ is not an integer number, there is no possibility to keep constant bin spacing for all elements of the MOMRS, because its elements $u_r(i)$ ($r = 1,2,\ldots,p$) have different numbers n_r of sine components. For large N ($N \gg p$) the number n_r for all MOMRS elements can be approximated by $\frac{N}{2p}$.

Table 2.3

	$\Delta^{o}_{\pi}(r,p)$				
p	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
2	3Ω	–	–	–	–
3	5Ω	3Ω	–	–	–
4	7Ω	5Ω	3Ω	–	–
5	9Ω	7Ω	5Ω	3Ω	–
6	11Ω	9Ω	7Ω	5Ω	3Ω

Values of the distance $\Delta^{o}_{\pi}(r,p)$ between the first-above and last-below Nyquist frequency for different numbers of MOMRS elements p and $r = 2,3,\ldots,p$ (N odd)

2.2.2 Nonergodic Case

Definitions

Consider a multivariate random time-series $\mathbf{u}(i)$ with the elements $u_r(i)$ ($r = 1,2,\ldots,p$ and $p > 1$) being scalar multisine random time-series for which the same relative frequency appears in all elements of the multivariate time-series. This implies nonergodicity of the multivariate time-series. Assuming additionally that constant components of all elements of the multivariate time-series are equal to 0 then thus obtained multivariate random time-series is orthogonal one taking into account ensemble averaging. This determines that in the sequel these time-series are called nonergodic multivariate orthogonal multisine random time-series (NMOMRS). The NMOMRS is defined in the time-domain by:

Definition 2.5 *The basic N-sample NMOMRS $\mathbf{u}^B(i)$ is defined in the time-domain by the p-dimensional multivariate time-series $\mathbf{u}^B(i) = \left[u^B_r(i)\right]_{r=1,2,\ldots,p}$, with the rth element given by:*

$$u^B_r(i) = \sum_{n=0}^{\frac{N}{2}} A_{r,n}\sin(\Omega ni + \phi_{r,n}), \tag{2.72}$$

where $\Omega = \frac{2\pi}{N}$ denotes the fundamental relative frequency, $n = 0,1,\ldots,\frac{N}{2}$ denotes consecutive harmonics of this frequency in the range $[0,\pi]$, $i = 0,1,\ldots,N-1$ denotes consecutive discrete time instants, $A_{r,n}$ are deterministic amplitudes of the sine components ($A_{r,n} \in \mathcal{R}$), $\phi_{r,n}$ are phase shifts, of which $\phi_{r,0}$ are deterministic and the remaining phase shifts are random, independent and:

- *uniformly distributed on $[0,2\pi)$ for $n = 1,2,\ldots,\frac{N}{2}-1$ and $r = 1,2,\ldots,p$,*
- *Bernoulli distributed $\mathcal{B}\left(\frac{1}{2},\{\alpha,\pi+\alpha\}\right)$ for $n = \frac{N}{2}$ and $r = 1,2,\ldots,p$.*

□

For $p = 1$ the NMOMRS becomes the weakly ergodic SMRS.

In the frequency-domain, the basic N-sample NMOMRS is given by the p-dimensional vector $\mathbf{U}^B(j\Omega m) = \left[U^B_r(j\Omega m)\right]_{r=1,2,\ldots,p}$ of finite discrete Fourier transforms with the rth element given by:

$$U^B_r(j\Omega m) = \frac{N}{2j}\sum_{n=0}^{\frac{N}{2}} A_{r,n}\left[e^{j\phi_{r,n}}\delta(m-n) - e^{-j\phi_{r,n}}\delta(m-(N-n))\right], \tag{2.73}$$

where $m = 0,1,\ldots,N-1$.

Elements of the basic NMOMRS inherit properties of the scalar multisine random time-series.

Similarly as the previous scalar and multivariate orthogonal multisine random time-series, the extended NMOMRS is periodic modulo N. The elements of the NMOMRS have common frequencies but the independence of its sine components random phase shifts under the assumption that all constant components are equal to 0 implies orthogonality of the elements for the ensemble averaging:

$$\mathcal{E}\{u_r(i)u_s(i)\} = 0, \tag{2.74}$$

where $r \neq s$ and $r, s = 0, 1, \ldots, p$.

Properties

The periodogram matrix of the basic N-sample NMOMRS is given by the lemma:

Lemma 2.7 *Consider the basic N-sample NMOMRS. Its periodogram matrix is $\Phi^B_{\mathbf{uu}}(j\Omega m) = \left[\Phi^B_{u_r u_s}(j\Omega m)\right]_{r,s=1,2,\ldots,p}$, where for $m = 0, 1, \ldots, N-1$:*

$$\Phi^B_{u_r u_s}(j\Omega m) = \frac{TN}{4}\left\{(4A_{r,0}A_{s,0}\sin\phi_{r,0}\sin\phi_{s,0} + j0)\delta(m) + \sum_{n=1}^{\frac{N}{2}-1} A_{r,n}A_{s,n}\left[(\cos(\phi_{r,n}-\phi_{s,n})\right.\right.$$
$$-j\sin(\phi_{r,n}-\phi_{s,n}))\delta(m-n) + (\cos(\phi_{r,n}-\phi_{s,n}) + j\sin(\phi_{r,n}-\phi_{s,n}))\,\delta(m-(N-n))]$$
$$\left. +(4A_{r,\frac{N}{2}}A_{s,\frac{N}{2}}\sin\phi_{r,\frac{N}{2}}\sin\phi_{s,\frac{N}{2}} + j0)\delta(m-\frac{N}{2})\right\}. \tag{2.75}$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.4.

It is worth to note that the diagonal elements of the NMOMRS periodogram matrix are deterministic functions of the frequency Ωm, which are invariant to the choice of random phase shifts while its off-diagonal elements are functions of the random phase shifts $\phi_{r,n}$. It follows from Lemma 2.7 that the expected value of $\Phi^B_{\mathbf{uu}}(j\Omega m)$ is the matrix $\mathcal{E}\{\Phi^B_{\mathbf{uu}}(j\Omega m)\} = \left[\mathcal{E}\{\Phi^B_{u_r u_s}(j\Omega m)\}\right]_{r,s=1,2,\ldots,p}$, where:

- its diagonal elements are:

$$\mathcal{E}\{\Phi^B_{u_r u_r}(j\Omega m)\} = \frac{NT}{4}\left\{(4A^2_{r,0}\sin^2\phi_{r,0} + j0)\delta(m) + \sum_{n=1}^{\frac{N}{2}-1}(A^2_{r,n} + j0)\left[\delta(m-n)\right.\right.$$
$$\left. +\delta(m-(N-n))] + (4A^2_{r,\frac{N}{2}}\sin^2\alpha + j0)\delta(m-\frac{N}{2})\right\}, \tag{2.76}$$

for $r = 1, 2, \ldots, p$;

- its off-diagonal elements are

$$\mathcal{E}\{\Phi^B_{u_r u_s}(j\Omega m)\} = \frac{NT}{4}\left\{(4A_{r,0}4A_{s,0}\sin\phi_{r,0}\sin\phi_{s,0} + j0)\delta(m) + \sum_{n=1}^{\frac{N}{2}-1}(0 + j0)\left[\delta(m-n)\right.\right.$$
$$\left. +\delta(m-(N-n))] + (0 + j0)\delta(m-\frac{N}{2})\right\}, \tag{2.77}$$

for $r, s = 1, 2, \ldots, p$ and $r \neq s$.

If $A_{r,0} = 0$ or $\phi_{r,0} = 0$ $(r = 1, 2, \ldots, p)$ then $\mathcal{E}\{\Phi^B_{\mathbf{uu}}(j\Omega m)\}$ is:

- a diagonal matrix for all frequencies $\Omega n \in \left\{\Omega, 2\Omega, \ldots, \frac{N}{2}\Omega\right\}$ but
- the zero matrix for the constant component $\Omega n = 0$.

The properties of NMOMRS resulting from the ensemble averaging are given by the lemma:

Lemma 2.8 *Consider the extended NMOMRS. For each time instant $i = 0, 1, \ldots, \infty$:*

1. *its expected value vector is $\mathcal{E}\{\mathbf{u}(i)\} = [\mathcal{E}\{u_r(i)\}]_{r=1,2,\ldots,p}$, where:*

$$\mathcal{E}\{u_r(i)\} = A_{r,0}\sin\phi_{r,0}. \tag{2.78}$$

2. *its correlation function matrix is $\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\} = [\mathcal{E}\{u_r(i)u_s(i-\tau)\}]_{r,s=1,2,\ldots,p}$, where for $\tau = 0, 1, \ldots, \infty$:*

$$\mathcal{E}\{u_r(i)u_s(i-\tau)\} = \begin{cases} \mathcal{E}\{u_r(i)u_r(i-\tau)\} & \text{if } r = s \\ A_{r,0}A_{s,0}\sin\phi_{r,0}\sin\phi_{s,0} & \text{if } r \neq s \end{cases} \tag{2.79}$$

$\mathcal{E}\{u_r(i)u_r(i-\tau)\}$ is the autocorrelation function of the rth NMOMRS element:

$$\mathcal{E}\{u_r(i)u_r(i-\tau)\} = A^2_{r,0}\sin^2\phi_{r,0} + \frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1}A^2_{r,n}\cos(\Omega n\tau) + (-1)^\tau A^2_{r,\frac{N}{2}}\sin^2\alpha. \tag{2.80}$$

3. *its variance matrix is $\mathcal{E}\{(\mathbf{u}(i)-\mathcal{E}\{\mathbf{u}(i)\})(\mathbf{u}(i)-\mathcal{E}\{\mathbf{u}(i)\})^T\} = diag\,[\rho_{u_r u_r}]_{r=1,2,\ldots,p}$, where:*

$$\rho_{u_r u_r} = \mathcal{E}\{(u_r(i) - \mathcal{E}\{u_r(i)\})^2\} = \frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1}A^2_{r,n} + A^2_{r,\frac{N}{2}}\sin^2\alpha. \tag{2.81}$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.2.

It follows from this lemma that the extended NMOMRS is a wide-sense stationary multivariate random process.

When the time-domain averaging on any particular extended NMOMRS is analysed, the following lemma can be formulated:

Lemma 2.9 *Consider the extended NMOMRS.*

1. *Its mean value vector is $\mathcal{M}\{\mathbf{u}(i)\} = [\mathcal{M}\{u_r(i)\}]_{r=1,2,\ldots,p}$, where:*

$$\mathcal{M}\{u_r(i)\} = A_{r,0}\sin\phi_{r,0}. \tag{2.82}$$

2. *Its correlation function matrix is $\mathbf{R}_{\mathbf{uu}}(\tau) = [R_{u_r u_s}(\tau)]_{r,s=1,2,\ldots,p}$, where for $\tau = 0, 1, \ldots, \infty$:*

$$R_{u_r u_s}(\tau) = A_{r,0}A_{s,0}\sin\phi_{r,0}\sin\phi_{s,0}$$
$$+\frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1}A_{r,n}A_{s,n}\cos(\Omega n\tau + \phi_{r,n} - \phi_{s,n}) + (-1)^\tau A_{r,\frac{N}{2}}A_{s,\frac{N}{2}}\sin\phi_{r,\frac{N}{2}}\sin\phi_{s,\frac{N}{2}}. \tag{2.83}$$

3. *Its variance matrix is $\sigma^2_{\mathbf{uu}} = [\sigma^2_{u_r u_s}]_{r,s=1,2,\ldots,p}$, where:*

$$\sigma^2_{u_r u_s} = \frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1}A_{r,n}A_{s,n}\cos(\phi_{r,n} - \phi_{s,n}) + A_{r,\frac{N}{2}}A_{s,\frac{N}{2}}\sin\phi_{r,\frac{N}{2}}\sin\phi_{s,\frac{N}{2}}. \tag{2.84}$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.3.

$\mathbf{R_{uu}}(\tau)$ is a periodic function with the period N. Its diagonal elements are deterministic functions, which are invariant to the choice of random phase shifts but its off-diagonal elements are random phase shifts dependent. Comparison of Lemma 2.8 and 2.9 allows us to say that the extended NMOMRS is a nonergodic multivariate random process. This nonergodicity is introduced by identical sets of sine components frequencies present in the NMOMRS elements.

The expected value of $\mathbf{R_{uu}}(\tau)$ is the matrix $\mathcal{E}\{\mathbf{R_{uu}}(\tau)\} = [\mathcal{E}\{R_{u_ru_s}(\tau)\}]_{r,s=1,2,\ldots,p}$, where:

- its diagonal elements are:

$$\mathcal{E}\{R_{u_ru_r}(\tau)\} = A_{r,0}^2\sin^2\phi_{r,0} + \frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1} A_{r,n}^2\cos(\Omega n\tau) + (-1)^{\tau}A^2_{r,\frac{N}{2}}\sin\phi_{r,\frac{N}{2}}\sin\phi_{s,\frac{N}{2}}. \tag{2.85}$$

for $r = 1, 2, \ldots, p$;

- its off-diagonal elements are:

$$\mathcal{E}\{R_{u_ru_s}(\tau)\} = A_{r,0}A_{s,0}\sin\phi_{r,0}\sin\phi_{s,0} \tag{2.86}$$

for $r, s = 1, 2, \ldots, p$ and $r \neq s$.

It is worth to note that $\mathcal{E}\{\mathbf{R_{uu}}(\tau)\} = \mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\}$. Additionally, if $A_{r,0} = 0$ or $\phi_{r,0} = 0$ $(r = 1, 2, \ldots, p)$ then $\mathcal{E}\{\mathbf{R_{uu}}(\tau)\}$ and $\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\}$ are diagonal matrices.

2.3 MULTIVARIATE NONORTHOGONAL MULTISINE RANDOM TIME-SERIES

Definitions

Let any nonergodic multivariate orthogonal random time-series be used as an excitation of a multi-input linear, discrete-time system with a transfer function matrix which off-diagonal elements are not all equal to $0 + j0$. When the system reaches steady-state conditions, its multivariate response vector is a nonergodic multivariate nonorthogonal multisine random time-series (NMMRS). The NMMRS is defined in the time-domain by:

Definition 2.6 *The basic N-sample NMMRS is defined in the time-domain by the p-dimensional multivariate time-series $\mathbf{u}^B(i) = \left[u_r^B(i)\right]_{r=1,2,\ldots,p}$ with the rth element given by:*

$$u_r^B(i) = \sum_{t=1}^{p}\sum_{n=0}^{\frac{N}{2}} A_{r,t,n}\sin(\Omega ni + \phi_{t,n} + \varphi_{r,t,n}), \tag{2.87}$$

where $\Omega = \frac{2\pi}{N}$ denotes the fundamental relative frequency, $n = 0, 1, \ldots, \frac{N}{2}$ denotes consecutive harmonics of this frequency in the range $[0, \pi]$, $i = 0, 1, \ldots, N-1$ denotes consecutive discrete time instants, $A_{r,t,n}$ are deterministic amplitudes of the sine components $(A_{r,t,n} \in \mathcal{R})$, $\phi_{t,n}$ and $\varphi_{r,t,n}$ are phase shifts, of which $\phi_{t,0}$ and $\varphi_{r,t,n}$ are deterministic and the remaining phase shifts $\phi_{t,n}$ are random, independent and:

- *uniformly distributed on $[0, 2\pi)$ for $n = 1, 2, \ldots, \frac{N}{2} - 1$ and $t = 1, 2, \ldots, p$,*
- *Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$ for $n = \frac{N}{2}$ and $t = 1, 2, \ldots, p$.*

□

The basic N-sample NMMRS is defined in the frequency-domain for the (relative) frequency range $[0, 2\pi)$ by the p-dimensional vector $\mathbf{U}^B(j\Omega m) = \left[U_r^B(j\Omega m)\right]_{r=1,2,\ldots,p}$ of finite discrete Fourier transforms with rth element given by:

$$U_r^B(j\Omega m) = \frac{N}{2j}\sum_{t=1}^{p}\sum_{n=0}^{\frac{N}{2}} A_{r,t,n}\left[e^{j(\phi_{t,n}+\varphi_{r,t,n})}\delta(m-n) - e^{-j(\phi_{t,n}+\varphi_{r,t,n})}\delta(m-(N-n))\right], \tag{2.88}$$

where $m = 0, 1, \ldots, N-1$.

Properties

The periodogram matrix of the basic N-sample NMMRS is given by the lemma:

Lemma 2.10 *Consider the basic N-sample NMMRS. Its periodogram matrix is $\mathbf{\Phi^B_{uu}}(j\Omega m) = \left[\Phi^B_{u_ru_s}(j\Omega m)\right]_{r,s=1,2,\ldots,p}$, where for $m = 0, 1, \ldots, N-1$:*

$$\begin{aligned}\Phi^B_{u_ru_s}(j\Omega m) = \frac{NT}{4}\sum_{t=1}^{p}\sum_{\mu=1}^{p}\Bigg\{&(4A_{r,t,0}A_{s,\mu,0}\sin(\phi_{t,0}+\varphi_{r,t,0})\sin(\phi_{\mu,0}+\varphi_{s,\mu,0}) + j0)\delta(m)\\ &+ \sum_{n=1}^{\frac{N}{2}-1} A_{r,t,n}A_{s,\mu,n}\Big[e^{j(\varphi_{r,t,n}-\varphi_{s,\mu,n})}(\cos(\phi_{t,n}-\phi_{\mu,n}) - j\sin(\phi_{t,n}-\phi_{\mu,n}))\delta(m-n)\\ &+ e^{-j(\varphi_{r,t,n}-\varphi_{s,\mu,n})}(\cos(\phi_{t,n}-\phi_{\mu,n}) + j\sin(\phi_{t,n}-\phi_{\mu,n}))\delta(m-(N-n))\Big]\\ &+ (4A_{r,t,\frac{N}{2}}A_{s,\mu,\frac{N}{2}}\sin(\alpha+\varphi_{r,t,\frac{N}{2}})\sin(\alpha+\varphi_{s,\mu,\frac{N}{2}}) + j0)\delta(m-\frac{N}{2})\Bigg\}.\end{aligned} \tag{2.89}$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.4.

It follows from the above lemma that the periodogram matrix $\mathbf{\Phi^B_{uu}}(j\Omega m)$ can be written as:

$$\mathbf{\Phi^B_{uu}}(j\Omega m) = \mathbf{K}(j\Omega m)\mathbf{\Phi}^B_{\beta\beta}(j\Omega m)\mathbf{K}^*(j\Omega m), \tag{2.90}$$

where:

- elements of the matrix $\mathbf{K}(j\Omega m) = [K_{u_ru_s}(j\Omega m)]_{r,s=1,2,\ldots,p}$ are given by:

$$K_{u_ru_s}(j\Omega m) = \begin{cases}\sqrt{NT}A_{r,s,0}\sin(\phi_{t,0}+\varphi_{r,s,0}) + j0 & \text{if } m = 0\\ \frac{\sqrt{NT}}{2}A_{r,s,m}e^{j\varphi_{r,s,m}} & \text{if } m = 1, 2, \ldots, \frac{N}{2}-1\\ \sqrt{NT}A_{r,s,\frac{N}{2}}\sin(\alpha+\varphi_{r,s,\frac{N}{2}}) + j0 & \text{if } m = \frac{N}{2}\end{cases}. \tag{2.91}$$

For harmonic frequencies from the range $(\pi, 2\pi)$, the following condition

$$K_{u_ru_s}(j(2\pi - \Omega m)) = K^*_{u_ru_s}(j\Omega m) \tag{2.92}$$

is satisfied;

- elements of the matrix $\mathbf{\Phi}^B_{\beta\beta}(j\Omega m) = \left[\Phi^B_{\beta_r\beta_s}(j\Omega m)\right]_{r,s=1,2,\ldots,p}$ are given by:

$$\Phi^B_{\beta_r\beta_s}(j\Omega m) = (1+j0)\delta(m) + \sum_{n=1}^{\frac{N}{2}-1}[(\cos(\phi_{r,n}-\phi_{s,n}) - j\sin(\phi_{r,n}-\phi_{s,n}))\delta(m-n)$$

$$+ (\cos(\phi_{r,n} - \phi_{s,n}) + j\sin(\phi_{r,n} - \phi_{s,n}))\,\delta(m - (N-n))] + (1 + j0)\delta(m - \frac{N}{2}). \quad (2.93)$$

The matrix $\Phi^B_{\beta\beta}(j\Omega m)$ is the periodogram matrix of a NMOMRS with amplitudes of its sine components chosen so that $\mathcal{E}\{\Phi^B_{\beta\beta}(j\Omega m)\} = I$.

The above spectral factorisation of the NMMRS periodogram matrix allows us to write the finite discrete Fourier transform $\mathbf{U}^B(j\Omega m)$ of NMMRS as:

$$\mathbf{U}^B(j\Omega m) = \mathbf{K}(j\Omega m)\beta(j\Omega m), \quad (2.94)$$

where $\beta(j\Omega m)$ is the finite discrete Fourier transform of a NMOMRS with $\mathcal{E}\{\Phi^B_{\beta\beta}(j\omega T)\} = I$.

Elements of the NMMRS periodogram matrix $\Phi^B_{\mathbf{uu}}(j\Omega m)$ are random phase shifts dependent. The expected value of $\Phi^B_{\mathbf{uu}}(j\Omega m)$ is the matrix $\mathcal{E}\{\Phi^B_{\mathbf{uu}}(j\Omega m)\} = \left[\mathcal{E}\{\Phi^B_{u_r u_s}(j\Omega m)\}\right]_{r,s=1,2,\ldots,p}$, where for $r, s = 1, 2, \ldots, p$:

$$\mathcal{E}\{\Phi^B_{u_r u_s}(j\Omega m)\} = \frac{NT}{4}\sum_{t=1}^{p}\Bigg\{(4A_{r,t,0}A_{s,t,0}\sin(\phi_{t,0}+\varphi_{r,t,0})\sin(\phi_{t,0}+\varphi_{s,t,0})+j0)\delta(m)$$

$$+\sum_{n=1}^{\frac{N}{2}-1} A_{r,t,n}A_{s,t,n}\left[e^{j(\varphi_{r,t,n}-\varphi_{s,t,n})}\delta(m-n) + e^{-j(\varphi_{r,t,n}-\varphi_{s,t,n})}\delta(m-(N-n))\right]$$

$$+(4A_{r,t,\frac{N}{2}}A_{s,t,\frac{N}{2}}\sin(\alpha+\varphi_{r,t,\frac{N}{2}})\sin(\alpha+\varphi_{s,t,\frac{N}{2}}) + j0)\delta(m - \frac{N}{2})\Bigg\}. \quad (2.95)$$

It is obvious that:

$$\mathcal{E}\{\Phi^B_{\mathbf{uu}}(j\Omega m)\} = \mathbf{K}(j\Omega m)\mathbf{K}^*(j\Omega m). \quad (2.96)$$

The properties of NMMRS, which follow from the ensemble averaging, are given by:

Lemma 2.11 *Consider the extended NMMRS. For each time instant $i = 0, 1, \ldots, \infty$:*

1. *its expected value vector is $\mathcal{E}\{\mathbf{u}(i)\} = [\mathcal{E}\{u_r(i)\}]_{r=1,2,\ldots,p}$, where for $r = 1, 2, \ldots, p$:*

$$\mathcal{E}\{u_r(i)\} = \sum_{t=1}^{p} A_{r,t,0}\sin(\phi_{t,0} + \varphi_{r,t,0}); \quad (2.97)$$

2. *its correlation function matrix is $\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\} = [\mathcal{E}\{u_r(i)u_s(i-\tau)\}]_{r,s=1,2,\ldots,p}$, where for $\tau = 0, 1, \ldots, \infty$:*

$$\mathcal{E}\{u_r(i)u_s(i-\tau)\} = \sum_{t=1}^{p}\Bigg[A_{r,t,0}A_{s,t,0}\sin(\phi_{r,0}+\varphi_{r,t,0})\sin(\phi_{s,0}+\varphi_{s,t,0})$$

$$+\frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1} A_{r,t,n}A_{s,t,n}\cos(\Omega n\tau + \varphi_{r,t,n} - \varphi_{s,t,n})$$

$$+(-1)^{\tau}A_{r,t,\frac{N}{2}}A_{s,t,\frac{N}{2}}\sin(\alpha+\varphi_{r,t,\frac{N}{2}})\sin(\alpha+\varphi_{s,t,\frac{N}{2}})\Bigg]. \quad (2.98)$$

3. *its variance matrix is $\mathcal{E}\{(\mathbf{u}(i)-\mathcal{E}\{\mathbf{u}(i)\})(\mathbf{u}(i)-\mathcal{E}\{\mathbf{u}(i)\})^T\} = [\rho_{u_r u_s}]_{r,s=1,2,\ldots,p}$, where:*

$$\rho_{u_r u_s} = \sum_{t=1}^{p}\Bigg[\frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1} A_{r,t,n}A_{s,t,n}\cos(\varphi_{r,t,n} - \varphi_{s,t,n})$$

$$+ A_{r,t,\frac{N}{2}}A_{s,t,\frac{N}{2}}\sin(\alpha+\varphi_{r,t,\frac{N}{2}})\sin(\alpha+\varphi_{s,t,\frac{N}{2}})\Bigg]. \quad (2.99)$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.2.

It follows from this lemma that the extended NMMRS is a wide-sense stationary multivariate random process.

When the time-domain averaging on any particular extended NMMRS is analysed, the following lemma can be formulated:

Lemma 2.12 *Consider the extended NMMRS.*

1. *Its mean value vector is $\mathcal{M}\{\mathbf{u}(i)\} = [\mathcal{M}\{u_r(i)\}]_{r=1,2,\ldots,p}$, where:*

$$\mathcal{M}\{u_r(i)\} = \sum_{t=1}^{p} A_{r,t,0}\sin(\phi_{t,0} + \varphi_{r,t,0}). \quad (2.100)$$

2. *Its correlation function matrix is $\mathbf{R}_{\mathbf{uu}}(\tau_\nu) = [R_{u_r u_s}(\tau_\nu)]_{r,s=1,2,\ldots,p}$, where for $\tau = 0, 1, \ldots, \infty$:*

$$R_{u_r u_s}(\tau) = \sum_{t=1}^{p}\sum_{\mu=1}^{p}\Bigg[A_{r,t,0}A_{s,\mu,0}\sin(\phi_{t,0}+\varphi_{r,t,0})\sin(\phi_{t,0}+\varphi_{s,\mu,0})$$

$$+\frac{1}{2}\sum_{n=1}^{\frac{N}{2}-1} A_{r,t,n}A_{s,\mu,n}\cos(\Omega n\tau + \phi_{t,n} - \phi_{\mu,n} + \varphi_{r,t,n} - \varphi_{s,\mu,n})$$

$$+(-1)^{\tau}A_{r,t,\frac{N}{2}}A_{s,\mu,\frac{N}{2}}\sin(\alpha+\varphi_{r,t,\frac{N}{2}})\sin(\alpha+\varphi_{s,\mu,\frac{N}{2}})\Bigg]. \quad (2.101)$$

3. *Its variance matrix is $\sigma^2_{\mathbf{uu}} = [\sigma^2_{u_r u_s}]_{r,s=1,2,\ldots,p}$, where:*

$$\sigma^2_{u_r u_s} = \frac{1}{2}\sum_{t=1}^{p}\sum_{\mu=1}^{p}\Bigg[\sum_{n=1}^{\frac{N}{2}-1} A_{r,t,n}A_{s,\mu,n}\cos(\phi_{t,n} - \phi_{\mu,n} + \varphi_{r,t,n} - \varphi_{s,\mu,n})$$

$$+2A_{r,t,\frac{N}{2}}A_{s,\mu,\frac{N}{2}}\sin(\alpha+\varphi_{r,t,\frac{N}{2}})\sin(\alpha+\varphi_{s,\mu,\frac{N}{2}})\Bigg]. \quad (2.102)$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.3.

$\mathbf{R}_{\mathbf{uu}}(\tau)$ is a periodic function with the period $\frac{N}{2}$. Its values are random phase shifts dependent. Comparison of Lemma 2.11 and 2.12 allows us to say that the extended NMMRS is a nonergodic multivariate random process.

The expected value of $\mathbf{R_{uu}}(\tau)$ is the matrix: $\mathcal{E}\{\mathbf{R_{uu}}(\tau)\} = [\mathcal{E}\{R_{u_r u_s}(\tau)\}]_{r,s=1,2,\ldots,p}$, where:

$$\mathcal{E}\{R_{u_r u_s}(\tau)\} = \frac{1}{2}\sum_{t=1}^{p}\Bigg[2A_{r,t,0}A_{s,t,0}\sin(\phi_{t,0}+\varphi_{r,t,0})\sin(\phi_{t,0}+\varphi_{s,\mu,0})$$

$$+\sum_{n=1}^{\frac{N}{2}-1} A_{r,t,n}A_{s,t,n}\cos(\Omega n\tau+\varphi_{r,t,n}-\varphi_{s,t,n})+(-1)^{\tau}2A_{r,t,\frac{N}{2}}A_{s,t,\frac{N}{2}}\sin(\alpha+\varphi_{r,t,\frac{N}{2}})\sin(\alpha+\varphi_{s,\mu,\frac{N}{2}})\Bigg]. \tag{2.103}$$

2.4 SYNTHESIS AND SIMULATION

A single N-sample realisation of the random sine sequence

$$u_n^B(i) = A_n \sin(\omega_n Ti + \phi_n) \tag{2.104}$$

for the time instants $i = 0, 1, \ldots, N-1$, any frequency $\omega_n T$ from the set $\left\{\Omega n;\ n = 1, 2, \ldots, \frac{N}{2}\right\}$ and a realisation of the random phase shift ϕ_n can obviously be numerically calculated by using the time-domain definition (2.104). This N-sample sequence could also be calculated by transforming the corresponding realisation of its frequency-domain representation back into the time-domain by the inverse finite discrete Fourier transform. For a single sine this approach seems artificial. Things change however if realisations of a multisine random time-series consisting of a sum of hundreds or thousands sine components should be obtained. To calculate the multisine random time-series realisation for large values of N, a good starting point is offered by its frequency-domain representation. This approach gains a lot from the numerical efficiency of Fast Fourier Transform algorithms [72].

For a given set of multisine random time-series amplitudes, phase shifts for constant components, and parameters of Nyquist frequency phase shifts distributions, the procedure of simulating the basic N-sample multisine random time-series consists of two steps:

- step 1: synthesis of the corresponding multisine random time-series spectrum;
- step 2: transformation of the synthesised spectrum back into the time-domain by the inverse finite discrete Fourier transform. It results in the basic N-sample multisine random time-series.

For example, if the amplitudes $\left\{A_0, A_1, \ldots, A_{\frac{N}{2}}\right\}$ and two phase shifts $\{\phi_0, \alpha\}$ for a SMRS are given, the basic N-sample SMRS $u^B(i)$ may be simulated as the inverse discrete Fourier transform of the corresponding synthesised spectrum $U^N(j\Omega m)$:

- for $m = 0$:

$$U^N(j0) = NA_0 \sin\phi_0 + j0; \tag{2.105}$$

- for $m = 1, 2, \ldots, \frac{N}{2} - 1$:

$$Re\left\{U^N(j\Omega m)\right\} = \frac{N}{2}A_m \sin\phi_m, \tag{2.106}$$

$$Im\left\{U^N(j\Omega m)\right\} = -\frac{N}{2}A_m \cos\phi_m, \tag{2.107}$$

  where ϕ_m are random, independent and uniformly distributed on $[0, 2\pi)$;

- for $m = \frac{N}{2}$:

$$U^N(j\pi) = NA_{\frac{N}{2}} \sin\phi_{\frac{N}{2}} + j0, \tag{2.108}$$

  where $\phi_{\frac{N}{2}}$ is random, independent and Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi+\alpha\}\right)$;

- for $N - m = N-1, N-2, \ldots, N - (\frac{N}{2} - 1)$:

$$U^N(j\Omega(N-m)) = Re\left\{U^N(j\Omega m)\right\} - jIm\left\{U^N(j\Omega m)\right\}; \tag{2.109}$$

Synthesis and simulation of basic MOMRS's, NMOMRS's and NMMRS's can be performed in the same way as the synthesis and simulation of scalar multisine random time-series.

Sets of multisine random time-series amplitudes, phase shifts for constant components, and the parameter of Nyquist frequency components distribution are important degrees of freedom for different multisine random time-series synthesis and simulation. Their choice allows us to control the expected value vector of the extended multisine random time-series. Additionally, in spite of random phase shifts, the periodogram and correlation function matrices for weakly ergodic multisine random time-series or expected values of periodogram and correlation function matrices for nonergodic multisine random time-series are deterministic, real-valued functions. They are uniquely defined by the sets of multisine random time-series amplitudes, phase shifts for constant components, and the parameter of Nyquist frequency components distribution. It implies that the multisine periodogram (or expected value of the periodogram) matrix elements can be fitted to the corresponding power spectral density function matrix elements of a wide-sense stationary multivariate random process. This fitting is behind the proposed synthesis and simulation method [24] of wide-sense stationary multivariate random processes defined by their power spectral densities given by nonparametric representations, e.g. as diagrams or table, where:

- *synthesis* means the determination of the spectrum of a multisine random time-series based on the corresponding power spectral density of a wide-sense stationary random process to be simulated,
- *simulation* means the generation the corresponding multisine random process approximation by performing the inverse finite discrete Fourier transform of the synthesised spectrum.

Sample realisations of the synthesised and simulated multisine random process approximation may be obtained by replacing the sequence of random phase shifts by their respective realisations. The numerical complexity of generating the sample realisations can be reduced by using the FFT algorithms.

Chapter 3

Power Spectral Density Defined Multisine Random Processes

In this chapter, the synthesis of multisine random time-series defined by power spectral densities of wide-sense stationary random processes and their simulation with the inverse finite discrete Fourier transform is described. Statistical properties of obtained multisine random process approximations are established. Asymptotic Gaussianess and ergodicity of the synthesised time-series are discussed.

This chapter is finished with an extension of the proposed random process synthesis and simulation method to generation of wide-sense stationary continuous-time band-limited random processes, given also by their power spectral densities.

3.1 SYNTHESIS

In the sequel, it is assumed that:

- $\mathbf{v}(i)$ is a wide-sense stationary, real-valued multivariate (orthogonal or nonorthogonal) random process given by the power spectral density matrix $\mathbf{\Phi}_{\mathbf{vv}}(j\omega T) = [\Phi_{v_r v_s}(j\Omega m)]_{r,s=1,2,\ldots,p}$, which satisfies, for $\omega T \in [0, 2\pi)$, the following conditions:

$$\mathbf{\Phi}_{\mathbf{vv}}(j\omega T) = \mathbf{\Phi}_{\mathbf{vv}}(j(2\pi - \omega T)) \tag{3.1}$$

and:

$$\|\mathbf{\Phi}_{\mathbf{vv}}(j\omega T)\| < \infty, \tag{3.2}$$

where:

$$\|\mathbf{\Phi}_{\mathbf{vv}}(j\omega T)\| = \sqrt{\sum_{r=1}^{p}\sum_{s=1}^{p} |\Phi_{v_r v_s}(j\Omega m)|^2}; \tag{3.3}$$

- the autocorrelation function $\mathbf{R}_{\mathbf{vv}}(\tau)$ of $\mathbf{v}(i)$ for lags $|\tau| > \tau_0$ satisfies the condition:

$$\mathbf{R}_{\mathbf{vv}}(\tau) = \mathbf{o}. \tag{3.4}$$

The power spectral density matrix $\mathbf{\Phi}_{\mathbf{vv}}(j\omega T)$ of a multivariate wide-sense stationary random process with finite powers of its elements may be approximated by the periodogram of a multivariate multisine random time-series with amplitudes of its sine components chosen so as to make values of the periodogram matrix (or expected value of the periodogram matrix) equal to the corresponding values of power spectral density matrix of the original random process for some equally spaced frequencies (being approximation nodes) from the range $[0, 2\pi)$. This approximation criterion allows us to synthesise the spectrum $\mathbf{U}^B(j\Omega m)$ of the multisine random time-series. The corresponding time-series is simulated by the inverse finite discrete Fourier transform of the synthesised spectrum $\mathbf{U}^B(j\Omega m)$. It is worth

to note that the above approximation criterion can be interpreted as sampling of the power spectral density matrix in the frequency domain.

The power spectral density matrix $\mathbf{\Phi}_{\mathbf{vv}}(j\omega T)$ is approximated by a periodogram matrix of the corresponding multisine random time-series - it means that the power spectral densities $\Phi_{v_r v_s}(j\omega T)$ $(r, s = 1, 2, \ldots, p)$ are sampled in the frequency-domain choosing for each, n' sample points along the ωT axis at relative, equidistant frequencies from the range $[0, 2\pi)$. It does not produce aliasing [47] if the spacing Δ between the samples along the frequency axis is such that:

$$\Delta \leq \Delta_0 = \frac{2\pi}{2\tau_0}. \tag{3.5}$$

In this case the original power spectral densities $\Phi_{v_r v_s}(j\omega T)$ may be reconstructed from their sampled values (periodograms of approximating multisine random time-series) by using the sinc interpolation:

$$\Phi_{v_r v_s}(j\omega T) = \sum_{m=-\infty}^{\infty} \Phi_{v_r v_s}(j\Delta m)\,\mathrm{sinc}\left(\frac{\pi(\omega T - \Delta m)}{\Delta}\right). \tag{3.6}$$

The accuracy of the reconstruction is dependent of the number of terms used to perform the summation in (3.6).

The assumption (3.4) can be interpreted as a lower bound on the number of approximation nodes - samples of multisine random time-series to be simulated. When it is satisfied (the number of approximation nodes is two times greater than τ_0), the original power spectral density matrix may be reconstructed uniquely without producing aliasing.

For asymptotically uncorrelated random processes ($\lim_{\tau\to\infty} \mathbf{R}_{\mathbf{vv}}(\tau) = \mathbf{o}$) the assumption (3.4) can be satisfied only asymptotically for $n' \to \infty$. In this case, the finite number n' of approximation nodes implies aliasing in the shift-domain of the corresponding autocorrelation function. This aliasing may be made insignificant by selecting a sufficiently large τ_0 such that for all $\tau > \tau_0$ it is reasonably to assume that $\mathbf{R}_{\mathbf{vv}}(\tau)$ is a zero matrix.

In the sequel, for given power spectral density matrices of wide-sense stationary orthogonal and nonorthogonal multivariate random processes the synthesis of the corresponding multisine random time-series is discussed in details.

3.1.1 Ergodic case

Let $\mathbf{v}(i)$ be a wide-sense stationary, real-valued multivariate orthogonal random process with the power spectral density matrix $\mathbf{\Phi}_{\mathbf{vv}}(j\omega T) = \mathrm{diag}\,[\Phi_{v_r v_r}(\omega T) + j0]_{r=1,2,\ldots,p}$. The power spectral densities $\Phi_{v_r v_r}(\omega T)$ $(r = 1, 2, \ldots, p)$ are sampled in the frequency-domain (approximated by a periodogram of MOMRS) choosing for each, n_r' sample points along the ωT axis at relative frequencies from the set $\mathcal{N}_{r,p}^{c,1}$. It does not produce aliasing if the spacing Δ_r between the samples along the frequency axis is such that:

$$\max_{r=1,2,\ldots,p} \Delta_r = p\frac{2\pi}{N} = p\Omega < \Delta_0. \tag{3.7}$$

The approximation criterion:

$$\Phi_{v_r v_r}(\omega T)|_{\omega T \in \mathcal{N}_{r,p}^{c,1}} = \Phi_{u_r u_r}^{B}(\Omega m)|_{\Omega m \in \mathcal{N}_{r,p}^{c,1}} \tag{3.8}$$

for $r = 1, 2, \ldots, p$ allows us to synthesise the rth element $U_r(j\Omega m)$ of the MOMRS finite discrete Fourier transform $\mathbf{U}^B(j\Omega m) = \left[U_r^B(j\Omega m)\right]_{r=1,2,\ldots,p}$ as:

- for $(m = 0) \wedge (\Omega m \in \mathcal{N}_{r,p}^{c,1})$:

$$U_r^B(j0) = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(0)} + j0; \tag{3.9}$$

- for $(m = 1, 2, \dots, \frac{N}{2} - 1) \wedge (\Omega m \in \mathcal{N}^{c,1}_{r,p})$:

$$Re\left\{U^B_r(j\Omega m)\right\} = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(\Omega m)}\sin\phi_m, \tag{3.10}$$

$$Im\left\{U^B_r(j\Omega m)\right\} = -\sqrt{\frac{N}{T}\Phi_{v_r v_r}(\Omega m)}\cos\phi_m, \tag{3.11}$$

where ϕ_m are random, independent and uniformly distributed on $[0, 2\pi)$;

- for $(m = \frac{N}{2}) \wedge (\Omega m \in \mathcal{N}^{c,1}_{r,p})$:

$$U^B_r(j\pi) = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(\pi)}\sin\phi_{\frac{N}{2}} + j0, \tag{3.12}$$

where $\phi_{\frac{N}{2}}$ is random, independent and Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$;

- for $(m = 0, 1, \dots, \frac{N}{2}) \wedge (\Omega m \notin \mathcal{N}^{c,1}_{r,p})$:

$$U^B_r(j\Omega m) = 0 + j0; \tag{3.13}$$

- for $N - m = N - 1, N - 2, \dots, N - (\frac{N}{2} - 1)$:

$$U^B_r(j\Omega(N - m)) = Re\left\{U^B_r(j\Omega m)\right\} - jIm\left\{U^B_r(j\Omega m)\right\}. \tag{3.14}$$

Accuracy of the multisine random process approximation defined by the criterion (3.8) may be discussed in the shift-domain of its autocorrelation function.

Let us assume that for the given power spectral density $\Phi_{vv}(\omega T)$ a scalar multisine random time-series $u(i)$ was synthesised and simulated. The approximation error $\epsilon(\tau)$ is defined as:

$$\epsilon(\tau) = R_{vv}(\tau) - R_{uu}(\tau), \tag{3.15}$$

where $\tau = 0, 1, \dots, N - 1$. It follows from the approximation criterion (3.8) and Lemma 2.2 that:

$$\epsilon(\tau) = \frac{1}{2\pi T}\int_0^{2\pi}\Phi_{vv}(\omega T)\cos(\omega T\tau)d(\omega T) - \frac{1}{2NT}\sum_{n=0}^{\frac{N}{2}}\Phi_{vv}(\Omega n)\cos(\Omega n\tau)$$

$$= \frac{1}{2\pi T}\int_0^{2\pi}\Phi_{vv}(\omega T)\cos(\omega T\tau)d(\omega T) - \frac{1}{2\pi T}\sum_{n=0}^{N-1}\Phi_{vv}(\Omega n)\cos(\Omega n\tau)\Omega. \tag{3.16}$$

For $N \to \infty$ the product $(N - 1)\Omega$ tends to 2π. It implies under Riemann's definition of the integral that the approximation error $\epsilon(\tau)$ declines with $\frac{1}{N}$ (because $\Omega = \frac{2\pi}{N}$).

For the given number N of approximation nodes, values of the corresponding approximation error $\epsilon(\tau)$ depend on the shape of the power spectral density $\Phi_{vv}(\omega T)$. Analysis of the expression (3.16) leads to the conclusion, that the approximation error $\epsilon(\tau)$ is equal to zero for lags $\tau = 0, 1, \dots, N - 1$ when a white noise with the power spectral density $\Phi_{vv}(\omega T) = \lambda^2$ is synthesised (see Chapter 4 for details).

Example 3.1 *Let $v(i)$ be the following third-order scalar AR time-series:*

$$v(i) = \frac{1.00}{1.00 - 2.00z^{-1} + 1.45z^{-2} - 0.35z^{-3}}e(i), \tag{3.17}$$

where $e(i)$ is a hypothetical white noise.

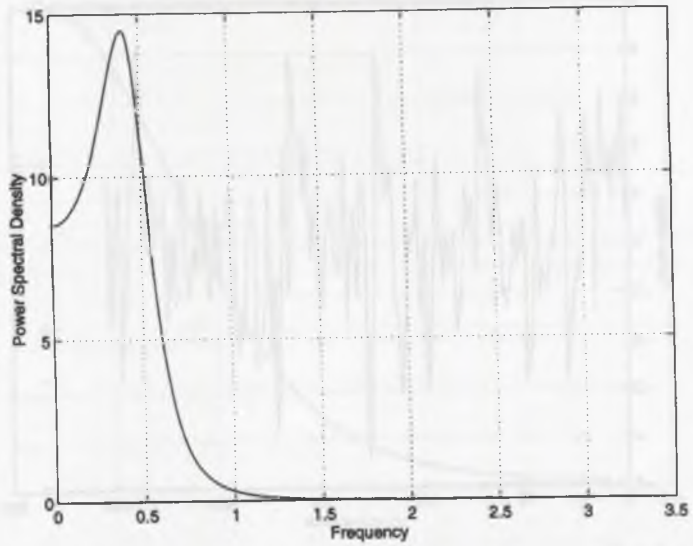

Fig. 3.1. Power spectral density of the third-order AR time-series

Table 3.1

	Parameter Estimates	
Parameter	$N = 128$	$N = 256$
−2.00	−2.00 (0.012)	−2.00 (0.006)
1.45	1.45 (0.019)	1.45 (0.010)
−0.35	−0.35 (0.011)	−0.35 (0.006)

Mean values and standard deviations (in parentheses) of the third-order AR time-series model parameter estimates obtained for 100 simulation experiments using the Least Squares identification method

The $v(i)$ with variance equal to 1 was simulated by using its frequency-domain representation as the power spectral density diagram (Fig. 3.1), which was approximated by the periodogram of a scalar multisine random time-series.

Each simulated N-sample third-order AR time-series realisation ($N = 128$ and $N = 256$) was identified using the Least Squares identification method [13]. The mean values and standard deviations (in parentheses) of the estimated parameters for a third-order AR model in 100 simulation experiments are presented in Tab. 3.1.

The mean values of the estimated parameters do not differ from the true values but its standard deviations show that the autoregressive time-series simulated with multisine random time-series very precisely reconstruct spectral and correlation properties of the original random process for finite number of samples.

Example 3.2 *A time-series $v(i)$ with the nonrational power spectral density:*

$$S_{vv}(\omega T) = e^{2\cos(\omega T)} \tag{3.18}$$

was simulated by using the proposed approach: the power spectral density $S_{vv}(\omega T)$ was approximated by the periodogram of a scalar multisine random time-series. The number N of approximation nodes was 256. An example of the simulated $N = 256$-sample multisine random time-series realisation $u^{256}(i)$ is shown in Fig. 3.2.

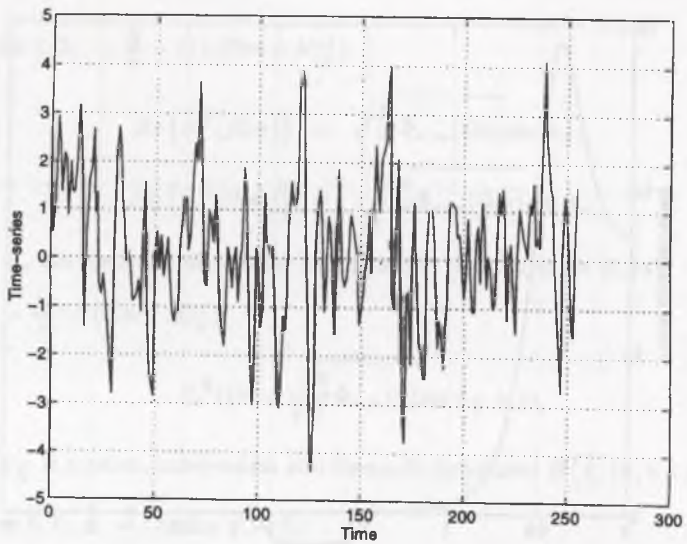

Fig. 3.2. Nonrational multisine random time-series realisation - $N = 256$

For this realisation the rational AR, MA and ARMA approximations were identified by using the Least Squares and Recursive Prediction Error methods [55], respectively. Application of the AIC criterion [56] for the model order selection resulted in:

- *the AR(5) model:*

$$u^{256}(i) = \frac{1.000}{1.000 - 0.993z^{-1} + 0.494z^{-2} - 0.161z^{-3} + 0.050z^{-4} - 0.016z^{-5}} e(i); \tag{3.19}$$

- *the MA(4) model:*

$$u^{256}(i) = \left(1.000 + 0.987z^{-1} + 0.492z^{-2} + 0.169z^{-3} + 0.042z^{-4}\right) e(i); \tag{3.20}$$

- *the ARMA(2,1) model:*

$$u^{256}(i) = \frac{1.000 + 0.369z^{-1}}{1.000 - 0.621z^{-1} + 0.130z^{-2}} e(i), \tag{3.21}$$

where $e(i)$ is a hypothetical white noise. One-step prediction error variances for the above models were all equal to 1.006.

The original power spectral density (3.18), unwindowed periodogram for the simulated time-series realisation and power spectral densities for identified rational AR(5), MA(4) and ARMA(2,1) approximations are compared in Fig. 3.3.

The time-domain identification methods allow us to identify rational approximations of $v(i)$ based on the corresponding multisine random time-series realisation $u^{256}(i)$. The power spectral densities calculated by using these rational approximations very precisely reconstruct the original one.

The proposed approach based on approximating the power spectral density by the periodogram of a multisine random time-series allows us to simulate nonrational time-series without calculating any parametric approximation.

Example 3.3 *The following bivariate orthogonal* **AR** *time-series* $\mathbf{v}(i)$*:*

$$A(z^{-1})\mathbf{v}(i) = \mathbf{e}(i) \tag{3.22}$$

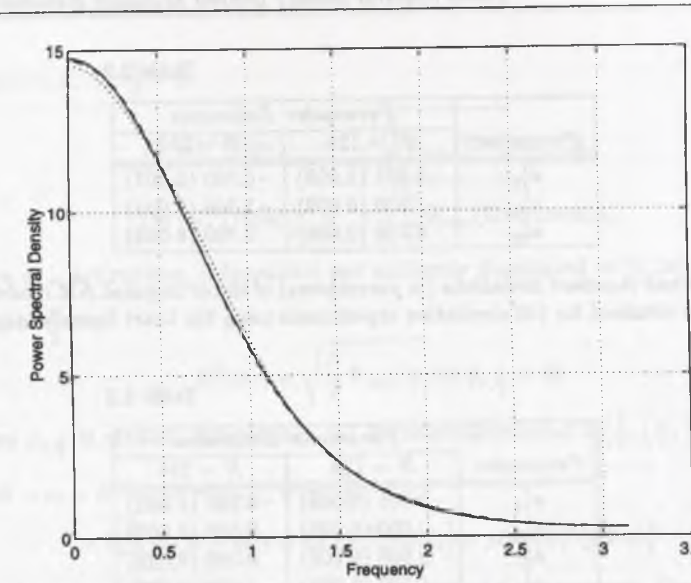

Fig. 3.3. Comparison of the original power spectral density (solid line), unwindowed periodogram for $N = 256$-sample multisine random time-series realisation (solid line) and power spectral density for indentified: AR(5) (dotted line), MA(4) (dashdot line) and ARMA(2,1) (dashed line) models

with

$$A(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} -0.80 & 0.00 \\ 0.00 & -1.50 \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.70 \end{bmatrix} z^{-2} \tag{3.23}$$

and a unit variance matrix of the white noise $\mathbf{e}(i)$ was simulated by using its frequency-domain representation as the power spectral density matrix

$$\Phi_{\mathbf{vv}}(j\omega T) = \begin{bmatrix} \dfrac{1.00 + j0}{1.64 - 1.60\cos\omega T} & 0 + j0 \\ 0 + j0 & \dfrac{1.00 + j0}{3.74 - 5.10\cos\omega T + 1.40\cos 2\omega T} \end{bmatrix}, \tag{3.24}$$

which was approximated by the periodogram matrix of a multivariate (bivariate) orthogonal multisine random time-series.

Each simulated N-sample **AR** *time-series realisation ($N = 128$ and $N = 256$) was identified using the Least Squares identification method [13]. The mean values and standard deviations (in parentheses) of parameters estimated in 100 simulation experiments for the orthogonal* **AR** *model with a structure of the matrix $A(z^{-1})$ chosen as*

$$A(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & 0.00 \\ 0.00 & a_{22}^1 \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^2 \end{bmatrix} z^{-2} \tag{3.25}$$

are presented in Tab. 3.2.

The corresponding results for a nonorthogonal **AR** *model with the following structure of the matrix $A(z^{-1})$*

$$A(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^2 \end{bmatrix} z^{-2} \tag{3.26}$$

are presented in Tab. 3.3.

Table 3.2

Parameter	Parameter Estimates	
	$N = 128$	$N = 256$
a_{11}^1	−0.801 (0.006)	−0.800 (0.002)
a_{22}^1	−1.500 (0.009)	−1.500 (0.004)
a_{22}^2	0.700 (0.008)	0.700 (0.003)

Mean values and standard deviations (in parentheses) of the orthogonal **AR** model parameter estimates obtained for 100 simulation experiments using the Least Squares identification method

Table 3.3

Parameter	Parameter Estimates	
	$N = 128$	$N = 256$
a_{11}^1	-0.801 (0.006)	-0.800 (0.002)
a_{12}^1	0.000 (0.009)	0.000 (0.003)
a_{21}^1	0.000 (0.009)	0.000 (0.003)
a_{22}^1	-1.500 (0.009)	-1.500 (0.003)
a_{22}^2	0.700 (0.008)	0.700 (0.003)

Mean values and standard deviations (in parentheses) of the nonorthogonal **AR** model parameter estimates obtained for 100 simulation experiments using the Least Squares identification method

The mean values of the estimated parameters do not differ from the true values but their standard deviations show that autoregressive multivariate orthogonal time-series simulated using weakly ergodic bivariate orthogonal multisine random time-series precisely approximates properties of the original multivariate orthogonal random process.

3.1.2 Nonergodic case

Multivariate orthogonal multisine time-series

Similarly as for the previous ergodic case, let $\mathbf{v}(i)$ be a wide-sense stationary, real-valued multivariate orthogonal random process with the power spectral density matrix $\mathbf{\Phi_{vv}}(j\omega T) = \text{diag}\left[\Phi_{v_r v_r}(\omega T) + j0\right]_{r=1,2,\ldots,p}$. It is assumed that the number of approximation nodes N is chosen so that $\frac{N}{2} > \tau_0$. This implies that choosing the maximum spacing $\Delta = \Omega$ between the samples along the frequency axis, the original power spectral densities $\Phi_{v_r v_r}(\omega T)$ for $r = 1, 2, \ldots, p$ may be recovered from their samples $(\Phi_{v_r v_r}(\Omega m))$ without producing aliasing.

The approximation criterion:

$$\Phi_{v_r v_r}(\omega T)|_{\omega T = \Omega m} = \Phi^B_{u_r u_r}(\Omega m) \tag{3.27}$$

for $m = 0, 1, \ldots, \frac{N}{2}$ and $r = 1, 2, \ldots, p$ allows us to synthesise the rth element $U_r(j\Omega m)$ of the NMOMRS finite discrete Fourier transform $\mathbf{U}^B(j\Omega m) = \left[U_r^B(j\Omega m)\right]_{r=1,2,\ldots,p}$ as:

- for $m = 0$:

$$U_r^B(j0) = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(0)} + j0; \tag{3.28}$$

- for $m = 1, 2, \ldots, \frac{N}{2} - 1$:

$$Re\left\{U_r^B(j\Omega m)\right\} = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(\Omega m)}\sin\phi_{r,m}, \tag{3.29}$$

$$Im\left\{U_r^B(j\Omega m)\right\} = -\sqrt{\frac{N}{T}\Phi_{v_r v_r}(\Omega m)}\cos\phi_{r,m}, \tag{3.30}$$

where $\phi_{r,m}$ are random, independent and uniformly distributed on $[0, 2\pi)$;

- for $m = \frac{N}{2}$:

$$U_r^B(j\pi) = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(\pi)}\sin\phi_{r,\frac{N}{2}} + j0, \tag{3.31}$$

where $\phi_{r,\frac{N}{2}}$ is random, independent and Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}\right)$;

- for $N - m = N - 1, N - 2, \ldots, N - (\frac{N}{2} - 1)$:

$$U_r^B(j\Omega(N - m)) = Re\left\{U_r^B(j\Omega m)\right\} - jIm\left\{U_r^B(j\Omega m)\right\}. \tag{3.32}$$

Example 3.4 *The simulation experiment from Example 3.3 was repeated. The power spectral density matrix (3.24) was approximated by the expected value of the periodogram matrix of a nonergodic multivariate (bivariate) orthogonal multisine random time-series. Similarly as for the previous case, each simulated N-sample* **AR** *time-series realisation (N = 128, N = 256 and N = 1024) was identified using the Least Squares identification method [13]. The mean values and standard deviations (in parentheses) of parameters estimated in 100 simulation experiments for the orthogonal* **AR** *model with the structure of the matrix* $A(z^{-1})$:

$$A(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & 0.00 \\ 0.00 & a_{22}^1 \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^2 \end{bmatrix} z^{-2} \tag{3.33}$$

are presented in Tab. 3.4.

Table 3.4

Parameter	Parameter Estimates		
	$N = 128$	$N = 256$	$N = 1024$
a_{11}^1	−0.800 (0.006)	−0.800 (0.002)	0.800 (0.0003)
a_{22}^1	−1.499 (0.010)	−1.500 (0.003)	−1.500 (0.0006)
a_{22}^2	0.699 (0.008)	0.700 (0.003)	0.700 (0.0008)

Mean values and standard deviations (in parentheses) of the orthogonal **AR** model parameter estimates obtained for 100 simulation experiments using the Least Squares identification method

The corresponding results for a nonorthogonal **AR** *model with the following structure of the matrix* $A(z^{-1})$

$$A(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^2 \end{bmatrix} z^{-2} \tag{3.34}$$

are presented in Tab. 3.5.

It follows from Tabs 3.4 and 3.5 that the synthesised and simulated nonergodic multivariate orthogonal multisine random time-series very precisely reconstruct correlation properties of the original random process. Additionally, when the value of N grew up from N = 128 to N = 1024, the results of identification for any NMOMRS realisation approached the results of the ensemble averaging.

Table 3.5

Parameter	Parameter Estimates		
	$N = 128$	$N = 256$	$N = 1024$
a_{11}^1	−0.797 (0.009)	−0.798 (0.005)	0.800 (0.001)
a_{12}^1	−0.007 (0.031)	−0.004 (0.022)	0.001 (0.010)
a_{21}^1	−0.017 (0.051)	−0.012 (0.040)	0.002 (0.019)
a_{22}^1	−1.495 (0.012)	−1.498 (0.005)	−1.500 (0.001)
a_{22}^2	0.698 (0.015)	0.700 (0.005)	0.700 (0.001)

Mean values and standard deviations (in parentheses) of the nonorthogonal **AR** model parameter estimates obtained for 100 simulation experiments using the Least Squares identification method

Multivariate nonorthogonal multisine time-series

Let $\mathbf{v}(i)$ be a wide-sense stationary, real-valued multivariate nonorthogonal random time-series with the power spectral density matrix $\mathbf{\Phi_{vv}}(j\omega T)$. The $\mathbf{\Phi_{vv}}(j\omega T)$ may be reconstructed from its approximation by the expected value of a NMMRS periodogram matrix if the number N of approximation nodes is chosen so that $\frac{N}{2} > \tau_0$. The approximation criterion:

$$\mathbf{\Phi_{vv}}(j\omega T)|_{\omega T=\Omega m} = \mathcal{E}\{\mathbf{\Phi}^B_{\mathbf{uu}}(j\Omega m)\} = \mathbf{K}(j\Omega m)\mathbf{K}^*(j\Omega m). \tag{3.35}$$

allows us to synthesise the NMMRS finite discrete Fourier transform $\mathbf{U}^B(j\Omega m)$ using the following two-step procedure:

- step 1: synthesis of the finite discrete Fourier transform $\beta^B(j\Omega m) = \left[\beta^B_r(j\Omega m)\right]_{r=1,2,\ldots,p}$ of NMOMRS which approximates multivariate orthogonal white noise with the periodogram matrix equal to $\boldsymbol{I}$:

  – for $m = 0$:

$$\beta^B_r(j0) = \sqrt{\frac{N}{T}} + j0; \tag{3.36}$$

  – for $m = 1, 2, \ldots, \frac{N}{2} - 1$:

$$Re\left\{\beta^B_r(j\Omega m)\right\} = \sqrt{\frac{N}{T}} \sin\phi_{r,m}, \tag{3.37}$$

$$Im\left\{\beta^B_r(j\Omega m)\right\} = -\sqrt{\frac{N}{T}} \cos\phi_{r,m}, \tag{3.38}$$

  where $\phi_{r,m}$ are random, independent and uniformly distributed on $[0, 2\pi)$;

  – for $m = \frac{N}{2}$:

$$\beta^B_r(j\pi) = \sqrt{\frac{N}{T}} \sin\phi_{r,\frac{N}{2}} + j0, \tag{3.39}$$

  where $\phi_{r,\frac{N}{2}}$ is random, independent and Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}\right)$;

  – for $N - m = N - 1, N - 2, \ldots, N - (\frac{N}{2} - 1)$:

$$\beta^B_r(j\Omega(N-m)) = Re\left\{\beta^B_r(j\Omega m)\right\} - jIm\left\{\beta^B_r(j\Omega m)\right\}, \tag{3.40}$$

  where $r = 1, 2, \ldots, p$

- step 2: for each frequency Ωm ($m = 0, 1, \ldots, N-1$) the matrix $\mathbf{K}(j\Omega m)$ is chosen so that the following spectral factorisation equation is satisfied:

$$\mathbf{\Phi_{vv}}(j\Omega m) = \mathbf{K}(j\Omega m)\mathbf{K}^*(j\Omega m). \tag{3.41}$$

  The finite discrete Fourier transform $\mathbf{U}^B(j\Omega m)$ is calculated as:

$$\mathbf{U}^B(j\Omega m) = \mathbf{K}(j\Omega m)\beta(j\Omega m). \tag{3.42}$$

3.2 ASYMPTOTIC PROPERTIES

The synthesised and simulated multisine random process approximations of wide-sense stationary random processes with specified power spectral densities turn asymptotically for the number of approximation nodes $N \to \infty$ into Gaussian random processes. Additionally, nonergodic multisine random time-series become asymptotically ergodic. This is summarised in the sequel.

3.2.1 Ergodic case

Lemma 3.1 *Assuming that:*

1. $\mathbf{\Phi_{vv}}(j\omega T) = diag[\Phi_{v_r v_r}(\Omega m) + j0]_{r=1,2,\ldots,p}$ *is the power spectral density matrix of a wide-sense stationary, orthogonal, real-valued multivariate random process with zero mean vector and the variance matrix* $\boldsymbol{\sigma}^2_{\mathbf{vv}} = diag\left[\sigma^2_{v_r v_r}\right]_{r=1,2,\ldots,p}$, *where:*

$$\sigma^2_{v_r v_r} = \frac{1}{2\pi T}\int_0^{2\pi} \Phi_{v_r v_r}(\omega T)d(\omega T); \tag{3.43}$$

2. A_n *converges to 0 for* $N \to \infty$ *in such a way that for* $r = 1, 2, \ldots, p$:

$$\frac{NTA_n^2}{4} = \Phi_{v_r v_r}(\Omega n), \tag{3.44}$$

   where $n = 1, 2, \ldots, \frac{N}{2} - 1$ *and* $\Omega n \in \mathcal{N}^{c,1}_{r,p}$;

3. $A_0 = A_{\frac{N}{2}} = 0$ *or* $\phi_0 = \alpha = 0$;

then the corresponding extended MOMRS $\mathbf{u}(i)$ *with the consecutively circularly ordered frequencies converges in distribution for* $N \to \infty$ *to a Gaussian multivariate orthogonal multisine random time-series of type 1 (GMOMRS1)* $\mathbf{g}(i) = [g_r(i)]_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\frac{1}{p}\boldsymbol{\sigma}^2_{\mathbf{vv}}$:

$$\mathbf{g}(i) \in As\mathcal{N}(\mathbf{o}, \frac{1}{p}\boldsymbol{\sigma}^2_{\mathbf{vv}}). \tag{3.45}$$

Additionally the correlation function matrix of the GMOMRS1 converges to:

$$\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\} = \mathbf{R_{gg}}(\tau) = \frac{1}{2\pi pT}\int_0^{2\pi} \mathbf{\Phi_{vv}}(j\omega T)\cos(\omega T\tau)d(\omega T) = \frac{1}{p}\mathbf{R_{vv}}(\tau), \tag{3.46}$$

where $\tau = 0, 1, \ldots, \infty$.

□

Proof: *The uniform distribution of the independent, random phase shifts* ϕ_n *on* $[0, 2\pi)$ *for each frequency* Ωn ($n = 1, 2, \ldots, \frac{N}{2} - 1$) *implies that for any time instant* i *the random vector* $\mathbf{l}_n(i) = [l_{r,n}(i)]_{r=1,2,\ldots,p}$, *with elements:*

$$l_{r,n}(i) = \begin{cases} \frac{\sqrt{NT}A_n}{2}\sin(\Omega n i + \phi_n) & \text{if } \Omega n \in \mathcal{N}^{c,1}_{r,p} \\ 0 & \text{if } \Omega n \notin \mathcal{N}^{c,1}_{r,p} \end{cases}, \tag{3.47}$$

is characterised by the expected value $\mathcal{E}\{\mathbf{l}_n(i)\} = \mathbf{o}$. *Its variance matrix is* $\mathcal{E}\{\mathbf{l}_n(i)\mathbf{l}_n^T(i)\} = diag\left[\mathcal{E}\{l_{r,n}^2(i)\}\right]_{r=1,2,\ldots,p}$, *where:*

$$\mathcal{E}\{l_{r,n}^2(i)\} = \begin{cases} \frac{NTA_n^2}{8} & \text{if } \Omega n \in \mathcal{N}_{r,p}^{c,1} \\ 0 & \text{if } \Omega n \notin \mathcal{N}_{r,p}^{c,1} \end{cases}. \tag{3.48}$$

It follows from (3.47) that:

$$\mathbf{S}_N = \sum_{n=1}^{\frac{N}{2}-1} \mathbf{l}_n(i) = \frac{\sqrt{NT}}{2}\mathbf{u}^B(i) \tag{3.49}$$

and for each time instant i:

$$\mathcal{E}\{\mathbf{u}^B(i)\} = 0. \tag{3.50}$$

The corresponding variance matrix is:

$$\varSigma_N^2 = \sum_{n=1}^{\frac{N}{2}-1} \mathcal{E}\{\mathbf{l}_n(i)\mathbf{l}_n^T(i)\} = \frac{NT}{4}\mathrm{diag}\left[\sigma_{rr,N}^2\right]_{r=1,2,\ldots,p}, \tag{3.51}$$

where:

$$\sigma_{rr,N}^2 = \frac{2}{2\pi T} \sum_{\Omega n \in \mathcal{N}_{r,p}^{c,1} \setminus \{0,\pi\}} \Phi_{v_r v_r}(\Omega n)\Omega = \frac{2}{2\pi T p} \sum_{n=0}^{n_r-1} \Phi_{v_r v_r}(\Omega p n + (r-1)\Omega)\Omega p. \tag{3.52}$$

For $N \to \infty$ *the product* $(r-1)\Omega$ *tends to* 0 *and* $(n_r-1)\Omega p$ *tends to* π. *It implies under Riemann's definition of the integral that*

$$\lim_{N\to\infty} \sigma_{rr,N}^2 = \frac{1}{2\pi T p}\int_0^{2\pi} \Phi_{v_r v_r}(\omega T)d(\omega T) = \frac{\sigma_{v_r v_r}^2}{p}. \tag{3.53}$$

Let:

$$\|\mathbf{l}_n(i)\|_2 = \sqrt{\sum_{r=1}^{p} l_{r,n}^2(i)} \tag{3.54}$$

denotes the Euclidean norm of the vector $\mathbf{l}_n(i)$. *It follows from the properties of sine function that for each time instant* i *the sequence of random vectors* $\mathbf{l}_n(i)$ $(n = 1, 2, \ldots, \frac{N}{2} - 1)$ *is a uniformly bounded sequence [51], i.e. there exists a constant* c *such that*

$$P\{\|\mathbf{l}_n(i)\|_2 \le c\} = 1 \tag{3.55}$$

holds for $n = 1, 2, \ldots, \frac{N}{2} - 1$. *It implies that for every* $\varepsilon > 0$, *the extended Lindeberg condition:*

$$\lim_{N\to\infty} \frac{1}{\frac{N}{2}-1} \sum_{n=1}^{\frac{N}{2}-1} \mathcal{E}\left\{\|\mathbf{l}_n(i)\|_2^2; \left\{\|\mathbf{l}_n(i)\|_2 \ge \varepsilon\sqrt{\frac{N}{2}-1}\right\}\right\} = 0 \tag{3.56}$$

is satisfied by the sequence of $\mathbf{l}_n(i)$ $(n = 1, 2, \ldots, \frac{N}{2} - 1)$ *for each time instant* i. *It follows from an extension of the Lindeberg-Feller central limit theorem to the multivariate case [83] that for each time instant* i *the random variable*

$$\frac{2}{\sqrt{NT}}\mathbf{S}_N = \mathbf{u}^B(i) \tag{3.57}$$

converges in distribution for $N \to \infty$ *to a Gaussian multivariate orthogonal random variable* $\mathbf{g}(i)$ *with zero mean vector and the variance matrix* $\frac{1}{p}\boldsymbol{\sigma}_{\mathbf{vv}}^2$.

The proof of the property 3.46 follows from Riemann's definition of the integral applied for $N \to \infty$ *to correlation function matrix elements of the power spectral density defined MOMRS.*

□

3.2.2 Nonergodic case

Multivariate orthogonal multisine time-series

The properties of NMOMRS for the ensemble averaging under $N \to \infty$ are given by the lemma:

Lemma 3.2 *Assuming that:*

1. $\boldsymbol{\Phi}_{\mathbf{vv}}(j\omega T) = diag[\Phi_{v_r v_r}(\omega T) + j0]_{r=1,2,\ldots,p}$ *is the power spectral density matrix of a wide-sense stationary, orthogonal, real-valued multivariate random process with zero mean vector and the variance matrix* $\boldsymbol{\sigma}_{\mathbf{vv}}^2 = diag\left[\sigma_{v_r v_r}^2\right]_{r=1,2,\ldots,p}$, *where:*

$$\sigma_{v_r v_r}^2 = \frac{1}{2\pi T}\int_0^{2\pi} \Phi_{v_r v_r}(\omega T)d(\omega T); \tag{3.58}$$

2. $A_{r,n}$ *converges to* 0 *for* $N \to \infty$ *in such a way that for* $r = 1, 2, \ldots, p$:

$$\frac{NTA_{r,n}^2}{4} = \Phi_{v_r v_r}(\Omega n), \tag{3.59}$$

where $n = 1, 2, \ldots, \frac{N}{2} - 1$;

3. $A_{r,0} = A_{r,\frac{N}{2}} = 0$ *or* $\phi_{r,0} = \alpha = 0$ *for* $r = 1, 2, \ldots, p$;

then the corresponding extended NMOMRS $\mathbf{u}(i)$ *converges in distribution for* $N \to \infty$ *to a Gaussian multivariate orthogonal multisine random time-series of type 2 (GMOMRS2)* $\mathbf{g}(i) = [g_r(i)]_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\boldsymbol{\sigma}_{\mathbf{vv}}^2$:

$$\mathbf{g}(i) \in As\mathcal{N}(\boldsymbol{O}, \boldsymbol{\sigma}_{\mathbf{vv}}^2). \tag{3.60}$$

Additionally the correlation function matrix $\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\}$ *of the GMOMRS2 converges to:*

$$\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\} = \frac{1}{2\pi T}\int_0^{2\pi T} \boldsymbol{\Phi}_{\mathbf{vv}}(j\omega T)\cos(\omega T\tau)d(\omega T), \tag{3.61}$$

where $\tau = 0, 1, \ldots, \infty$.

□

Proof: *When for each frequency* Ωn $(n = 1, 2, \ldots, \frac{N}{2} - 1)$ *and any time instant* i *elements the random vector* $\mathbf{l}_n(i) = [l_{r,n}(i)]_{r=1,2,\ldots,p}$ *are defined as:*

$$l_{r,n}(i) = \frac{\sqrt{NT}A_{r,n}}{2}\sin(\Omega n i + \phi_{r,n}), \tag{3.62}$$

the proof proceeds similarly as the proof of previous lemma. It should only be noticed that for the r*th element of the GMOMRS1:*

$$\sigma_{rr,N}^2 = \frac{1}{2\pi T}\left[\sum_{n=0}^{N-1} \Phi_{v_r v_r}(\Omega n)\Omega - \Phi_{v_r v_r}(0)\Omega - \Phi_{v_r v_r}(\pi)\Omega\right], \tag{3.63}$$

where $\sigma_{rr,N}^2$ *corresponds to* $\sigma_{rr,N}^2$ *in equation (3.52). It follows for* $N \to \infty$ *under Riemann's definition of the integral that*

$$\lim_{N\to\infty} \sigma_{rr,N}^2 = \frac{1}{2\pi T}\int_0^{2\pi} \Phi_{v_r v_r}(\omega T)d(\omega T) = \sigma_{v_r v_r}^2. \tag{3.64}$$

□

The properties of NMOMRS obtained for the time-domain averaging under $N \to \infty$ are given by the lemma:

Lemma 3.3 *Consider the GMOMRS2.*

1. *Elements of its periodogram matrix* $\Phi^B_{\mathbf{gg}}(j\Omega m) = \left[\Phi^B_{g_r g_s}(j\Omega m)\right]_{r,s=1,2,\ldots,p}$ *for* $m = 0, 1, \ldots, N-1$ *under* $N \to \infty$ *converge to:*

$$\Phi^B_{g_r g_s}(j\Omega m) = (0 + j0)\delta(m)$$

$$+ \sum_{n=1}^{\frac{N}{2}-1} \sqrt{\Phi_{v_r v_r}(\Omega n)\Phi_{v_s v_s}(\Omega n)}\,[(\cos(\phi_{r,n} - \phi_{s,n}) - j\sin(\phi_{r,n} - \phi_{s,n}))\delta(m-n)$$

$$+ (\cos(\phi_{r,n} - \phi_{s,n}) + j\sin(\phi_{r,n} - \phi_{s,n}))\delta(m-(N-n))] + (0 + j0)\delta(m - \frac{N}{2}), \quad (3.65)$$

where $r, s = 1, 2, \ldots, p$.

2. *The mean value vector* $\mathcal{M}\{\mathbf{g}(i)\}$ *is equal to* $\mathbf{o}$ *for* $N \to \infty$.

3. *Elements of the correlation function matrix* $\mathbf{R_{gg}}(\tau) = [R_{g_r g_s}(\tau)]_{r,s=1,2,\ldots,p}$ *for* $\tau = 0, 1, \ldots, \infty$ *under* $N \to \infty$ *converge to:*

$$R_{g_r g_s}(\tau) = \begin{cases} R_{v_r v_r}(\tau) & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \quad a.s. \quad (3.66)$$

4. *Elements of the variance matrix* $\sigma^2_{\mathbf{gg}} = \left[\sigma^2_{g_r g_s}\right]_{r,s=1,2,\ldots,p}$ *under* $N \to \infty$ *converge to:*

$$\sigma^2_{g_r g_s} = \begin{cases} \sigma_{v_r v_r} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \quad a.s. \quad (3.67)$$

Proof: □

1. *It follows from Lemma 2.7 when* $A_{r,n}$ *calculated from equation (3.59) for* $n = 1, 2, \ldots, \infty$ *are used.*

2. *It follows from Lemma 2.9 and Lemma 3.2.*

3. *It follows from Lemma 2.9 and equation (3.59) that the elements of the correlation function matrix for* $\tau = 0, 1, \ldots, \infty$ *are given by:*

$$R_{u_r u_s}(\tau) = \frac{2}{N}\sum_{n=1}^{\frac{N}{2}-1} \sqrt{\Phi_{v_r v_r}(\Omega n)\Phi_{v_s v_s}(\Omega n)}\cos(\Omega n\tau + \phi_{r,n} - \phi_{s,n}), \quad (3.68)$$

where $r, s = 1, 2, \ldots, p$. *For the case of* $r = s$ *the elements* $R_{u_r u_r}(\tau)$ *are deterministic functions and:*

$$\lim_{N\to\infty} R_{u_r u_r}(\tau) = \lim_{N\to\infty} \frac{2}{N}\sum_{n=1}^{\frac{N}{2}-1} \Phi_{v_r v_r}(\Omega n)\cos(\Omega n\tau) = R_{v_r v_r}(\tau), \quad (3.69)$$

while for $r \neq s$ *and* $\tau = 0, 1, .., N-1$ *the components*

$$r^n_{rs}(\tau) = \sqrt{\Phi_{v_r v_r}(\Omega n)\Phi_{v_s v_s}(\Omega n)}\cos(\Omega n\tau + \phi_{r,n} - \phi_{s,n}) \quad (3.70)$$

are independent random variables with $\mathcal{E}\{r^n_{rs}(\tau)\} = 0$ *and* $\mathcal{E}\left\{(r^n_{rs}(\tau))^2\right\} < \infty$ *for all* n $(n = 1, 2, \ldots, \frac{N}{2} - 1)$. *Additionally:*

$$\lim_{N\to\infty} \sum_{n=1}^{\frac{N}{2}-1} \frac{\mathcal{E}\left\{(r^n_{rs}(\tau))^2\right\}}{n^2} < \infty. \quad (3.71)$$

It implies under the strong law of large numbers [51] that

$$\lim_{N\to\infty} R_{u_r u_s}(\tau) = \lim_{N\to\infty} \frac{2}{N}\sum_{n=1}^{\frac{N}{2}-1} r^n_{rs}(\tau) = 0 \quad a.s. \quad (3.72)$$

4. *It follows immediately from the variance matrix definition that:*

$$\sigma^2_{\mathbf{gg}} = \mathbf{R_{gg}}(0) \quad a.s. \quad (3.73)$$

□

It follows from Lemma 3.2 and 3.3 that the power spectral density defined NMOMRS converges asymptotically for $N \to \infty$ to a Gaussian multivariate random process which is ergodic.

Multivariate nonorthogonal multisine time-series

The extended NMMRS obtained from application of the approximation criterion (3.35) to the power spectral density matrix $\Phi_{\mathbf{vv}}(j\omega T)$ $(\omega T \in [0, 2\pi))$ of a wide-sense stationary multivariate nonorthogonal random process $\mathbf{v}(i)$ turns also asymptotically for $N \to \infty$ into an ergodic Gaussian multivariate nonorthogonal multisine random time-series:

Lemma 3.4 *Assuming that:*

1. $\Phi_{\mathbf{vv}}(j\omega T) = [\Phi_{v_r v_s}(j\omega T)]_{r,s=1,2,\ldots,p}$ *is the power spectral density matrix of a wide-sense stationary, real-valued multivariate nonorthogonal random time-series with zero mean vector and the variance matrix* $\sigma^2_{\mathbf{vv}} = [\sigma^2_{v_r v_s}]_{r,s=1,2,\ldots,p}$, *where:*

$$\sigma^2_{v_r v_s} = \frac{1}{2\pi T}\int_0^{2\pi} \Phi_{v_r v_s}(j\omega T)d(\omega T); \quad (3.74)$$

2. *for* $r, t = 1, 2, \ldots, p$ *values of* $A_{r,t,n}$ *(amplitudes of the sine components of the extended NMMRS* $\mathbf{u}(i)$*) converge to 0 for* $N \to \infty$ *in such a way that:*

$$\mathcal{E}\{\Phi^B_{\mathbf{uu}}(j\Omega n)\} = \Phi_{\mathbf{vv}}(j\Omega n), \quad (3.75)$$

where $n = 1, 2, \ldots, \frac{N}{2} - 1$;

3. $A_{r,t,0} = A_{r,t,\frac{N}{2}} = 0$ *or* $\phi_{r,0} = \alpha = 0$ *for* $r, t = 1, 2, \ldots, p$;

then the corresponding extended NMMRS $\mathbf{u}(i)$ *converges in distribution for* $N \to \infty$ *to a Gaussian multivariate nonorthogonal multisine random time-series (GNMMRS)* $\mathbf{g}(i) = [g_r(i)]_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\sigma^2_{\mathbf{vv}}$:

$$\mathbf{g}(i) \in As\mathcal{N}(O, \sigma^2_{\mathbf{vv}}). \quad (3.76)$$

Additionally the correlation function matrix $\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\}$ *of the GNMMRS converges to:*

$$\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\} = \frac{1}{2\pi T}\int_0^{2\pi} \Phi_{\mathbf{vv}}(j\omega T)\cos(\omega T\tau)d(\omega T), \quad (3.77)$$

where $\tau = 0, 1, \ldots, \infty$.

□

Proof: *When for each frequency Ωn $(n = 1,2,\ldots,\frac{N}{2}-1)$ and any time instant i elements the random vector $\mathbf{l}_n(i) = [l_{r,n}(i)]_{r=1,2,\ldots,p}$ are defined as:*

$$l_{r,n}(i) = \frac{\sqrt{NT}}{2}\sum_{t=1}^{p} A_{r,t,n}\sin(\Omega n i + \phi_{t,n} + \varphi_{r,t,n}), \tag{3.78}$$

the proof proceeds similarly as for Lemma 3.2. It should only be noticed that:

$$\Sigma_N^2 = \sum_{n=1}^{\frac{N}{2}-1} \mathcal{E}\{\mathbf{l}_n(i)\mathbf{l}_n^T(i)\} = \frac{NT}{4}\frac{2}{N}\sum_{n=1}^{\frac{N}{2}-1}\Phi_{\mathbf{vv}}(j\Omega n). \tag{3.79}$$

□

The time-domain averaging on any GNMMRS realisation under $N \to \infty$ results in the following lemma:

Lemma 3.5 *Consider the GNMMRS.*

1. *Its periodogram matrix $\Phi^B_{\mathbf{gg}}(j\Omega m)$ for $m = 0,1,\ldots,N-1$ under $N \to \infty$ converges to:*

$$\Phi^B_{\mathbf{gg}}(j\Omega m) = \mathbf{K}(j\Omega m)\Phi^B_{\beta\beta}(j\Omega m)\mathbf{K}^*(j\Omega m), \tag{3.80}$$

where elements of the matrix $\Phi^B_{\beta\beta}(j\Omega m) = \left[\Phi^B_{\beta_r\beta_s}(j\Omega m)\right]_{r,s=1,2,\ldots,p}$ for $r,s = 1,2,\ldots,p$ are given by:

$$\Phi^B_{\beta_r\beta_s}(j\Omega m) = (0+j0)\delta(m) + \sum_{n=1}^{\frac{N}{2}-1}[(\cos(\phi_{r,n}-\phi_{s,n}) - j\sin(\phi_{r,n}-\phi_{s,n}))\delta(m-n)$$
$$+(\cos(\phi_{r,n}-\phi_{s,n}) + j\sin(\phi_{r,n}-\phi_{s,n}))\delta(m-(N-n))] + (0+j0)\delta(m-\frac{N}{2}). \tag{3.81}$$

2. *Elements of its correlation function matrix: $\mathbf{R_{gg}}(\tau) = [R_{g_rg_s}(\tau)]_{r,s=1,2,\ldots,p}$ for $\tau = 0,1,\ldots,\infty$ under $N \to \infty$ converge to:*

$$\lim_{N\to\infty} R_{g_rg_s}(\tau) = R_{v_rv_s}(\tau) \qquad a.s.. \tag{3.82}$$

3. *Its mean value vector $\mathcal{M}\{\mathbf{g}(i)\}$ is equal to $\mathbf{o}$.*

4. *Elements of its variance matrix $\sigma^2_{\mathbf{gg}} = \left[\sigma^2_{g_rg_s}\right]_{r,s=1,2,\ldots,p}$ under $N \to \infty$ converge to:*

$$\lim_{N\to\infty} \sigma^2_{g_rg_s} = \sigma_{v_rv_s} \qquad a.s. \tag{3.83}$$

□

Proof of the above lemma proceeds similarly as for Lemma 3.3.

It follows from Lemma 3.4 and 3.5 that the power spectral density defined NMMRS converges asymptotically for $N \to \infty$ to an ergodic, Gaussian multivariate random process.

To summarise the asymptotic properties of the power spectral density defined multisine random time-series it is worth to emphasise that:

1. The results of lemmas (3.1 ÷ 3.5) still hold if:
   - the zero-frequency phase shifts and Nyquist-frequency distribution parameter α are equal to $\frac{\pi}{2}$, and
   - the sine component amplitudes for constant and Nyquist frequency components are assumed to be chosen so as the approximation criterion is satisfied,

   because amplitudes of the constant and Nyquist frequency components tend to zero for $N \to \infty$.
2. For a given power spectral density matrix of an orthogonal random process the corresponding synthesised MOMRS exhibits an interesting property: its periodogram matrix is a consistent estimator of the true power spectral density matrix.

3.3 CONTINUOUS-TIME RANDOM PROCESS GENERATION

Consider a wide-sense stationary continuous-time band-limited multivariate random process $\mathbf{s}(t)$ with the power spectral density matrix $\Phi_{\mathbf{ss}}(j\omega)$ ($\|\Phi_{\mathbf{ss}}(j\omega)\| < \infty$ for $0 \le \omega < \infty$) and the corresponding autocorrelation function $\mathbf{R_{ss}}(\iota)$ for which:

$$\lim_{\iota\to\infty}\mathbf{R_{uu}}(\iota) = \mathbf{o}. \tag{3.84}$$

In the sequel, the ι_0 denotes a lag beyond which the autocorrelation function matrix $\mathbf{R_{ss}}(\iota)$ may be assumed to be a zero matrix.

The band-limited property of $\mathbf{s}(t)$ implies that there exists such a frequency ω_{max} that for all $\omega > \omega_{max}$ the following assumption is satisfied:

$$\Phi_{\mathbf{ss}}(j\omega) = \mathbf{o}. \tag{3.85}$$

The proposed approach to the continuous random process $\mathbf{s}(t)$ generation is based on the previously defined synthesis and simulation of random time-series.

The power spectral density matrix $\Phi_{\mathbf{ss}}(j\omega)$ is sampled in the frequency-domain (approximated by the periodogram of the corresponding multivariate multisine random time-series) so as to avoid aliasing in the shift-domain of its autocorrelation function. Reversing the sampling theorem it is evident that $\Phi_{\mathbf{ss}}(j\omega)$ can be sampled without producing aliasing if the spacing δ between the samples along the ω axis is such that:

$$\delta \le \frac{\pi}{2\iota_0 T}. \tag{3.86}$$

Choosing the spacing $\delta = \frac{2\pi}{2\iota_0 T}$, the resulting discrete-frequency power spectral density lines $\Phi_{\mathbf{ss}}(j\delta m)$ are then given by:

$$\Phi_{ss}(j\delta m) = \Phi_{ss}(j\omega)|_{\omega=\delta m}. \tag{3.87}$$

The synthesis and simulation of continuous-time random processes follows closely what has been done while synthesising and simulating multisine random time-series $\mathbf{u}(i)$ with specified power spectral densities. The approximation criterion:

$$\Phi_{\mathbf{ss}}(j\delta T m) = \Phi^B_{\mathbf{uu}}(j\Omega m), \tag{3.88}$$

for $m = 0,1,\ldots,\frac{N}{2}$ results in the finite discrete Fourier transform $\mathbf{U}^B(j\Omega m)$ of the corresponding basic N-sample multisine random time-series provided the sampling interval T and the number N of approximation nodes in the relative frequency range $[0,2\pi)$ are properly determined.

It follows from the given bandwidth ω_{max} of $\Phi_{\mathbf{ss}}(j\omega)$ under the sampling theorem that the sampling interval T for $\mathbf{R_{ss}}(\iota)$ is constrained by:

$$T \le \frac{\pi}{\omega_{max}}. \tag{3.89}$$

The sampling interval of the continuous-time random process $\mathbf{s}(t)$ is, also of course, equal to T. The corresponding number N of approximation nodes (the number of discrete-frequency power spectral lines) should be chosen so as to satisfy the assumption (3.4).

Transforming the synthesised spectrum $\mathbf{U}^B(j\Omega m)$ back into the time-domain an N-sample multisine approximation $\mathbf{u}^B(i)$ of the random signal $\mathbf{s}(t)$ sampled with the sampling interval T is obtained. The spectrum of the time-series $\mathbf{u}^B(i)$ is given by periodic repetitions of the continuous-time random process spectrum. These periodic repetitions do nor overlap and accurate reconstruction of continuous-time random process is possible. Additional $\mathbf{s}(t)$ values in between the sampling intervals (needed e. g. for numerical integration) can be calculated by using the sinc interpolation. This interpolation can be interpreted as a filtering, of a series of rectangular pulses spaced T seconds apart, with the area under each pulse equal to the amplitude of the corresponding sample, by an analog reconstruction filter with the impulse response:

$$k(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}}. \tag{3.90}$$

This reconstruction filter is noncausal and therefore physically unrealizable in real time. Its good approximations are high-orders analog low-pass filters with sharp cutoff characteristics. Such reconstruction is called band-limited interpolation [61].

It should be noticed that continuous-time multisine approximations of wide-sense stationary band-limited continuous-time random processes inherit properties of the corresponding multisine random time-series - their stationarity, ergodicity or nonergodicity, asymptotic Gaussianess and ergodicity.

Chapter 4

Multisine White Noise Approximation

This chapter is devoted to multisine white noise approximations obtained by using the proposed random process synthesis and simulation method. The following cases are discussed:

- weakly ergodic scalar and bivariate white multisine random time-series for which whiteness holds for finite N-sample representations. Its pseudo-white and asymptotically Gaussian cases are introduced, too;
- weakly ergodic multivariate orthogonal asymptotically Gaussian and white multisine random time-series which is obtained by approximating the power spectral density matrix of multivariate white noise by the periodogram matrix of a weakly ergodic multivariate multisine random time-series with the number of elements greater than 2;
- nonergodic multivariate orthogonal pseudo-white multisine random time-series which is asymptotically ergodic and Gaussian. It is synthesised using the corresponding nonergodic multivariate orthogonal multisine random time-series.

4.1 SCALAR WHITE MULTISINE RANDOM TIME-SERIES

N-lag white multisine random time-series

When the power spectral density of a scalar white noise is approximated, the corresponding extended SMRS can be turned into an extended white SMRS [32] for which the autocorrelation function behaves for lags $0, 1, \ldots, N-1$ as a pure white noise autocorrelation function. This time-series is called N-lag white multisine random time-series (WSMRS):

Definition 4.1 *An extended scalar multisine random time-series $x(i)$ is said to be N-lag white if its autocorrelation function for lags $\tau = 0, 1, \ldots, N-1$ is the same as the white noise autocorrelation function, i. e.:*

$$\mathcal{E}\{x(i)x(i-\tau)\} = R_{xx}(\tau) = \begin{cases} \Gamma^2 & \text{if } \tau = 0 \\ 0 & \text{if } \tau = 1, 2, \ldots, N-1 \end{cases}. \tag{4.1}$$

□

For any real random process simulation with WSMRS, the N-lag white multisine random time-series seems to be as good as the pure white noise series, because there exists a possibility to establish the necessary length N for which the autocorrelation function values have to be equal to 0.

The statistical properties of the WSMRS are given by the following lemma:

Lemma 4.1 *Assuming that:*

1. $\Phi_{vv}(\omega T) = \lambda^2$ $(\omega T \in [0, 2\pi))$ *is the power spectral density of a real-valued white noise;*

2. $A_n = A$ *for* $n = 1, 2\ldots, \frac{N}{2} - 1$ *and the value of* A *is chosen so that:*

$$\frac{NTA^2}{4} = \lambda^2; \tag{4.2}$$

3. $A_0 = A_{\frac{N}{2}} = \frac{A}{2}$ *and* $\phi_0 = \alpha = \frac{\pi}{2}$;

then the corresponding extended SMRS is a white multisine random time-series (WSMRS) and:

1. *its periodogram is:*

$$\Phi^B_{uu}(\Omega m) = \lambda^2 \sum_{n=0}^{N-1} \delta(m-n), \tag{4.3}$$

where $m = 0, 1, \ldots, N-1$.

2. *its mean value is:*

$$\mathcal{M}\{u(i)\} = \mathcal{E}\{u(i)\} = \sqrt{\frac{1}{NT}}\lambda^2. \tag{4.4}$$

3. *its autocorrelation function is:*

$$\mathcal{E}\{u(i)u(i-\tau)\} = R_{uu}(\tau) = \begin{cases} \frac{\lambda^2}{T} & \text{if } \tau = 0, N, \ldots \\ 0 & \text{otherwise} \end{cases}. \tag{4.5}$$

4. *its variance is:*

$$\sigma^2_{uu} = \frac{N-1}{N}\frac{\lambda^2}{T}. \tag{4.6}$$

□

Proof:

1. *It follows immediately from the assumptions 2, 3 and from Lemma 2.2.*
2. *It follows immediately from Definition 2.1.*
3. *Application of (2.33) and (4.3) results in:*

$$\mathcal{E}\{u(i)u(i-\tau)\} = R_{uu}(\tau) = \frac{A^2}{4}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\delta(m-n)e^{j\Omega m\tau} = \frac{A^2}{4}\sum_{n=0}^{N-1} e^{j\Omega n\tau}. \tag{4.7}$$

It ends the proof when (2.5) is taken into account.

4. *It follows from the WSMRS autocorrelation function.*

□

It is surprising that the WSMRS has a constant mean value and its autocorrelation function is equal to 0 for all lags τ different from $\tau = 0, N, \ldots$. This is a consequence of choosing the WSMRS constant component amplitude $A_0 = \frac{A}{2}$ and the corresponding phase shift $\phi_0 = \frac{\pi}{2}$.

It should be emphasised that the property of whiteness (4.5) depends upon the fact that the WSMRS has the constant frequency bin Ω throughout the entire frequency range

$[0, 2\pi)$. It is also worth to note that whiteness of the WSMRS holds for finite N-sample time-series but Gaussianess of the time-series is an asymptotic property for $N \to \infty$.

It follows from Lemma 4.1 that for $N \to \infty$, the corresponding extended WSMRS $u(i)$ converges in distribution to a Gaussian WSMRS (GWSMRS) $g(i)$ with zero mean and the variance $\frac{\lambda^2}{T}$:

$$g(i) \in As\mathcal{N}(0, \frac{\lambda^2}{T}). \tag{4.8}$$

N-lag pseudo-white multisine random time-series

The choice of the WSMRS mean value as equal to zero ($A_0 = 0$) influences behaviour of its autocorrelation function resulting in an N-lag pseudo-white multisine random time-series (PWSMRS).

Definition 4.2 *An extended scalar multisine random time-series* $x(i)$ *is said to be* N*-lag pseudo-white if its autocorrelation function for lags* $\tau = 0, 1, \ldots, N-1$ *satisfies the condition:*

$$\mathcal{E}\{x(i)x(i-\tau)\} = R_{xx}(\tau) = \begin{cases} \Gamma^2 & \text{if } \tau = 0 \\ \gamma(\tau)\Gamma^2 & \text{if } \tau = 1, 2, \ldots, N-1 \end{cases}, \tag{4.9}$$

where $|\gamma(\tau)| \ll 1$.

□

The following lemma can be formulated:

Lemma 4.2 *Assuming that:*

1. $\Phi_{vv}(\omega T) = \lambda^2$ $(\omega T \in [0, 2\pi))$ *is the power spectral density of a real-valued white noise;*

2. $A_n = A$ *for* $n = 1, 2\ldots, \frac{N}{2} - 1$ *and the value of* A *is chosen so that:*

$$\frac{NTA^2}{4} = \lambda^2; \tag{4.10}$$

3. $A_0 = 0$ *or* $\phi_0 = 0$, $A_{\frac{N}{2}} = \frac{A}{2}$ *and* $\alpha = \frac{\pi}{2}$;

then the corresponding extended SMRS is a pseudo-white multisine random time-series of type 1 (PWSMRS1) and:

1. *Its periodogram is:*

$$\Phi_{uu}(\Omega m) = \lambda^2 \sum_{n=1}^{N-1} \delta(m-n), \tag{4.11}$$

where $m = 0, 1, \ldots, N-1$.

2. *Its mean value is* $\mathcal{M}\{u(i)\} = \mathcal{E}\{u(i)\} = 0$.

3. *Its autocorrelation function is:*

$$\mathcal{E}\{u(i)u(i-\tau)\} = R_{uu}(\tau) = \begin{cases} \frac{N-1}{N}\frac{\lambda^2}{T} & \text{if } \tau = 0, N, \ldots \\ -\frac{1}{N}\frac{\lambda^2}{T} & \text{otherwise} \end{cases}. \tag{4.12}$$

4. *Its variance is:*

$$\sigma^2_{uu} = \frac{N-1}{N}\frac{\lambda^2}{T}. \tag{4.13}$$

□

Proof of the above lemma proceeds similarly as for Lemma 4.1.

Another type of pseudo-white multisine random time-series may be defined by taking additionally $A_{\frac{N}{2}} = 0$ or $\alpha = 0$.

Lemma 4.3 *Assuming that:*

1. $\Phi_{vv}(\omega T) = \lambda^2$ $(\omega T \in [0, 2\pi))$ *is the power spectral density of a real-valued white noise;*
2. $A_n = A$ *for* $n = 1, 2\ldots, \frac{N}{2} - 1$ *and the value of* A *is chosen so that:*

$$\frac{NTA^2}{4} = \lambda^2; \tag{4.14}$$

3. $A_0 = A_{\frac{N}{2}} = 0$ *or* $\phi_0 = \alpha = 0$,

then the corresponding extended SMRS is a pseudo-white multisine random time-series of type 2 (PWSMRS2) and:

1. *Its periodogram is:*

$$\Phi_{uu}(\Omega m) = \lambda^2 \sum_{n=1}^{\frac{N}{2}-1} [\delta(m-n) + \delta(m-(N-n))], \tag{4.15}$$

where $m = 0, 1, \ldots, N-1$.

2. *Its mean value is* $\mathcal{M}\{u(i)\} = \mathcal{E}\{u(i)\} = 0$.
3. *Its autocorrelation function is given by:*

$$\mathcal{E}\{u(i)u(i-\tau)\} = R_{uu}(\tau) = \begin{cases} \frac{N-2}{N}\frac{\lambda^2}{T} & \text{if } \tau = 0, N, \ldots \\ -\frac{1}{N}[1+(-1)^\tau]\frac{\lambda^2}{T} & \text{otherwise} \end{cases}. \tag{4.16}$$

4. *Its variance is:*

$$\sigma_{uu}^2 = \frac{N-2}{N}\frac{\lambda^2}{T}. \tag{4.17}$$

□

Proof of the above lemma proceeds similarly as for Lemma 4.1.

Mean values of the PWSMRS1 and PWSMRS2 are both equal to 0 but their autocorrelation functions are not equal to 0 for all lags different from integer multiplicity of N. For these lags:

- the autocorrelation function of PWSMRS1 has a small constant value and
- the autocorrelation function of PWSMRS2 exhibits oscillatory behaviour with small amplitudes.

When $N \to \infty$:

- either the PWSMRS1 and PWSMRS2 autocorrelation function values tend to zero for all lags $\tau > 0$;
- either the PWSMRS1 and PWSMRS2 converge in distribution for $N \to \infty$ to the Gaussian WSMRS (GWSMRS) $g(i)$ with zero mean and the variance $\frac{\lambda^2}{T}$:

$$g(i) \in As\mathcal{N}(0, \frac{\lambda^2}{T}). \tag{4.18}$$

Example 4.1 *Using a standard linear congruential random number generator with shuffling [82], 100 different 128-sample equally distributed on* $[0, 2\pi)$ *white noise time-series realisations* $\phi_k^{128}(i)$, $(k = 1, 2, \ldots, 100)$ *were generated, see Fig. 4.1 for an example. The unbiased estimate of the autocorrelation function and unwindowed periodogram of this time-series realisation with removed mean value are shown respectively in Figs 4.2 and 4.3.*

Autoregressive models:

$$\phi_k^{128}(i) = \frac{1}{1 + a_{k,1}z^{-1}} e_k(i), \tag{4.19}$$

where $e_k(i)$ *is a hypothetical white noise with the variance* σ_k^2, *were fitted to all 128-sample time-series realisations* $\phi_k^{128}(i)$ *using the Least Squares identification method [13]. The mean values and standard deviations of the* $\hat{a}_{k,1}$ *and* $\hat{\sigma}_k^2$ *estimates are shown in Tab. 4.1.*

Table 4.1

Time-series	*Parameter Estimates* $\hat{a}_{k,1}$	$\hat{\sigma}_k$
True white noise	*0.000*	*1.000*
$\phi_k^{128}(i)$	$-4.63 \cdot 10^{-3}$ $(9.32 \cdot 10^{-2})$	0.990 $(1.30 \cdot 10^{-1})$
$u_k^{128}(i)$	$-6.71 \cdot 10^{-4}$ $(0.83 \cdot 10^{-2})$	1.008 $(1.76 \cdot 10^{-4})$
$x_k^{128}(i)$	$-6.50 \cdot 10^{-3}$ $(0.77 \cdot 10^{-2})$	0.998 $(2.24 \cdot 10^{-4})$
$g_k^{128}(i)$	$-2.46 \cdot 10^{-4}$ $(0.85 \cdot 10^{-2})$	0.992 $(1.77 \cdot 10^{-4})$

Mean values and standard deviations (in parentheses) for 100 parameter estimations of AR models for an equally distributed on $[0, 2\pi)$ white noise generator $\phi_k^{128}(i)$, WSMRS $u_k^{128}(i)$, PWSMRS1 $x_k^{128}(i)$ and PWSMRS2 $g_k^{128}(i)$; $k = 1, 2, \ldots, 100$

Table 4.2

Time-series	*Parameter Estimates* $\hat{a}_{k,1}$	$\hat{a}_{k,2}$	$\hat{\sigma}_k$
Simulated	*-1.500*	*0.700*	*1.000*
$y_k^{128}(i)$	-1.499 $(1.20 \cdot 10^{-2})$	0.699 $(1.34 \cdot 10^{-2})$	1.007 $(8.22 \cdot 10^{-4})$
$s_k^{128}(i)$	-1.496 $(1.20 \cdot 10^{-2})$	0.703 $(1.35 \cdot 10^{-2})$	0.998 $(8.42 \cdot 10^{-4})$
$t_k^{128}(i)$	-1.499 $(1.18 \cdot 10^{-2})$	0.703 $(1.30 \cdot 10^{-2})$	0.991 $(8.14 \cdot 10^{-4})$

Mean values and standard deviations (in parentheses) for 100 parameter estimations of second-order AR models excited by WSMRS $(y_k^{128}(i))$, PWSMRS1 $(s_k^{128}(i))$ and PWSMRS2 $(t_k^{128}(i))$; $k = 1, 2, \ldots, 100$

This standard linear congruential random number generator with shuffling was used to generate 63-sample time-series realisations applied as phase shifts to synthesise hundred 128-sample WSMRS realisations $u_k^{128}(i)$, $(k = 1, 2, \ldots, 100)$ *with the variance* 1. *An example of the resulting 128-sample WSMRS realisation, the unbiased estimate of its autocorrelation function and unwindowed periodogram are shown in Figs 4.1, 4.2 and 4.3, respectively.*

Autoregressive models were fitted to all 128-sample time-series realisations $u_k^{128}(i)$ *using the Least Squares identification method and the AIC criterion [56] to determine the models order. The results of this simulated identifications are presented in Tabs 4.1 and 4.2.*

All WSMRS realisations were used to generate 100 second-order AR time-series realisations $y_k^{128}(i)$ $(k = 1, 2, \ldots, 100)$:

$$y_k^{128}(i) = \frac{1.000}{1.000 - 1.500z^{-1} + 0.700z^{-2}} u_k^{128}(i). \tag{4.20}$$

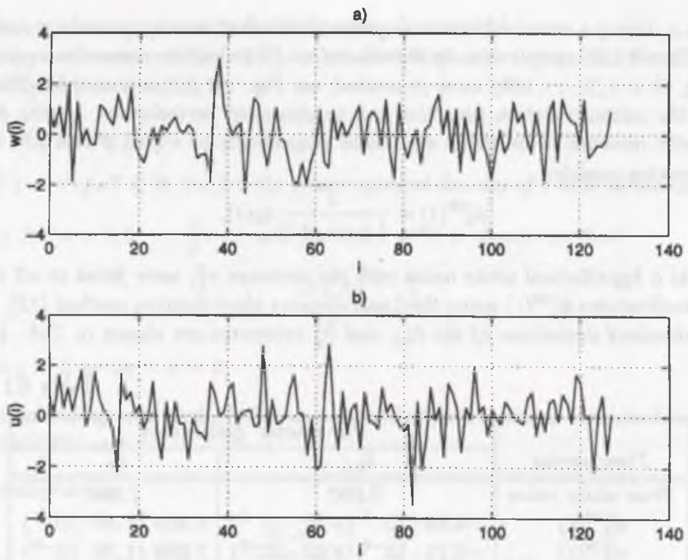

Fig. 4.1. A 128-sample white noise time-series realisation generated by using a standard white noise generator (a) and a WSMRS (b)

The models:

$$y_k^{128}(i) = \frac{1.000}{1.000 + a_{k,1}z^{-1} + a_{k,2}z^{-2}} e_k^{'}(i) \tag{4.21}$$

with $e_k^{'}(i)$ being another hypothetical white noise with the variance $\sigma^2_{1,k}$ were fitted to the time-series using the Least Squares identification method [13]. The mean values and standard deviations for estimates of the model parameters $\hat{a}_{k,1}$, $\hat{a}_{k,2}$ and $\hat{\sigma}^2_k$ are shown in Tab. 4.2.

This simulation and identification experiments were repeated for the 128-sample realisations of PWSMRS1 and PWSMRS2 with variances equal to 1. Examples of unbiased estimates of their autocorrelation functions (Fig. 4.2) show that correlation properties of the resulting PWSMRS1 and PWSMRS2 are indistinguishable from the properties of the WSMRS. The presence of correlations is apparent from the results of fitting first-order AR models to the 100 realisations of PWSMRS1 or PWSMRS2 (Tab. 4.1). All 128-sample PWSMRS1 and PWSMRS2 realisations were also used to generate the corresponding hundred second-order AR time-series realisations with parameters as in (4.20). The obtained results (Tab. 4.2) differ only slightly from those achieved for WSMRS.

The Gaussianess is an asymptotic property of WSMRS amplitudes for any given time instant i which holds for $N \to \infty$. This feature of GWSMRS was tested by finding biased normalised autocorrelation estimates for WSMRS realisations at the time instant $i = 0$ for the case of finite N. In order not to invoke the ergodicity assumption, averaging was not performed in time but in the sample space.

Normalised biased autocorrelation function estimates for 100 segments of M ($M = 32, 64, 128, 256, 512$) realisations of the $N = M$-sample WSMRS for the time instant $i = 0$ have been calculated. In each segment the samples were numbered from 0 to $M - 1$, its lag being the shift between the sample numbers. In Tab. 4.3 the mean square values of autocorrelation estimates for lags 1 and 2 calculated on the basis of this 100 data segments are compared with the values of the hypothetical variance for white noise. For real-valued white noise time-series of the length M such tests result in the autocorrelation functions for all lags asymptotically normally distributed with zero mean and with the variance $\frac{1}{M}$.

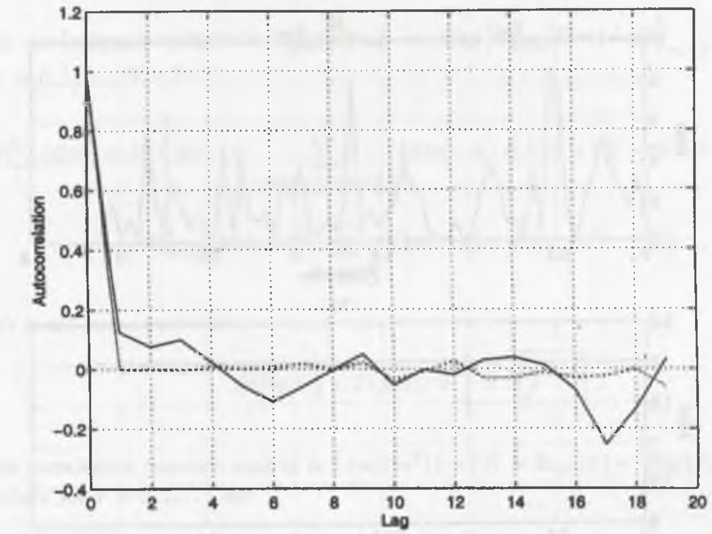

Fig. 4.2. Unbiased estimates of the autocorrelation function for 128-sample white noise time-series realisations generated with a standard white noise generator (solid line), WSMRS, PWSMRS1 and PWSMRS2 (dotted lines)

Table 4.3

M	$\frac{1}{M}$	*Variance of autocorrelation*	
		Lag1	*Lag2*
32	0.0312	0.0370	0.0298
64	0.0156	0.0152	0.0146
128	0.0078	0.0069	0.0071
256	0.0039	0.0037	0.0040
512	0.0019	0.0016	0.0017

Mean square values for biased autocorrelation function estimates of WSMRS realisations for the time instant $i = 0$ for 100 simulation experiments

The small difference between the calculated and hypothetical values should be noted. It implies that Gaussianess is very good approximated for even small ones as well as for large number of samples by white multisine random time-series.

4.2 MULTIVARIATE MULTISINE WHITE NOISE

4.2.1 Ergodic Case

Bivariate $\frac{N}{2}$-lag white multisine random time-series

When the power spectral density matrix of a bivariate white noise is approximated by the periodogram matrix of the extended BOMRS, a bivariate orthogonal white multisine random time-series [66] is obtained. It is characterised by the autocorrelation function matrix which for a number of lags behaves exactly like correlation function matrix of the bivariate white noise. This time-series is called bivariate $\frac{N}{2}$-lag white multisine random time-series (BOWMRS):

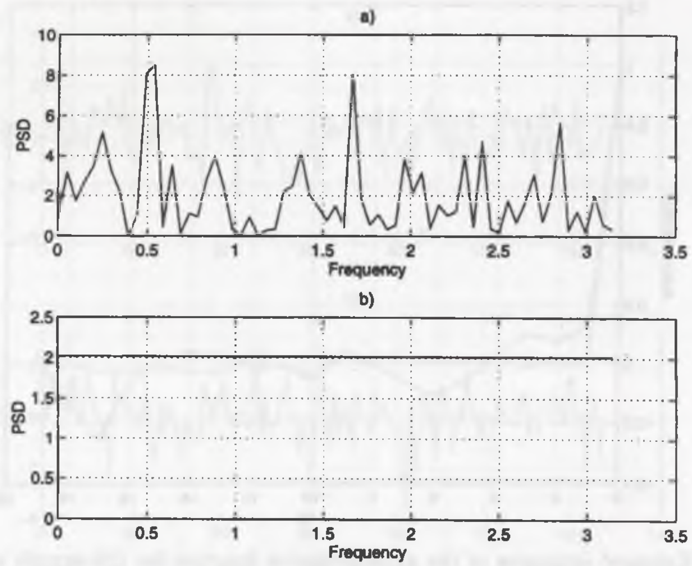

Fig. 4.3. Unwindowed periodograms for 128-sample white noise time-series realisations generated with a standard white noise generator (a) and WSMRS (b)

Definition 4.3 *An extended bivariate orthogonal multisine random time-series* $\mathbf{x}(i)$ *is said to be* $\frac{N}{2}$*-lag white if its correlation function matrix* $\mathcal{E}\{\mathbf{x}(i)\mathbf{x}^T(i-\tau)\} = \mathbf{R}_{\mathbf{xx}}(\tau) = [R_{x_rx_s}(\tau)]_{r,s=1,2}$ *for lags* $\tau = 0, 1, \ldots, \frac{N}{2} - 1$ *is the same as for bivariate white noise correlation function matrix – its elements satisfy the conditions:*

$$R_{x_1x_1}(\tau) = R_{x_2x_2}(\tau) = \begin{cases} \Gamma & \text{if } \tau = 0 \\ 0 & \text{if } \tau = 1, 2, \ldots, \frac{N}{2} - 1 \end{cases}, \tag{4.22}$$

$$R_{x_1x_2}(\tau) = R_{x_2x_1}(\tau) = 0. \tag{4.23}$$

□

The spectral and correlation properties of the BOWMRS are:

Lemma 4.4 *Assuming that:*

1. $\Phi_{\mathbf{vv}}(j\omega T) = \lambda^2 I$ $(\omega T \in [0, 2\pi))$ *is the power spectral density matrix of a real-valued bivariate white noise;*
2. $A_n = A$ *for* $n = 1, 2, \ldots, \frac{N}{2} - 1$ *and the value of* A *is chosen so that:*

$$\frac{NTA^2}{4} = \lambda^2; \tag{4.24}$$

3. $A_0 = A_{\frac{N}{2}} = \frac{A}{2}$ *and* $\phi_0 = \alpha = \frac{\pi}{2}$;

then the corresponding extended BOMRS is a bivariate $\frac{N}{2}$*-lag white multisine random time-series (BOWMRS) and:*

1. *its periodogram matrix is* $\Phi^B_{\mathbf{uu}}(j\Omega m) = diag\left[\Phi^B_{u_ru_r}(\Omega m) + j0\right]_{r=1,2}$, *where for* $m = 0, 1, \ldots, N-1$:

$$\Phi^B_{u_1u_1}(\Omega m) = \lambda^2 \left\{ \delta(m) + \sum_{\Omega n \in \mathcal{N}^{c,1}_{1,2} \setminus \{0, \Omega\frac{N}{2}\}} [\delta(m-n) + \delta(m-(N-n))] + \delta(m - \frac{N}{2}) \right\}, \tag{4.25}$$

$$\Phi^B_{u_2u_2}(\Omega m) = \lambda^2 \sum_{\Omega n \in \mathcal{N}^{c,1}_{2,2}} [\delta(m-n) + \delta(m-(N-n))], \tag{4.26}$$

2. *its mean value vector is*

$$\mathcal{M}\{\mathbf{u}(i)\} = \mathcal{E}\{\mathbf{u}(i)\} = \begin{bmatrix} \sqrt{\frac{1}{NT}\lambda^2} \\ 0 \end{bmatrix}. \tag{4.27}$$

3. *its correlation function matrix is* $\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\} = \mathbf{R}_{\mathbf{uu}}(\tau) = diag[R_{u_ru_r}(\tau)]_{r=1,2}$, *where for* $\tau = 0, 1, \ldots, \infty$:

$$R_{u_1u_1}(\tau) = \begin{cases} \frac{\lambda^2}{2T} & \text{if } \tau = 0, \frac{N}{2}, \ldots \\ 0 & \text{otherwise} \end{cases}, \tag{4.28}$$

$$R_{u_2u_2}(\tau) = \begin{cases} \frac{\lambda^2}{2T} & \text{if } \tau = 0, N, \ldots \\ -\frac{\lambda^2}{2T} & \text{if } \tau = \frac{N}{2}, \frac{3N}{2}, \ldots \\ 0 & \text{otherwise} \end{cases}. \tag{4.29}$$

4. *its variance matrix is given by:*

$$\sigma^2_{\mathbf{uu}} = \frac{\lambda^2}{2T} diag\left[\frac{N-2}{N}, 1\right]. \tag{4.30}$$

□

Proof: *This lemma can be proven similarly as Lemma 4.1 when it is noticed that for* n_e *zero or even holds:*

$$\sum_{n_e=0}^{N-2} e^{j\Omega n_e \tau} = \begin{cases} \frac{N}{2} & \text{if } \tau = 0, \frac{N}{2}, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{4.31}$$

and that for n_o *odd holds:*

$$\sum_{n_o=1}^{N-1} e^{j\Omega n_o \tau} = \begin{cases} \frac{N}{2} & \text{if } \tau = 0, N, \ldots \\ \frac{-N}{2} & \text{if } \tau = \frac{N}{2}, \frac{3N}{2}, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{4.32}$$

so that:

$$\sum_{n=0}^{N-1} e^{j\Omega n \tau} = \sum_{n_o=1}^{N-1} e^{j\Omega n_o \tau} + \sum_{n_e=0}^{N-2} e^{j\Omega n_e \tau}. \tag{4.33}$$

□

When $N \to \infty$, the variance matrix of the BOWMRS converges to $\frac{\lambda^2}{2T} I$ and its mean value vector tends to a zero vector.

Bivariate $\frac{N}{2}$-lag pseudo-white multisine random time-series

Each of the BOWMRS elements is an $\frac{N}{2}$-lag WSMRS. When values of its zero- and (or) Nyquist-frequency amplitudes or the corresponding phase shifts are chosen as equal to 0, the resulting BOMRS becomes an $\frac{N}{2}$-lag orthogonal pseudo-white multisine random time-series.

Definition 4.4 *An extended bivariate orthogonal multisine random time-series $\mathbf{x}(i)$ is said to be $\frac{N}{2}$-lag pseudo-white if elements of its correlation function matrix $\mathcal{E}\{\mathbf{x}(i)\mathbf{x}^T(i-\tau)\} = \mathbf{R}_{\mathbf{xx}}(\tau) = [R_{x_r x_s}(\tau)]_{r,s=1,2}$ for lags $\tau = 0,1,\ldots,\frac{N}{2}-1$ satisfy the conditions:*

$$R_{x_r x_r}(\tau) = \begin{cases} \Gamma^2 & \text{if } \tau = 0 \\ \gamma_1(\tau)\Gamma^2 & \text{if } \tau = 1,2,\ldots,\frac{N}{2}-1 \end{cases}, \tag{4.34}$$

where $r = 1,2$, $|\gamma_r(\tau)| \ll 1$ and:

$$R_{x_1x_2}(\tau) = R_{x_2x_1}(\tau) = 0. \tag{4.35}$$

□

The properties of the BOWMRS for the case of $A_0 = 0$ or $\phi_0 = 0$ are given by:

Lemma 4.5 *Assuming that:*

1. *$\Phi_{\mathbf{vv}}(j\omega T) = \lambda^2 I$ ($\omega T \in [0,2\pi)$) is the power spectral density matrix of a real-valued bivariate white noise;*
2. *$A_n = A$ for $n = 1,2,\ldots,\frac{N}{2}-1$ and the value of A is chosen so that:*
$$\frac{NTA^2}{4} = \lambda^2; \tag{4.36}$$
3. *$A_0 = 0$ or $\phi_0 = 0$ and $A_{\frac{N}{2}} = \frac{A}{2}$ and $\alpha = \frac{\pi}{2}$,*

then the corresponding extended BOMRS is a bivariate $\frac{N}{2}$-lag pseudo-white multisine random time-series of type 1 (BOPWMRS1) and:

1. *its periodogram matrix is $\Phi^B_{\mathbf{uu}}(j\Omega m) = diag\left[\Phi^B_{u_r u_r}(\Omega m) + j0\right]_{r=1,2}$, where for $m = 0,1,\ldots,N-1$:*
$$\Phi^B_{u_1u_1}(\Omega m) = \lambda^2 \sum_{\Omega n \in \mathcal{N}^{c,1}_{1,2}\setminus\{0,\pi\}} [\delta(m-n) + \delta(m-(N-n))] + \lambda^2\delta(m-\frac{N}{2}), \tag{4.37}$$
$$\Phi^B_{u_2u_2}(\Omega m) = \lambda^2 \sum_{\Omega n \in \mathcal{N}^{c,1}_{2,2}} [\delta(m-n) + \delta(m-(N-n))], \tag{4.38}$$
2. *its mean value vector is $\mathcal{M}\{\mathbf{u}(i)\} = \mathcal{E}\{\mathbf{u}(i)\} = \mathbf{o}$.*
3. *its correlation function matrix is $\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\} = \mathbf{R}_{\mathbf{uu}}(\tau) = diag[R_{u_r u_r}(\tau)]_{r=1,2}$, where for $\tau = 0,1,\ldots,\infty$:*
$$R_{u_1u_1}(\tau) = \begin{cases} \frac{N-2}{N}\frac{\lambda^2}{2T} & \text{if } \tau = 0,\frac{N}{2},\ldots \\ -\frac{1}{N}\frac{\lambda^2}{T} & \text{otherwise} \end{cases}, \tag{4.39}$$
$$R_{u_2u_2}(\tau) = \begin{cases} \frac{\lambda^2}{2T} & \text{if } \tau = 0,N,\ldots \\ -\frac{\lambda^2}{2T} & \text{if } \tau = \frac{N}{2},\frac{3N}{2},\ldots \\ 0 & \text{otherwise} \end{cases}. \tag{4.40}$$

4. *its variance matrix is given by:*
$$\sigma^2_{\mathbf{uu}} = \frac{\lambda^2}{2T} diag\left[\frac{N-2}{N}, 1\right]. \tag{4.41}$$

□

Proof of the above lemma proceeds similarly as for Lemma 4.4.

A second type of bivariate pseudo-white multisine random time-series can be obtained assuming additionally that $A_{\frac{N}{2}} = 0$ or $\alpha = 0$:

Lemma 4.6 *Assuming that:*

1. *$\Phi_{\mathbf{vv}}(j\omega T) = \lambda^2 I$ ($\omega T \in [0,2\pi)$) is the power spectral density matrix of a real-valued bivariate white noise;*
2. *$A_n = A$ for $n = 1,2,\ldots,\frac{N}{2}-1$ and the value of A is chosen so that:*
$$\frac{NTA^2}{4} = \lambda^2; \tag{4.42}$$
3. *$A_0 = A_{\frac{N}{2}} = 0$ or $\phi_0 = \alpha = 0$;*

then the corresponding extended BOMRS is a bivariate $\frac{N}{2}$-lag pseudo-white multisine random time-series of type 2 (BOPWMRS2) and:

1. *its periodogram matrix is $\Phi^B_{\mathbf{uu}}(j\Omega m) = diag\left[\Phi^B_{u_r u_r}(\Omega m) + j0\right]_{r=1,2}$, where for $m = 0,1,\ldots,N-1$:*
$$\Phi^B_{u_1u_1}(\Omega m) = \lambda^2 \sum_{\Omega n \in \mathcal{N}^{c,1}_{1,2}\setminus\{0,\pi\}} [\delta(m-n) + \delta(m-(N-n))], \tag{4.43}$$
$$\Phi^B_{u_2u_2}(\Omega m) = \lambda^2 \sum_{\Omega n \in \mathcal{N}^{c,1}_{2,2}} [\delta(m-n) + \delta(m-(N-n))], \tag{4.44}$$
2. *its mean value vector is $\mathcal{M}\{\mathbf{u}(i)\} = \mathcal{E}\{\mathbf{u}(i)\} = \mathbf{o}$.*
3. *its correlation function matrix is $\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\} = \mathbf{R}_{\mathbf{uu}}(\tau) = diag[R_{u_r u_r}(\tau)]_{r=1,2}$, for $\tau = 0,1,\ldots,\infty$:*
$$R_{u_1u_1}(\tau) = \begin{cases} \frac{N-4}{N}\frac{\lambda^2}{2T} & \text{if } \tau = 0,\frac{N}{2},\ldots \\ -\frac{1}{N}[1+(-1)^\tau]\frac{\lambda^2}{2T} & \text{otherwise} \end{cases}, \tag{4.45}$$
$$R_{u_2u_2}(\tau) = \begin{cases} \frac{\lambda^2}{2T} & \text{if } \tau = 0,N,\ldots \\ -\frac{\lambda^2}{2T} & \text{if } \tau = \frac{N}{2},\frac{3N}{2},\ldots \\ 0 & \text{otherwise} \end{cases}. \tag{4.46}$$
4. *its variance matrix is:*
$$\sigma^2_{\mathbf{uu}} = \frac{\lambda^2}{2T} diag\left[\frac{N-4}{N}, 1\right]. \tag{4.47}$$

□

Proof of the above lemma proceeds similarly as for Lemma 4.4.

When $N \to \infty$, either the BOPWMRS1 and BOPWMRS2 correlation function matrices tend to zero matrices for all lags $\tau > 0$.

Example 4.2 *To generate a bivariate orthogonal $\frac{N}{2}$-lag white multisine random time-series realisations, two series of random phase shifts realisations are necessary. However, the randomness of those series can rather be poor and they don't need to be orthogonal: the proposed approach is randomising the $[u_1^N(i), u_2^N(i)]^T$ outcome mainly thanks to the flatness of the spectra and orthogonalising it thanks to the choice of frequencies present in each time-series. This effect is particularly striking for short time-series.*

Table 4.4

Identification Method	*Parameter Estimates*	
	$\hat{a}_{k,1}^{\phi,u_1}$	$\hat{a}_{k,1}^{\phi,u_2}$
LS	$4.30\cdot10^{-3}$ (0.130)	$1.57\cdot10^{-3}$ (0.148)
AC	$4.45\cdot10^{-3}$ (0.130)	$1.51\cdot10^{-3}$ (0.145)

Mean values and standard deviations (in parenthesis) for 100 parameter estimations of AR models computed via the Least Squares (LS) and via normalised autocorrelation function (AC) for white noise realisations $\phi_{k,u_1}^{64}(n)$ and $\phi_{k,u_2}^{64}(n)$ $(k = 1, 2, \ldots, 100)$ generated by using the standard linear congruential random number generator Ran1

A standard random number generator Ran1 with shuffling [82], was used to generate 100 *different 128-sample equally distributed on* $[0, 2\pi)$ *white noise time-series realisations, the first half to be used as the* $\phi_{k,u_1}^{64}(n)$ *phase shifts, the second half to be used as the* $\phi_{k,u_2}^{64}(n)$ *phase shifts* $(k = 1, 2, \ldots, 100)$. *The diagrams of unbiased autocorrelation functions and cross-correlation function estimates for two 64-sample phase shifts realisations (Fig. 4.4) are presented in Figs 4.5 and 4.6, respectively.*

To each of the 100 *different 64-sample subseries the following autoregressive time-series*

$$\phi_k^{64u_1}(n) = \frac{1}{1 + a_{k,1}^{\phi,u_1} z^{-1}} e_k^{\phi,u_1}(n), \tag{4.48}$$

$$\phi_k^{64u_2}(n) = \frac{1}{1 + a_{k,1}^{\phi,u_2} z^{-1}} e_k^{\phi,u_2}(n) \tag{4.49}$$

with $e_k^{\phi,u_1}(n)$ *and* $e_k^{\phi,u_2}(n)$ *being hypothetical white noises were fitted using the Least Squares identification method [13]. The normalised autocorrelation approach (AC) was used in turn to determine another batch of estimates for the AR coefficients. The mean values and standard deviations of the* $a_{k,1}^{\phi,u_1}$ *and* $a_{k,1}^{\phi,u_2}$ $(k = 1, 2, \ldots, 100)$ *parameter estimates derived by both approaches are shown in Tab. 4.4. Both phase shift series were in turn used to generate* 100 *different 128-sample BOWMRS realisations* $\mathbf{u}_k^{128}(i)$ $(k = 1, 2, \ldots, 100)$.

Unbiased autocorrelation functions and cross-correlation function estimates for a realisation of the BOWMRS (Fig. 4.4) are shown in Figs 4.5 and 4.6, respectively.

To each of 100 *different 128-sample bivariate 64-lag white multisine random time-series realisations an autoregressive model was fitted using the Least Squares identification method and the AIC criterion [56] for order determination. It always resulted in first-order AR models:*

$$u_{k,1}^{128}(i) = \frac{1}{1 + a_{k,1}^{u_1} z^{-1}} e_k^{u_1}(i), \tag{4.50}$$

$$u_{k,2}^{128}(i) = \frac{1}{1 + a_{k,1}^{u_2} z^{-1}} e_k^{u_2}(i), \tag{4.51}$$

where $e_k^{u_1}(i)$ *and* $e_k^{u_2}(i)$ *are hypothetical white noises. The normalised autocorrelation approach (AC) was used in turn to determine another batch of estimates for the AR coefficients. The mean values and standard deviations of the* $a_{k,1}^{u_1}$ *and* $a_{k,1}^{u_2}$ $(k = 1, 2, \ldots, 100)$ *parameter estimates derived by both approaches are shown in Tab. 4.5.*

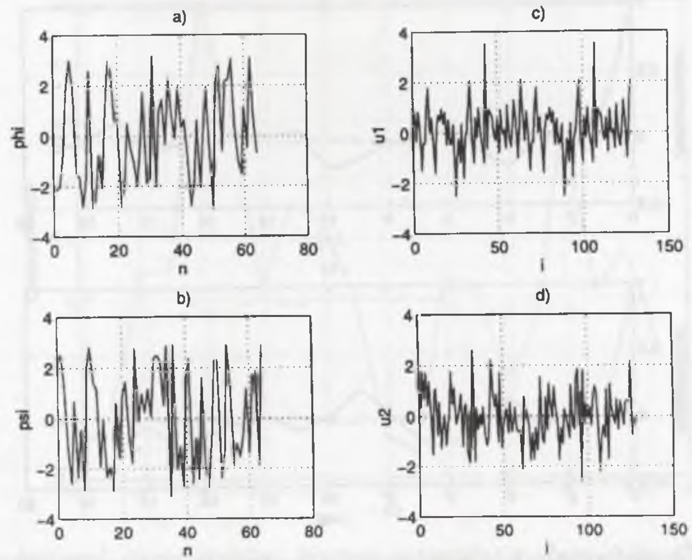

Fig. 4.4. Phase shifts time-series $\phi^{64,u_1}(n)$ (a), $\phi^{64,u_1}(n)$ (b) generated by using the Ran1 and BOWMRS elements $u_1^{128}(i)$ (c), $u_2^{128}(i)$ (d)

Table 4.5

Identification Method	*Parameter Estimates*	
	$\hat{a}_{k,1}^{u_1}$	$\hat{a}_{k,1}^{u_2}$
LS	$9.83\cdot10^{-6}$ (0.155)	$-5.67\cdot10^{-5}$ (0.113)
AC	$6.35\cdot10^{-5}$ (0.150)	$2.23\cdot10^{-4}$ (0.114)

Mean values and standard deviations (in parentheses) for 100 parameter estimations of AR models computed via the Least Squares (LS) and via normalised autocorrelation function (AC) for the BOWMRS elements $u_{k,1}^{128}(i)$ and $u_{k,2}^{128}(i)$ $(k = 1, 2, \ldots, 100)$ with phase shifts generated by the Ran1

The results of this simulation and identification experiments clearly demonstrate the dramatic improvement of the BOWMRS properties as compared with the original Ran1 phase shift time-series.

Biased autocorrelation function estimates were calculated for 100 *different 64-sample time-series realisations and lags* $0, 1, \ldots, 10$. *Mean values and mean square values for biased autocorrelation function estimates are presented in Tab. 4.6.*

It should be noticed that the variance of autocorrelation estimates is lag dependent. This variance groves with the increase of the lag. Mean values of the estimates are near equal to zero.

Gaussianess of the BOWMRS is an asymptotic $(N \to \infty)$ *property for any given time instant* i. *This property was tested for the time instant* $i = 0$. *In order not to invoke the ergodicity assumption, averaging was not performed in the time-domain but in the sample space.*

Let us determine $M = 100$ *realisations, each consisting of* $N = 64$ *samples of values* $u_1^{64}(0)$ *and* $u_2^{64}(0)$ *of a BOWMRS* $\mathbf{u}^{64}(i)$. *The samples were numbered from 0 to 63. For each realisation the normalised biased autocorrelation function was calculated, its lag being the shift between sample numbers. This was used in turn to calculate the mean value and mean square value of the autocorrelation functions. Tab. 4.7 shows results of those calculations*

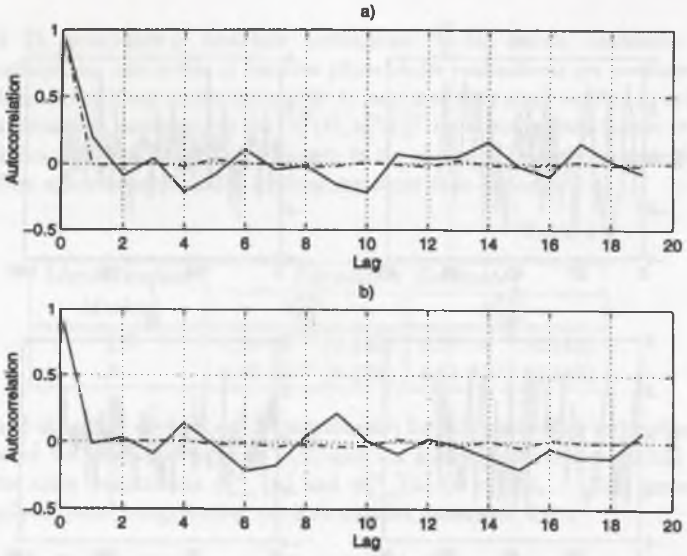

Fig. 4.5. Unbiased estimates of autocorrelation functions for phase shifts $\phi^{64,u_1}(n)$ (a), $\phi^{64,u_2}(n)$ (b) (solid lines) and BOWMRS elements $u_1^{128}(i)$ (a), $u_2^{128}(i)$ (b) (dash-dot lines)

for lags from 0 *to* 10.

For any time instant the biased autocorrelation function estimates for M real-valued white noise samples are for all lags asymptotically normally distributed with zero mean and with the variance $\frac{1}{M}$, which in our example is equal to 0.01 *for* $M = 100$.

The small difference between the calculated and theoretical values should be noted. It follows that BOWMRS is a very good approximation of Gaussian bivariate white noise even for small values of N.

Gaussian Multivariate White Noise

When the power spectral density matrix of a multivariate white noise is approximated by the periodogram matrix of an extended MOMRS, the corresponding time-series is an extended white multivariate orthogonal multisine random time-series (MOWMRS). For $p = 1, 2$ (WSMRS and BOWMRS) constant frequency bin spacings can be kept throughout the entire frequency range $[0, 2\pi)$ and whiteness holds for finite N-sample time-series. This property cannot be, unfortunately, extended for MOMRS having more than 2 elements. Correlation matrices of MOWMRS synthesised and simulated on the basis of the power spectral density of a p-variate white noise with the number of elements $p > 2$ coincide only asymptotically for $N \to \infty$ with the correlation matrices of a p-variate white noise. Asymptotically, the MOWMRS is a Gaussian multivariate orthogonal white multisine random time-series (GMOWMRS). Its spectral and correlation properties are given by the following lemma:

Lemma 4.7 *Assuming that:*

1. $\Phi_{\mathrm{vv}}(j\omega T) = \lambda^2 I$ ($\omega T \in [0, 2\pi)$) *is the power spectral density matrix of a real-valued multivariate white noise;*

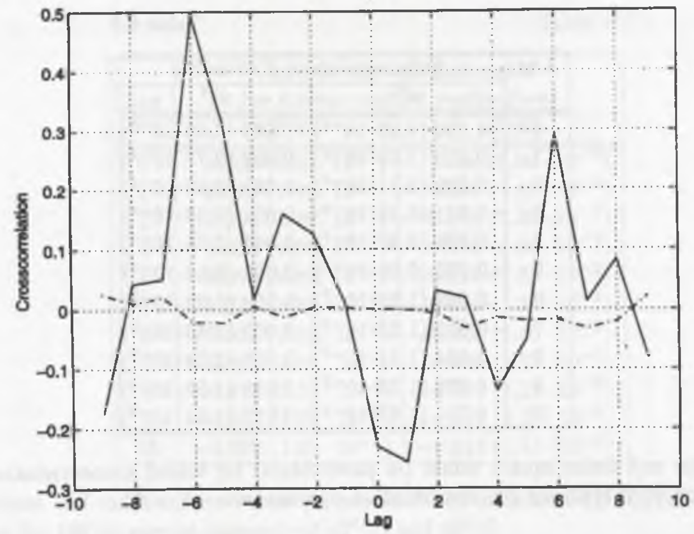

Fig. 4.6. Unbiased cross-correlation function estimates for phase shifts $\phi^{64,u_1}(n)$ and $\phi^{64,u_2}(n)$ (solid line), and BOWMRS elements $u_1^{128}(i)$ and $u_2^{128}(i)$ (dashdot line)

2. $A_n = A$ *for* $n = 1, 2, \ldots, \frac{N}{2} - 1$, $A_0 = A_{\frac{N}{2}} = \frac{A}{2}$, $\phi_0 = \alpha = \frac{\pi}{2}$ *and the value of* A *converges to* 0 *for* $N \to \infty$ *in such a way that*

$$\frac{NTA^2}{4} = \lambda^2; \tag{4.52}$$

then the corresponding extended MOMRS $\mathbf{u}(i)$ *with the consecutively circularly ordered frequencies converges in distribution for $N \to \infty$ to a Gaussian multivariate orthogonal white multisine random time-series of type 1 (GMOWMRS1)* $\mathbf{g}(i) = [g_r(i)]^T_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\frac{\lambda^2}{pT} I$*:*

$$\mathbf{g}(i) \in As\mathcal{N}(\mathbf{o}, \frac{\lambda^2}{pT} I). \tag{4.53}$$

and the GMOWMRS1 correlation function matrix converges to $\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\} = \mathbf{R}_{\mathbf{gg}}(\tau) = diag[R_{g_r g_r}(\tau)]_{r=1,2,\ldots,p}$, *where for* $\tau = 0, 1, \ldots, \infty$*:*

$$R_{g_r g_r}(\tau) = \begin{cases} \frac{\lambda^2}{pT} & \text{if } \tau = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{4.54}$$

□

Proof: *The asymptotic expressions for the correlation functions of MOMRS elements were derived for* $r = 1, 2, \ldots, p$ *as:*

$$R_{u_r u_r}(\tau) = \frac{\lambda^2}{NT} \sum_{\Omega n \in \mathcal{N}^{c,1}_{r,p} \setminus \{0,\pi\}} \left[e^{j\Omega n\tau} + e^{j\Omega(N-n)\tau} \right] + R_{0,\pi}(\tau)$$

$$= \frac{\lambda^2}{NT} e^{j\Omega(r-1)\tau} \sum_{\Omega n \in \mathcal{N}^{c,1}_{r,p} \setminus \{0,\pi\}} \left(e^{j\Omega p(n-r)\tau} + e^{j\Omega(N-n+r)\tau} \right) + R_{0,\pi}(\tau). \tag{4.55}$$

Table 4.6

Lag	*Autocorrelation Estimates*	
	u_1^{64}	u_2^{64}
0	1.000 ($0.00\cdot10^{-4}$)	1.000 ($0.00\cdot10^{-4}$)
1	−0.028 ($2.59\cdot10^{-4}$)	0.003 ($2.77\cdot10^{-4}$)
2	0.030 ($3.71\cdot10^{-4}$)	−0.065 ($3.59\cdot10^{-4}$)
3	−0.031 ($6.83\cdot10^{-4}$)	−0.002 ($7.33\cdot10^{-4}$)
4	0.026 ($8.30\cdot10^{-4}$)	−0.067 ($7.25\cdot10^{-4}$)
5	−0.035 ($9.00\cdot10^{-4}$)	−0.003 ($8.49\cdot10^{-4}$)
6	0.029 ($1.25\cdot10^{-3}$)	−0.058 ($1.22\cdot10^{-3}$)
7	−0.029 ($1.32\cdot10^{-3}$)	0.002 ($1.43\cdot10^{-3}$)
8	0.034 ($1.11\cdot10^{-3}$)	−0.055 ($1.24\cdot10^{-3}$)
9	−0.030 ($1.33\cdot10^{-3}$)	0.005 ($1.50\cdot10^{-3}$)
10	0.024 ($1.67\cdot10^{-3}$)	−0.057 ($1.94\cdot10^{-3}$)

Mean values and mean square values (in parentheses) for biased autocorrelation function estimates of BOWMRS for 100 identification experiments – $N = 64$

The $R_{u_r u_r}(\tau)$ can be approximated for $N \gg k$, $N \gg p$ and $N \to \infty$ by:

$$R_{g_r g_r}(\tau) = \lim_{N\to\infty} \frac{\lambda^2}{NT} \sum_{s=0}^{\frac{N}{p}} e^{j\Omega p s \tau}. \tag{4.56}$$

This implies, taking into account (2.5), that:

$$R_{g_r g_r}(\tau) = \begin{cases} \frac{\lambda^2}{pT} & \text{if } \tau = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{4.57}$$

The MOWMRS mean value vector is:

$$\mathcal{M}\{\mathbf{u}^B(i)\} = \left[\sqrt{\frac{1}{NT}}\lambda^2, 0, \ldots, 0\right]^T. \tag{4.58}$$

The corresponding variance matrix is given by:

$$\sigma^2_{\mathbf{uu}} = \frac{2\lambda^2}{NT} \mathrm{diag}\left[n_r'\right]_{r=1,2,\ldots,p}, \tag{4.59}$$

where n_r' is the number of elements of the set $\mathcal{N}^{c,1}_{r,p} \setminus \{0\}$. When $N \to \infty$, the mean value vector tends to a zero vector and n_r' approaches $\frac{N}{2p}$. It implies that $\sigma^2_{\mathbf{uu}}$ tends to $\frac{\lambda^2}{pT}I$ vector.

□

4.2.2 Nonergodic Case

N-lag pseudo-white multisine random time-series

When the power spectral density matrix of a multivariate white noise is approximated by the expected value of the periodogram matrix of an extended NMOMRS, a nonergodic multivariate orthogonal N-lag pseudo-white multisine random time-series (NMOPWMRS) may be synthesised:

Table 4.7

Lag	*Autocorrelation Estimates for 0-time instant realisations*	
	$u_1^{64}(0)$	$u_2^{64}(0)$
0	1.000 ($0.00\cdot10^{-2}$)	1.000 ($0.00\cdot10^{-2}$)
1	−0.046 ($1.42\cdot10^{-2}$)	−0.055 ($1.42\cdot10^{-2}$)
2	0.018 ($1.40\cdot10^{-2}$)	−0.018 ($1.38\cdot10^{-2}$)
3	0.020 ($1.60\cdot10^{-2}$)	0.018 ($1.62\cdot10^{-2}$)
4	−0.010 ($1.48\cdot10^{-2}$)	−0.017 ($1.29\cdot10^{-2}$)
5	−0.014 ($1.22\cdot10^{-2}$)	−0.011 ($1.29\cdot10^{-2}$)
6	−0.021 ($1.39\cdot10^{-2}$)	−0.023 ($1.40\cdot10^{-2}$)
7	−0.003 ($1.42\cdot10^{-2}$)	0.000 ($1.53\cdot10^{-2}$)
8	−0.013 ($1.32\cdot10^{-2}$)	−0.007 ($1.23\cdot10^{-2}$)
9	−0.007 ($1.23\cdot10^{-2}$)	−0.012 ($1.25\cdot10^{-2}$)
10	−0.022 ($1.50\cdot10^{-2}$)	−0.015 ($1.41\cdot10^{-2}$)

Mean values and mean square values (in parentheses) for biased autocorrelation function estimates for 100 64-sample segments of $u_1^{64}(0)$ and $u_2^{64}(0)$

Definition 4.5 *An extended nonergodic multivariate orthogonal multisine random time-series $\mathbf{x}(i)$ is said to be an N-lag pseudo-white if its correlation function matrix $\mathcal{E}\{\mathbf{x}(i)\mathbf{x}^T(i-\tau)\} = [\mathcal{E}\{x_r(i)x_s(i-\tau)\}]_{r,s=1,2,\ldots,p}$ for lags $\tau = 0, 1, \ldots, N-1$ satisfies the following conditions:*

- $$\mathcal{E}\{x_r(i)x_r(i-\tau)\} = \begin{cases} \Gamma^2 & \text{if } \tau = 0 \\ \gamma_r(\tau)\Gamma^2 & \text{if } \tau = 1, 2, \ldots, N-1 \end{cases} \tag{4.60}$$

  for $r = 1, 2, \ldots, p$ and $|\gamma_r(\tau)| \ll 1$;

- $$\mathcal{E}\{x_r(i)x_s(i-\tau)\} = 0, \tag{4.61}$$

  for $r, s = 1, 2, \ldots, p$ and $r \neq s$.

□

The properties of NMOPWMRS are given by the following lemmas:

Lemma 4.8 *Assuming that:*

1. *$\Phi_{\mathbf{vv}}(j\omega T) = \lambda^2 I$ ($\omega T \in [0, 2\pi)$) is the power spectral density matrix of a real-valued multivariate white noise;*

2. *$A_{r,n} = A$ for $n = 1, 2, \ldots, \frac{N}{2} - 1$ and $r = 1, 2, \ldots, p$ and the value of A is chosen so that:*

$$\frac{NTA^2}{4} = \lambda^2; \tag{4.62}$$

3. *$A_{r,0} = 0$ or $\phi_{r,0} = 0$ for $r = 1, 2, \ldots, p$;*

4. *$A_{r,\frac{N}{2}} = \frac{A}{2}$ and $\alpha = \frac{\pi}{2}$ for $r = 1, 2, \ldots, p$;*

then the corresponding extended NMOMRS $\mathbf{u}(i)$ is a nonergodic multivariate N-lag pseudo-white multisine random time-series (NMOPWMRS) and:

1. *its expected value vector is $\mathcal{E}\{\mathbf{u}(i)\} = \mathbf{o}$.*

2. its correlation function matrix is $\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\} = [\mathcal{E}\{u_r(i)u_s(i-\tau)\}]_{r,s=1,2,\ldots,p}$*, where for* $\tau = 0, 1, \ldots, \infty$*:*

$$\mathcal{E}\{u_r(i)u_s(i-\tau)\} = \begin{cases} \mathcal{E}\{u_r(i)u_r(i-\tau)\} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}. \tag{4.63}$$

$\mathcal{E}\{u_r(i)u_r(i-\tau)\}$ *is the autocorrelation function of the rth NMOPWMRS element:*

$$\mathcal{E}\{u_r(i)u_r(i-\tau)\} = \begin{cases} \frac{N-1}{N}\frac{\lambda^2}{T} & \text{if } \tau = 0, N, \ldots \\ -\frac{1}{N}\frac{\lambda^2}{T} & \text{otherwise} \end{cases} \tag{4.64}$$

3. its variance matrix is:

$$\mathcal{E}\{(\mathbf{u}(i) - \mathcal{E}\{\mathbf{u}(i)\})(\mathbf{u}(i) - \mathcal{E}\{\mathbf{u}(i)\})^T\} = \frac{N-1}{N}\frac{\lambda^2}{T}I. \tag{4.65}$$

□

Proof:

1. *It follows immediately from the assumption 3 and from Lemma 2.8.*
2. *It follows from Lemma 2.8 that* $\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\} = diag[\mathcal{E}\{u_r(i)u_r(i-\tau)\}]_{r=1,2,\ldots,p}$*. This ends proof when it is noticed that the rth* $(r = 1, 2, \ldots, p)$ *element* $u_r(i)$ *of NMOPWMRS* $\mathbf{u}(i)$ *is a PWSMRS1 (see Lemma 4.2).*
3. *It follows from the NMOPWMRS correlation function matrix.*

□

Lemma 4.9 *Consider the extended NMOPWMRS.*

1. Its periodogram matrix is $\Phi^B_{\mathbf{uu}}(j\Omega m) = \left[\Phi^B_{u_r u_s}(j\Omega m)\right]_{r,s=1,2,\ldots,p}$*, where for* $m = 0, 1, \ldots, N-1$*:*

$$\begin{aligned} \Phi^B_{u_r u_s}(j\Omega m) &= (0 + j0)\delta(m) \\ &+ \lambda^2 \sum_{n=1}^{\frac{N}{2}-1} [(\cos(\phi_{r,n} - \phi_{s,n}) - j\sin(\phi_{r,n} - \phi_{s,n}))\,\delta(m-n) + (\cos(\phi_{r,n} - \phi_{s,n}) \\ &+ j\sin(\phi_{r,n} - \phi_{s,n}))\,\delta(m - (N-n))] + (\lambda^2 \sin\phi_{r,\frac{N}{2}} \sin\phi_{s,\frac{N}{2}} + j0)\delta(m - \frac{N}{2}). \end{aligned} \tag{4.66}$$

2. Its mean value vector is $\mathcal{M}\{\mathbf{u}(i)\} = \mathbf{o}$*.*

3. Its correlation function matrix is $\mathbf{R_{uu}}(\tau) = [R_{u_r u_s}(\tau)]_{r,s=1,2,\ldots,p}$*, where for* $\tau = 0, 1, \ldots, \infty$*:*

$$R_{u_r u_s}(\tau) = \frac{\lambda^2}{NT}\left[\sum_{n=0}^{\frac{N}{2}-1} \cos(\Omega n\tau + \phi_{r,n} - \phi_{s,n}) + (-1)^\tau \sin\phi_{r,\frac{N}{2}} \sin\phi_{s,\frac{N}{2}}\right]. \tag{4.67}$$

4. Its variance matrix is $\sigma^2_{\mathbf{uu}} = [\sigma^2_{u_r u_s}]_{r,s=1,2,\ldots,p}$*, where:*

$$\sigma^2_{u_r u_s} = \frac{\lambda^2}{NT}\left[\sum_{n=1}^{\frac{N}{2}-1} \cos(\phi_{r,n} - \phi_{s,n}) + \sin\phi_{r,\frac{N}{2}} \sin\phi_{s,\frac{N}{2}}\right]. \tag{4.68}$$

□

Proof of the above lemma follows immediately from Lemma 2.7 and Lemma 2.9.

The expected value of $\mathbf{R_{uu}}(\tau)$ is the diagonal matrix:

$$\mathcal{E}\{\mathbf{R_{uu}}(\tau)\} = \begin{cases} \frac{N-1}{N}\frac{\lambda^2}{T}I & \text{if } \tau = 0, N, \ldots \\ \mathbf{o} & \text{otherwise} \end{cases}. \tag{4.69}$$

Gaussian multivariate white multisine random time-series

The extended NMOPWMRS turns asymptotically for $N \to \infty$ into a Gaussian multivariate white multisine random time-series, which is ergodic:

Lemma 4.10 *Assuming that:*

1. $\Phi_{\mathbf{vv}}(j\omega T) = \lambda^2 I$ $(\omega T \in [0, 2\pi))$ *is the power spectral density matrix of a real-valued multivariate white noise;*
2. $A_{r,n} = A$ *converges to 0 for* $N \to \infty$ *in such a way that for* $r = 1, 2, \ldots, p$*:*

$$\frac{NTA^2}{4} = \Phi_{v_r v_r}(\Omega n), \tag{4.70}$$

where $n = 1, 2, \ldots, \frac{N}{2} - 1$*;*

3. $A_{r,0} = A_{r,\frac{N}{2}} = 0$ *or* $\phi_{r,0} = \alpha = 0$ *for* $r = 1, 2, \ldots, p$*;*

then the extended NMOMRS $\mathbf{u}(i)$ *converges in distribution for* $N \to \infty$ *to an ergodic, Gaussian multivariate white multisine random time-series of type 2 (GMOWMRS2)* $\mathbf{g}(i) = [g_r(i)]_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\frac{\lambda^2}{T}I$*:*

$$\mathbf{g}(i) \in As\mathcal{N}(\mathbf{o}, \frac{\lambda^2}{T}I). \tag{4.71}$$

Additionally the correlation function matrix $\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\}$ *of the GMOWMRS2 converges to:*

$$\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\} = \mathbf{R_{gg}}(\tau) = \begin{cases} \frac{\lambda^2}{T}I & \text{if } \tau = 0 \\ \mathbf{o} & \text{if } \tau > 0 \end{cases}. \tag{4.72}$$

□

Proof of the above lemma follows immediately from Lemma 3.2 and Lemma 3.3.

Chapter 5

Simulation of Gaussian Random Processes

This chapter is concerned with simulation of Gaussian random processes. Simulation schemes based on the proposed approach to random process synthesis and simulation are discussed, including a proposition of simulation time-scale contraction. The proposed schemes are illustrated by simulation examples.

5.1 SIMULATION SCHEMES

It follows from the previous sections that the statistical properties of multivariate multisine random time-series synthesised based on the given power spectral density matrix of a random process to be simulated behave, asymptotically with the number of approximation nodes $N \to \infty$, exactly as these for the corresponding true Gaussian random process. In computer simulation experiments there is no possibility to perform simulations for an infinite N. To simulate random time-series, a finite value of N must be chosen. This choice influences the statistical properties of the synthesised multisine random process approximations. However, the original power spectral densities and autocorrelation functions are approximated very precisely by the corresponding properties of the synthesised multisine random process approximations, even for small values of N. The influence of finite N can be seen while variances of parameter estimates obtained in multiple repeated simulation experiments are compared with the corresponding theoretically calculated Cramer-Rao bounds for the true Gaussian random processes.

From the spectral factorisation theorem follows that results of parameter estimation for time-series obtained directly from the given power spectral density diagram and from the corresponding discrete-time filter excited by a multivariate orthogonal white multisine random time-series are comparable [24], [28]. It implies that discussion of the Cramer-Rao bounds for the results of parameter estimation for power spectral density defined multivariate multisine random processes may be done by analysing only the results for the corresponding multivariate orthogonal white multisine random time-series.

Let l be the number of consecutive samples taken from a synthesised WSMRS with the period N ($l < N$). It is well known [13] that, for a real-valued Gaussian white noise time-series of the length l, estimates of its normalised autocorrelation function for all lags are asymptotically normally distributed with zero mean and the variance $\frac{1}{l}$. For the WSMRS, variance of the normalised unbiased autocorrelation estimator

$$\frac{\hat{R}_{uu}(\tau)}{\hat{R}_{uu}(0)} = \frac{l}{l-\tau}\,\frac{\sum_{i=0}^{l-1} u(i)u(i-\tau)}{\sum_{i=0}^{l} u^2(i)} \tag{5.1}$$

is lag dependent. The smallest value of this variance is for the lag $\tau = 1$. The variance

$\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\}$ may be approximated by the following formula:

$$\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\} \cong \frac{1}{l + \frac{(l-1)^2}{N-l+1}}. \tag{5.2}$$

It is obvious that the above expression is also valid for each element of the NMOPWMRS. The variance $\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\}$ of the normalised autocorrelation function estimator (5.1) for all elements of MOWMRS is:

$$\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\} \cong \frac{1}{l + \frac{(l-1)^2}{\frac{N}{p}-l+1}}. \tag{5.3}$$

Analysis of the above expressions leads to the conclusion, that using multisine random time-series to simulate Gaussian random processes, the two simulation schemes may be proposed:

- case $l << N$ so that

$$l + f(l,N,p) \cong l, \tag{5.4}$$

where:

$$f(l,N,p) = \begin{cases} \frac{(l-1)^2}{N-l+1} & \text{for SMRS, NMOMRS and NMMRS} \\ \frac{(l-1)^2}{\frac{N}{p}-l+1} & \text{for MOMRS} \end{cases} \tag{5.5}$$

This implies that variances of the autocorrelation function estimator for elements of the power spectral density defined multivariate multisine random processes are comparable with the corresponding values of the Cramer-Rao bounds for the true Gaussian random processes. For the given length l of a random process realisation to be simulated, the period N of the corresponding multisine random time-series may be chosen using the following approximation:

$$N > \begin{cases} \frac{l}{k} & \text{for SMRS, NMOMRS and NMMRS} \\ \frac{pl}{k} & \text{for MOMRS} \end{cases}, \tag{5.6}$$

where k is the relative error, in the variance of normalised unbiased autocorrelation function estimator (5.1) for elements of the multivariate multisine random time-series, with respect to the Cramer-Rao bound, i.e.:

$$k = \left| \frac{\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\} - \frac{1}{l}}{\frac{1}{l}} \right| \cong \frac{1}{1 + \frac{l}{f(l,N,p)}}. \tag{5.7}$$

In this case an l-sample realisation of the power spectral density defined multisine random process $\mathbf{u}(i)$ may be obtained by using the non-destructive zoom FFT procedure [81]. Assuming that $\frac{N}{l}$ is an integer number, the l-sample multisine random process realisation may be calculated by performing $\frac{N}{l}$ l-sample inverse finite discrete Fourier transforms:

$$\mathrm{u}(i) = \sum_{q=0}^{\frac{N}{l}-1} e^{j\Omega q i} \sum_{m=0}^{l-1} \mathrm{U}(j\Omega(lm+q)) e^{j\Omega l m i}, \tag{5.8}$$

where $\mathrm{U}(\cdot)$ is a realisation of the spectrum of a multisine random process synthesised for a given power spectral density matrix of the random process to be simulated using rules presented in Chapter 3.

- case $l \approx N$ $(l < N)$ in which the variances of autocorrelation function matrix elements estimates for the power spectral density defined multivariate multisine random processes are always much smaller than the corresponding Cramer-Rao bounds for the true Gaussian random processes. The results of autocorrelation estimation behave as for the true Gaussian random process with the number of samples

$$l' = \frac{1}{\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\}} \cong l + f(l, N, p). \tag{5.9}$$

It means that to simulate an l'-sample time-series representation by using the classical Gaussian white noise random number generator you can simulate the corresponding l-sample $(l < l')$ multisine random process realisation with the same statistical properties. This is an interesting property of the power spectral density defined multisine random time-series which may be called simulation time-scale contraction. The simulation time-scale contraction allows us to reduce simulation effort radically. It is especially important in real-world experiments when test times are limited by the properties of system under tests.

5.2 EXAMPLES

The proposed approach to the synthesis and simulation of Gaussian random processes given by their power spectral densities is illustrated by the following examples:

Example 5.1 *$l = 256$-sample realisations of the following third-order AR time-series from Example 3.1:*

$$v(i) = \frac{1.00}{1.00 - 2.00z^{-1} + 1.45z^{-2} - 0.35z^{-3}} e(i) \tag{5.10}$$

with unit variance were simulated by using:

- *its time-domain representation as a discrete-time linear filter excited by the white noise $e(i)$ obtained from a standard Gaussian white noise generator (SGWNG) or by a white multisine random time-series WSMRS with the period $N = 256$;*
- *its frequency-domain representation as the power spectral density diagram (Fig. 3.1), which was approximated by the periodogram of a multisine random time-series with the period $N = 256, 262144$. For $N = 262144$ the non-destructive zoom FFT was used.*

The period $N = 256$ ($\frac{N}{l} = 1$ and the relative error $k \cong 0.996$) corresponds to the contracted time scale $l' \cong 65281$. For $N = 262144$ ($\frac{N}{l} = 1024$ and the relative error $k \cong 0.001$) the contracted time-scale is equal to the original one ($l' \cong 256$).

Each simulated $N = 256$-sample third-order AR time-series realisation was identified using the Least Squares identification method [13]. The mean values and standard deviations

Table 5.1

Parameter (CRB)	Parameter Estimates			
	N = 256			N = 262144
	SGWNG	WSMRS	SMRS	SMRS
−2.00 (0.058)	−1.99 (0.053)	−2.00 (0.006)	−2.00 (0.006)	−2.00 (0.058)
1.45 (0.104)	1.44 (0.095)	1.45 (0.012)	1.45 (0.010)	1.45 (0.099)
−0.35 (0.058)	−0.35 (0.053)	−0.35 (0.008)	−0.35 (0.006)	−0.35 (0.058)

Mean values and standard deviations (in parentheses) of the third-order AR time-series model parameter estimates obtained for 100 simulation experiments using the Least Squares identification method - $l = 256$

(in parentheses) of the estimated parameters for the third-order AR model in 100 simulation experiments are presented in Tab. 5.1.

It is worth to note that the method based on approximation of the power spectral density diagram by the SMRS periodogram gives results which are comparable in accuracy with those produced by the time-domain method with the WSMRS excitation.

Example 5.2 *The bivariate orthogonal random process $\mathbf{v}(i)$ given by the following power spectral density matrix*

$$\Phi_{\mathbf{vv}}(j\omega T) = \begin{bmatrix} \frac{1.00 + j0}{1.64 - 1.60\cos\omega T} & 0 + j0 \\ 0 + j0 & e^{2\cos(\omega T)} + j0 \end{bmatrix}, \tag{5.11}$$

was simulated by using the proposed approach. The $\Phi_{\mathbf{vv}}(j\omega T)$ was approximated by the expected value of NMOMRS periodogram matrix. Its $l = 1000$-sample realisations were obtained from the corresponding MOMRS with the period $N = 4096, 65536, 524288$. This choice of the period N corresponds to the contracted time-scale $l' \cong 1324, 1016, 1002$, respectively.

*The bivariate **AR** time-series:*

$$\mathbf{A}(z^{-1})\mathbf{v}(i) = \mathbf{e}(i) \tag{5.12}$$

*was identified for each simulated l-sample random process realisation using the Least Squares identification method [13], [56]. The mean values, standard deviations (in parentheses) of parameters estimated in 100 simulation experiments for the orthogonal **AR** model with the structure of the matrix $\mathbf{A}(z^{-1})$ chosen as*

$$\mathbf{A}(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & 0.00 \\ 0.00 & a_{22}^1 \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^2 \end{bmatrix} z^{-2} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^3 \end{bmatrix} z^{-3} \tag{5.13}$$

and the corresponding Cramer-Rao bounds (CRB) are presented in Tab. 5.2.

*The orthogonality of the simulated random process realisations was examined by identifying a nonorthogonal **AR** model with the following structure of the matrix $\mathbf{A}(z^{-1})$*

$$\mathbf{A}(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^2 \end{bmatrix} z^{-2} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^3 \end{bmatrix} z^{-3} \tag{5.14}$$

The results are presented in Tab. 5.3.

It follows from the above tables, that in simulation experiments for $l \approx N$ ($N = 4096$, $\frac{N}{l} = 4$) the standard deviations of the estimated parameters are much smaller than those which result from the Cramer-Rao bound. For interpretation of the identification results the simulation time-scale contraction proposition may be used. In the case of N equal to 65536 ($\frac{N}{l} = 65$ and the relative error $k \cong 0.015$) the variances of the estimated parameters are very close to the Cramer-Rao bounds. The same is true for the period $N = 524288$.

Table 5.2

Parameter (CRB)	Parameter Estimates $N = 4096$	$N = 65536$	$N = 524288$
a^1_{11} (0.019)	−0.798 (0.018)	−0.798 (0.017)	−0.798 (0.018)
a^1_{22} (0.031)	−0.993 (0.026)	−0.991 (0.035)	−0.999 (0.033)
a^2_{22} (0.042)	0.481 (0.034)	0.478 (0.041)	0.484 (0.041)
a^3_{22} (0.031)	−0.130 (0.024)	−0.124 (0.030)	−0.127 (0.031)

Mean values and standard deviations (in parentheses) of the orthogonal AR model parameter estimates obtained for 100 simulation experiments - $l = 1000$

Table 5.3

Parameter (CRB)	Parameter Estimates $N = 4096$	$N = 65536$	$N = 524288$
a^1_{11} (0.019)	−0.797 (0.018)	−0.797 (0.017)	−0.798 (0.018)
a^1_{12} (0.021)	0.001 (0.022)	0.000 (0.020)	0.000 (0.016)
a^1_{21} (0.019)	−0.001 (0.021)	−0.002 (0.020)	−0.001 (0.020)
a^1_{22} (0.031)	−0.992 (0.026)	−0.990 (0.035)	−0.998 (0.033)
a^2_{22} (0.042)	0.481 (0.034)	0.477 (0.041)	0.484 (0.044)
a^3_{22} (0.031)	−0.130 (0.024)	−0.124 (0.030)	−0.127 (0.030)

Mean values and standard deviations (in parentheses) of the nonorthogonal AR model parameter estimates obtained for 100 simulation experiments - $l = 1000$

It follows from the simulation and identification experiments that multivariate random process realisations simulated using multivariate multisine random time-series very precisely reconstruct the statistical properties of the original Gaussian random processes.

Chapter 6

Multidimensional Multisine Random Processes

In this chapter, an extension of multisine random time-series (1-D case) concepts presented in Chapter 2 to multidimensional (M-D) case is given. Scalar and multivariate M-D multisine random processes are formally defined and their independent variable- and frequency-domain properties are established. It is shown that M-D multisine random processes inherit properties of 1-D multisine random time-series.

The defined M-D multisine random processes are used to synthesise and simulate wide-sense stationary M-D random processes given by their power spectral densities. Asymptotic properties of the obtained M-D multisine random approximations are discussed.

6.1 FUNDAMENTALS

Definitions of M-D multisine random processes are closely related to the M-D discrete Fourier transform defined for a finite number of data.

Let $x(\boldsymbol{i}_\nu)$ $((\boldsymbol{i}_\nu) = (i_1, i_2, \ldots, i_M))$ be an M-D series represented by its $N_1 \cdot N_2 \cdots N_M$ values given for all independent variable M-tuples $(\boldsymbol{i}_\nu) \in \mathcal{X}^M$, where:

$$\mathcal{X}^M = \{0, 1, \ldots, N_1 - 1\} \times \{0, 1, \ldots, N_2 - 1\} \times \cdots \times \{0, 1, \ldots, N_M - 1\}. \tag{6.1}$$

The corresponding M-D finite discrete Fourier transform of $x(\boldsymbol{i}_\nu)$ is given by:

$$X(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_M=0}^{N_M-1} x(\boldsymbol{i}_\nu) e^{-j\sum_{\nu=1}^{M} \Omega_\nu m_\nu i_\nu}, \tag{6.2}$$

for all harmonic frequency M-tuples $(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = (\Omega_1 m_1, \Omega_2 m_2, \ldots, \Omega_M m_M) \in \mathcal{N}^M_{2\pi}$, where:

$$\mathcal{N}^M_{2\pi} = \{0, \Omega_1, \ldots, (N_1 - 1)\Omega_1\} \times \{0, \Omega_2, \ldots, (N_2 - 1)\Omega_2\} \times \cdots \times \{0, \Omega_M, \ldots, (N_M - 1)\Omega_M\}, \tag{6.3}$$

$(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = (j\Omega_1 m_1, j\Omega_2 m_2, \ldots, j\Omega_M m_M)$, and for $\nu = 1, 2, \ldots, M$:

- $\Omega_\nu = \frac{2\pi}{N_\nu}$ denote fundamental relative frequencies (bins),
- m_ν $(m_\nu = 0, 1, \ldots, N_\nu)$ denote the consecutive harmonics of these frequencies.

The inverse M-D finite discrete Fourier transform is:

$$x(\boldsymbol{i}_\nu) = \frac{1}{\prod_{\nu=1}^{M} N_\nu} \sum_{m_1=0}^{N_1-1} \cdots \sum_{m_M=0}^{N_M-1} X(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) e^{j\sum_{\nu=1}^{M} \Omega_\nu m_\nu i_\nu}. \tag{6.4}$$

The M-D finite discrete Fourier transform can be applied to synthesise and simulate $M-D$ random processes exactly like it was done in the 1-D case. The main building block in this is the M-D sine series:

$$x(i_\nu) = A_{(n_\nu)} \sin(\sum_{\nu=1}^{M} \Omega_\nu n_\nu i_\nu + \phi_{(n_\nu)}), \tag{6.5}$$

where $A_{(n_\nu)}$ ($(n_\nu) = (n_1, n_2, \ldots, n_M)$) is a deterministic amplitude of the $M-D$ sine series ($A_{(n_\nu)} \in \mathcal{R}$) and $\phi_{(n_\nu)}$ is a phase shift.

If the M-D sine series is represented by $N_1 N_2 \cdots N_M$ values given for all $(i_\nu) \in \mathcal{X}^M$, then its M-D finite discrete Fourier transform is calculated as follows:

$$X(j\Omega_\nu m_\nu) = \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_M=0}^{N_M-1} A_{(n_\nu)} \sin(\sum_{\nu=1}^{M} \Omega_\nu n_\nu i_\nu + \phi_{(n_\nu)}) e^{-j\sum_{\nu=1}^{M} \Omega_\nu m_\nu i_\nu}$$

$$= \frac{A_{(n_\nu)}}{2j} \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_M=0}^{N_M-1} \left[e^{j\phi(n_\nu)} e^{j\sum_{\nu=1}^{M}(\Omega_\nu n_\nu - \Omega_\nu m_\nu) i_\nu} - e^{-j\phi(n_\nu)} e^{-j\sum_{\nu=1}^{M}(\Omega_\nu n_\nu - \Omega_\nu m_\nu - 2\pi) i_\nu} \right]$$

$$= \frac{A_{(n_\nu)}(\prod_{\nu=1}^{M} N_\nu)}{2j} \left[e^{j\phi(n_\nu)} \prod_{\nu=1}^{M} \delta(m_\nu - n_\nu) - e^{-j\phi(n_\nu)} \prod_{\nu=1}^{M} \delta(m_\nu - (N_\nu - n_\nu)) \right], \tag{6.6}$$

where use has been made of:

$$\sum_{i_1=0}^{N_1-1} \cdots \sum_{i_M=0}^{N_M-1} e^{-j\sum_{\nu=1}^{M} \Omega_\nu k_\nu i_\nu} = \begin{cases} \prod_{\nu=1}^{M} N_\nu & \text{if } (k_\nu) \in \mathcal{X}_0^M \\ 0 & \text{otherwise} \end{cases} \tag{6.7}$$

and $\mathcal{X}_0^M = \{0, N_1, \ldots\} \times \{0, N_2, \ldots\} \times \cdots \times \{0, N_M, \ldots\}$.

The spectrum $X(j\Omega_\nu m_\nu)$ of the real-valued M-D sine series satisfies, for all harmonic frequency M-tuples $(\Omega_\nu m_\nu) \in \mathcal{N}_{2\pi}^M$, the following condition:

$$X(j(2\pi - \Omega_\nu m_\nu)) = X(-j\Omega_\nu m_\nu). \tag{6.8}$$

The M-D sine series is represented in the relative frequency range $[0, 2\pi)^M$ by two lines. This implies that to define a sum of the M-D sine series (an M-D multisine random process) having spectrum lines defined for all frequency M-tuples $(\Omega_\nu m_\nu) \in \mathcal{N}_{2\pi}^M$, a set $\mathcal{N}^M$ ($\mathcal{N}^M \subset \mathcal{N}_{2\pi}^M$) of the frequency M-tuples $(\Omega_\nu m_\nu)$ of the corresponding M-D sine series components included in the sum should be defined. If N_1, N_2, ..., N_M are all even then the set $\mathcal{N}^M$ with minimum number of the elements $(\Omega_\nu m_\nu)$ can be defined for:

- the 1-D case [28], [32], [34], [66] as:

$$\mathcal{N}^1 = \left\{0, \Omega_1, \ldots, \Omega_1 \frac{N_1}{2} = \pi\right\}. \tag{6.9}$$

  In the set $\mathcal{N}^1$ there are two special frequencies at which two spectral lines are placed. The set $\mathcal{N}_S^1$ of these frequencies is:

$$\mathcal{N}_S^1 = \{0, \pi\}. \tag{6.10}$$

- the 2-D case in two ways - either

$$\mathcal{N}^2 = \left\{\left\{0, \Omega_1, \ldots, \Omega_1(N_1-1)\right\} \times \left\{\Omega_2, 2\Omega_2, \ldots, \Omega_2(\frac{N_2}{2}-1)\right\} \cup \left\{0, \Omega_1, \ldots, \pi\right\} \times \left\{0, \pi\right\}\right\}, \tag{6.11}$$

  or

$$\mathcal{N}^2 = \left\{\left\{\Omega_1, 2\Omega_1, \ldots, \Omega_1(\frac{N_1}{2}-1)\right\} \times \left\{0, \Omega_2, \ldots, \Omega_2(N_2-1)\right\} \cup \left\{0, \pi\right\} \times \left\{0, \Omega_2, \ldots, \pi\right\}\right\}. \tag{6.12}$$

In the sequel, only the first definition of the set $\mathcal{N}^2$ is used. In this definition the last (the second) frequency variable covers the range $[0, \pi]$.

The set $\mathcal{N}^2$ can be constructed on the base of the corresponding set (6.9) for the 1-D case as:

$$\mathcal{N}^2 = \left\{\left\{0, \Omega_1, \ldots, \Omega_1(N_1-1)\right\} \times \left\{\Omega_2, 2\Omega_2, \ldots, \Omega_2(\frac{N_2}{2}-1)\right\} \cup \mathcal{N}^1 \times \left\{0, \pi\right\}\right\}. \tag{6.13}$$

The set $\mathcal{N}^2$ contains four special frequency 2-tuples at which two spectral lines are placed. Their set $\mathcal{N}_S^2$ is given by:

$$\mathcal{N}_S^2 = \{0, \pi\} \times \{0, \pi\} = \{0, \pi\}^2. \tag{6.14}$$

- the 3-D case in three ways. For the case when the last frequency variable (the third) covers the frequency range $[0, \pi]$, the set $\mathcal{N}^3$ can be obtained recursively by using the set $\mathcal{N}^2$ as:

$$\mathcal{N}^3 = \left\{\left\{0, \Omega_1, \ldots, \Omega_1(N_1-1)\right\} \times \left\{0, \Omega_2, \ldots, \Omega_2(N_2-1)\right\} \times \left\{\Omega_3, 2\Omega_3, \ldots, \Omega_3(\frac{N_3}{2}-1)\right\} \right.$$
$$\left. \cup \mathcal{N}^2 \times \left\{0, \pi\right\}\right\}. \tag{6.15}$$

The set $\mathcal{N}^3$ contains eight special frequency 3-tuples at which two spectral lines are placed:

$$\mathcal{N}_S^3 = \{0, \pi\} \times \{0, \pi\} \times \{0, \pi\} = \{0, \pi\}^3. \tag{6.16}$$

Generally, for the M-D case the set $\mathcal{N}^M$ can be constructed recursively using the corresponding set $\mathcal{N}^{M-1}$ for the $(M-1)$-D case as:

$$\mathcal{N}^M = \left\{\left\{0, \Omega_1, \ldots, \Omega_1(N_1-1)\right\} \times \cdots \times \left\{0, \Omega_{M-1}, \ldots, \Omega_{M-1}(N_{M-1}-1)\right\} \right.$$
$$\left. \times \left\{\Omega_M, 2\Omega_M, \ldots, \Omega_M(\frac{N_M}{2}-1)\right\} \cup \mathcal{N}^{M-1} \times \left\{0, \pi\right\}\right\}, \tag{6.17}$$

where:

$$\mathcal{N}^1 = \left\{0, \Omega_1, \ldots, \Omega_1 \frac{N_1}{2} = \pi\right\}. \tag{6.18}$$

In this set there are 2^M special frequency M-tuples at which there are two spectral lines. Their set is:

$$\mathcal{N}_S^M = \{0, \pi\}^M \tag{6.19}$$

Each frequency M-tuple $(\Omega_\nu m_\nu)$ is related to the absolute frequency M-tuple (ω_ν) by

$$(\Omega_\nu m_\nu) = (\omega_\nu T_\nu), \tag{6.20}$$

where T_ν is the sampling interval of the νth independent variable.

In the sequel, scalar and multivariate M-D multisine random series are defined using the sets $\mathcal{N}^M$ and $\mathcal{N}_S^M$. It should be emphasised that the defined M-D multisine random series inherit properties of the corresponding 1-D multisine random time-series.

6.2 M-D SCALAR MULTISINE RANDOM PROCESS

Definitions

The basic real-valued M-D scalar multisine random series (SMRS$^{M\text{-}D}$) is defined in the independent variable domain as:

Definition 6.1 *The basic SMRS$^{M\text{-}D}$ is defined by a sum of M-D sines including a constant component:*

$$u^B(i_\nu) = \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M} A_{(n_\nu)} \sin\Big(\sum_{\nu=1}^{M} \Omega_\nu n_\nu i_\nu + \phi_{(n_\nu)}\Big), \tag{6.21}$$

where $(i_\nu) \in \mathcal{X}^M$ and for $\nu = 1, 2, \ldots, M$:

- *$\Omega_\nu = \frac{2\pi}{N_\nu}$ denote fundamental relative frequencies,*
- *n_ν denote the consecutive harmonics of these frequencies,*

$A_{(n_\nu)}$ are deterministic amplitudes of the M-D sine components ($A_{(n_\nu)} \in \mathcal{R}$), $\phi_{(n_\nu)}$ are phase shifts, of which $\phi_{(o)}$ is deterministic and the remaining phase shifts are random, independent and:

- *uniformly distributed on $[0, 2\pi)$ for $(\Omega_\nu n_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M$,*
- *Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$ for $(\Omega_\nu n_\nu) \in \mathcal{N}_S^M \setminus \{(o)\}$.*

□

The basic SMRS$^{M\text{-}D}$ consists of $N_1 \cdot N_2 \cdots N_M$ samples. This SMRS$^{M\text{-}D}$ can be defined in the frequency-domain by its M-D finite discrete Fourier transform [5] as:

$$U^B(j\Omega_\nu m_\nu) = \frac{\prod_{\nu=1}^{M} N_\nu}{2j} \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M} A_{(n_\nu)} \Big[e^{j\phi_{(n_\nu)}} \prod_{\nu=1}^{M} \delta(m_\nu - n_\nu) - e^{-j\phi_{(n_\nu)}} \prod_{\nu=1}^{M} \delta(m_\nu - (N_\nu - n_\nu)) \Big], \tag{6.22}$$

where $(\Omega_\nu m_\nu) \in \mathcal{N}_{2\pi}^M$.

Expanding all independent variable ranges up to $i_\nu = 0, 1, \ldots, \infty$ for $\nu = 1, 2, \ldots, M$, an extended SMRS$^{M\text{-}D}$ is obtained, i.e. the extended SMRS$^{M\text{-}D}$ is defined for all independent variable M-tuples $(i_\nu) \in \mathcal{X}_\infty^M$, where:

$$\mathcal{X}_\infty^M = \{0, 1, \ldots, \infty\} \times \{0, 1, \ldots, \infty\} \times \cdots \times \{0, 1, \ldots, \infty\}. \tag{6.23}$$

The extended SMRS$^{M\text{-}D}$ is periodic with the period M-tuple $(N_\nu) = (N_1, N_2, \ldots, N_M)$. It implies that the extended SMRS$^{M\text{-}D}$ belongs to the space of M-D periodic signals [48] with the period M-tuple (N_ν) and the inner product:

$$\sum_{i_1=0}^{N_1-1} \cdots \sum_{i_M=0}^{N_M-1} u_1(i_\nu) u_2(i_\nu), \tag{6.24}$$

where $u_1(i_\nu), u_2(i_\nu)$ are two M-D periodic signals with the period M-tuple (N_ν).

The spectrum $U'(j\Omega'_\nu m'_\nu)$ of the first $q_1 N_1 \cdot q_2 N_2 \cdots q_M N_M$ samples of the extended MRS$^{M\text{-}D}$ is related to the $U^B(j\Omega_\nu m_\nu)$ by:

$$U'(j\Omega'_\nu m'_\nu) = \begin{cases} \prod_{\nu=1}^{M} q_\nu U^B(j\Omega_\nu m_\nu) & \text{if } (\Omega'_\nu m'_\nu) \in \mathcal{N}^M \\ 0 + j0 & \text{if } (\Omega'_\nu m'_\nu) \notin \mathcal{N}^M \end{cases}, \tag{6.25}$$

where $\Omega'_\nu = \frac{2\pi}{q_\nu N_\nu} = \frac{\Omega_\nu}{q_\nu}$ for $\nu = 1, 2, \ldots, M$ denote the relative fundamental frequencies for the $q_1 N_1 \cdot q_2 N_2 \cdots q_M N_M$-sample series and $m'_\nu = 0, 1, \ldots, q_\nu N_\nu - 1$ denote the consecutive harmonics of the νth fundamental frequency in the range $[0, 2\pi)$.

Properties

The spectral properties of the basic SMRS$^{M\text{-}D}$ are given by:

Lemma 6.1 *Consider the basic SMRS$^{M\text{-}D}$. Its periodogram is given by:*

$$\begin{aligned} \Phi_{uu}^B(\Omega_\nu m_\nu) = \frac{\prod_{\nu=1}^{M} N_\nu T_\nu}{4} \Bigg\{ & 4A_{(o)}^2 \sin^2\phi_{(o)} \prod_{\nu=1}^{M} \delta(m_\nu) \\ & + \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M\setminus\mathcal{N}_S^M} A_{(n_\nu)}^2 \Big[\prod_{\nu=1}^{M} \delta(m_\nu - n_\nu) + \prod_{\nu=1}^{M} \delta(m_\nu - (N_\nu - n_\nu)) \Big] \\ & + 4 \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_S^M\setminus\{(o)\}} A_{(n_\nu)}^2 \sin^2\alpha \prod_{\nu=1}^{M} \delta(m_\nu - n_\nu) \Bigg\}, \end{aligned} \tag{6.26}$$

where $(\Omega_\nu m_\nu) \in \mathcal{N}_{2\pi}^M$.

□

Proof: *The proof of this lemma proceeds in the same way as for Lemma 2.1. It follows from the periodogram definition [60] that:*

$$\Phi_{uu}^B(\Omega_\nu m_\nu) = \mathcal{E}\left\{ \frac{\prod_{\nu=1}^{M} T_\nu}{\prod_{\nu=1}^{M} N_\nu} |U^B(j\Omega_\nu m_\nu)|^2 \right\} = \frac{\prod_{\nu=1}^{M} T_\nu}{\prod_{\nu=1}^{M} N_\nu} |U^B(j\Omega_\nu m_\nu)|^2 \tag{6.27}$$

is a deterministic function.

□

The statistical properties of the extended SMRS$^{M\text{-}D}$ obtained for the ensemble averaging are given by the following lemma:

Lemma 6.2 *Consider the extended SMRS$^{M\text{-}D}$. For each M-tuple $(i_\nu) \in \mathcal{X}_\infty^M$:*

1. *its expected value is*
$$\mathcal{E}\{u(i_\nu)\} = A_{(o)} \sin\phi_{(o)}. \tag{6.28}$$
2. *its autocorrelation function is:*
$$\begin{aligned} \mathcal{E}\{u(i_\nu) u(i_\nu - \tau_\nu)\} = {} & A_{(o)}^2 \sin^2\phi_{(o)} \\ & + \frac{1}{2} \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M\setminus\mathcal{N}_S^M} A_{(n_\nu)}^2 \cos\Big(\sum_{\nu=1}^{M} \Omega_\nu n_\nu \tau_\nu\Big) + \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_S^M\setminus\{(o)\}} (-1)^{\sum_{\nu=1}^{M} \tau_\nu \eta(\Omega_\nu n_\nu)} A_{(n_\nu)}^2 \sin^2\alpha, \end{aligned} \tag{6.29}$$

where $(\tau_\nu) \in \mathcal{X}_\infty^M$ and:

$$\eta(\Omega_\nu n_\nu) = \begin{cases} 1 & \text{if } \Omega_\nu n_\nu = \pi \\ 0 & \text{if } \Omega_\nu n_\nu = 0 \end{cases}. \tag{6.30}$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.2.

It follows from this lemma that the extended SMRS$^{M\text{-}D}$ is a wide-sense stationary (homogeneous) [50] multidimensional random process, i.e. its expected value does not depend on the location of (i_ν) and the corresponding autocorrelation function depends only on the vector (its oriention and lenght) joining the two points (i_ν) and $(i_\nu - \tau_\nu)$. Any change of the assumption about distributions of the random phase shifts $\phi_{(n_\nu)}$ in the extended SMRS$^{M\text{-}D}$ definition would result in an extended SMRS$^{M\text{-}D}$ for which the expected value and autocorrelation function will depend on the independent variable (i_ν).

The following lemma presents properties of the extended SMRS$^{M\text{-}D}$ obtained for the independent variable domain averaging on any particular series:

Lemma 6.3 *Consider the extended SMRS$^{M\text{-}D}$.*

1. *Its mean value is*

$$\mathcal{M}\{u(i_\nu)\} = A_{(\mathbf{o})} \sin \phi_{(\mathbf{o})}. \tag{6.31}$$

2. *Its autocorrelation function is:*

$$R_{uu}(\tau_\nu) = A^2_{(\mathbf{o})} \sin^2 \phi_{(\mathbf{o})}$$

$$+ \frac{1}{2} \sum_{(\Omega_\nu n_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M} A^2_{(n_\nu)} \cos\left(\sum_{\nu=1}^{M} \Omega_\nu n_\nu \tau_\nu\right) + \sum_{(\Omega_\nu n_\nu) \in \mathcal{N}_S^M \setminus \{(\mathbf{o})\}} (-1)^{\sum_{\nu=1}^{M} \tau_\nu \eta(\Omega_\nu n_\nu)} A^2_{(n_\nu)} \sin^2 \alpha, \tag{6.32}$$

where $(\tau_\nu) \in \mathcal{X}_\infty^M$ and $\eta(\Omega_\nu n_\nu)$ is given by (6.30).

□

Proof: *The proof of this lemma proceeds similarly as for Lemma 2.3, where:*

- *the mean value of $u(i_\nu)$ is calculated as:*

$$\mathcal{M}\{u(i_\nu)\} = \lim_{q_1 \to \infty} \cdots \lim_{q_M \to \infty} \frac{1}{\prod_{\nu=1}^{M} q_\nu N_\nu} \sum_{i_1=0}^{q_1 N_1 - 1} \cdots \sum_{i_M=0}^{q_M N_M - 1} u(i_\nu)$$

$$= \frac{1}{\prod_{\nu=1}^{M} N_\nu} \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_M=0}^{N_M-1} u(i_\nu) = \frac{1}{\prod_{\nu=1}^{M} N_\nu} U^B(j\mathbf{o}) = A_{(\mathbf{o})} \sin \phi_{(\mathbf{o})}. \tag{6.33}$$

- *the independent variable domain averaged autocorrelation function of $u(i_\nu)$ is defined as:*

$$R_{uu}(\tau_\nu) = \lim_{q_1 \to \infty} \cdots \lim_{q_M \to \infty} \frac{1}{\prod_{\nu=1}^{M} q_\nu N_\nu} \sum_{i_1=0}^{q_1 N_1 - 1} \cdots \sum_{i_M=0}^{q_M N_M - 1} u(i_\nu) u(i_\nu - \tau_\nu)$$

$$= \frac{1}{\prod_{\nu=1}^{M} N_\nu T_\nu} \sum_{m_1=0}^{N_1-1} \cdots \sum_{m_M=0}^{N_M-1} \Phi^B_{uu}(\Omega_\nu m_\nu) e^{j \sum_{\nu=1}^{M} \Omega_\nu m_\nu \tau_\nu}. \tag{6.34}$$

□

It follows from Lemma 6.2 and Lemma 6.3 that the extended SMRS$^{M\text{-}D}$ inherit properties of the 1-D SMRS, i.e. the extended SMRS$^{M\text{-}D}$ is a weakly ergodic multidimensional random process.

6.3 M-D MULTIVARIATE ORTHOGONAL MULTISINE RANDOM PROCESS

6.3.1 Ergodic Case

Definitions

Following the MOMRS definition, each element $u_r(i_\nu)$ $(r = 1, 2, \ldots, p)$ of an M-D multivariate orthogonal multisine random series (MOMRS$^{M\text{-}D}$) is a sum of some of SMRS$^{M\text{-}D}$ M-D sine components with the constraint that the same frequency M-tuple may not appear in more than one MOMRS$^{M\text{-}D}$ element and each SMRS$^{M\text{-}D}$ M-D sine component belongs to one and only one MOMRS$^{M\text{-}D}$ element. It is formalised by the following definition:

Definition 6.2 *The basic MOMRS$^{M\text{-}D}$ is defined by the p-dimensional multivariate series $\mathbf{u}^B(i_\nu) = \left[u_r^B(i_\nu)\right]_{r=1,2,\ldots,p}$, where the rth MOMRS$^{M\text{-}D}$ element is given by:*

$$u_r^B(i_\nu) = \sum_{(\Omega_\nu n_\nu) \in \mathcal{N}_{r,p}^M} A_{(n_\nu)} \sin\left(\sum_{\nu=1}^{M} \Omega_\nu n_\nu i_\nu + \phi_{(n_\nu)}\right), \tag{6.35}$$

$\mathcal{N}_{r,p}^M$ is the set of all frequency M-tuples $(\Omega_\nu n_\nu)$ present in the rth MOMRS$^{M\text{-}D}$ element $u_r(i_\nu)$ and:

$$\mathcal{N}_{1,p}^M \cup \mathcal{N}_{2,p}^M \cup \cdots \cup \mathcal{N}_{p,p}^M = \mathcal{N}^M. \tag{6.36}$$

These sets are pairwise disjoint:

$$\mathcal{N}_{s,p}^M \cap \mathcal{N}_{t,p}^M = \emptyset \tag{6.37}$$

for $s \neq t$, $s, t = 1, 2, \ldots, p$. Additionally, $(i_\nu) \in \mathcal{X}^M$ and for $\nu = 1, 2, \ldots, M$:

- *$\Omega_\nu = \frac{2\pi}{N_\nu}$ denote fundamental relative frequencies,*
- *n_ν denote the consecutive harmonics of these frequencies,*

$A_{(n_\nu)}$ are deterministic amplitudes of the M-D sine components ($A_{(n_\nu)} \in \mathcal{R}$), $\phi_{(n_\nu)}$ are phase shifts, of which $\phi_{(\mathbf{o})}$ is deterministic and the remaining phase shifts are random, independent and:

- *uniformly distributed on $[0, 2\pi)$ for $(\Omega_\nu n_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M$,*
- *Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$ for $(\Omega_\nu n_\nu) \in \mathcal{N}_S^M \setminus \{(\mathbf{o})\}$.*

□

The spectrum of the basic MOMRS$^{M\text{-}D}$ is given in the frequency-domain for the (relative) frequency range $[0, 2\pi)^M$ by the p-dimensional vector $\mathbf{U}^M(j\Omega_\nu m_\nu) = \left[U_r^B(j\Omega_\nu m_\nu)\right]_{r=1,2,\ldots,p}$ of M-D finite discrete Fourier transforms with the rth element given by:

$$U_r^B(j\Omega_\nu m_\nu) = \frac{\prod_{\nu=1}^{M} N_\nu}{2j} \sum_{(\Omega_\nu n_\nu) \in \mathcal{N}_{r,p}^M} A_{(n_\nu)} \left[e^{j\phi_{(n_\nu)}} \prod_{\nu=1}^{M} \delta(m_\nu - n_\nu) - e^{-j\phi_{(n_\nu)}} \prod_{\nu=1}^{M} \delta(m_\nu - (N_\nu - n_\nu)) \right], \tag{6.38}$$

where $(\Omega_\nu m_\nu) \in \mathcal{N}_{2\pi}^M$.

Elements of the basic MOMRS$^{M\text{-}D}$ can be regarded as scalar real-valued SMRS$^{M\text{-}D}$. The fact that elements of the MOMRS$^{M\text{-}D}$ have no common frequency M-tuples implies orthogonality of its elements for the ensemble averaging:

$$\mathcal{E}\{u_r(i_\nu) u_s(i_\nu)\} = 0 \tag{6.39}$$

as well as the independent variable domain averaging:

$$\frac{1}{\prod_{\nu=1}^{M} q_\nu N_\nu} \sum_{i_1=0}^{q_1 N_1 - 1} \cdots \sum_{i_M=0}^{q_M N_M - 1} u_r(i_\nu) u_s(i_\nu) = 0, \tag{6.40}$$

where $r \neq s$, $r, s = 1, 2, \ldots, p$ and $q_1, q_2, \ldots, q_M = 1, 2, \ldots, \infty$.

Properties

The periodogram matrix of the basic MOMRS$^{M\text{-}D}$ is given by the lemma:

Lemma 6.4 *Consider the basic MOMRS$^{M\text{-}D}$. Its periodogram matrix is $\Phi^B_{\mathbf{uu}}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \left[\Phi^B_{u_r u_s}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\right]_{r,s=1,2,\dots,p}$, where for $(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) \in \mathcal{N}^M_{2\pi}$:*

$$\Phi^B_{u_r u_s}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \begin{cases} \Phi^B_{u_r u_r}(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) + j0 & \text{if } r = s \\ 0 + j0 & \text{if } r \neq s \end{cases}. \tag{6.41}$$

$\Phi^B_{u_r u_r}(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)$ is the periodogram of the rth MOMRS$^{M\text{-}D}$ element:

$$\Phi^B_{u_r u_r}(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \Phi^0_{u_r u_r}(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) +$$

$$\frac{\prod_{\nu=1}^M N_\nu T_\nu}{4} \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_{r,p} \setminus \mathcal{N}^M_S} A^2_{(\boldsymbol{n}_\nu)} \left[\prod_{\nu=1}^M \delta(m_\nu - n_\nu) + \prod_{\nu=1}^M \delta(m_\nu - (N_\nu - n_\nu))\right]$$

$$+ \prod_{\nu=1}^M N_\nu T_\nu \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_{r,p} \cap \mathcal{N}^M_S \setminus \{(\mathbf{o})\}} A^2_{(\boldsymbol{n}_\nu)} \sin^2\alpha \prod_{\nu=1}^M \delta(m_\nu - n_\nu), \tag{6.42}$$

where:

$$\Phi^0_{u_r u_r}(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \begin{cases} \prod_{\nu=1}^M N_\nu T_\nu A_{(\mathbf{o})} \sin^2\phi_{(\mathbf{o})} \prod_{\nu=1}^M \delta(m_\nu) & \text{if } (\mathbf{o}) \in \mathcal{N}^M_{r,p} \\ 0 & \text{if } (\mathbf{o}) \in \mathcal{N}^M_{r,p} \end{cases}. \tag{6.43}$$

□

Proof: *The proof of the above lemma proceeds similarly as for Lemma 6.1, when it is noticed that the periodogram matrix of MOMRS$^{M\text{-}D}$ is:*

$$\Phi^N_{\mathbf{uu}}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \mathcal{E}\left\{\frac{\prod_{\nu=1}^M T_\nu}{\prod_{\nu=1}^M N_\nu} \mathbf{U}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\mathbf{U}^{T,B}(-j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\right\}$$

$$= \frac{\prod_{\nu=1}^M T_\nu}{\prod_{\nu=1}^M N_\nu} \mathbf{U}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\mathbf{U}^{T,B}(-j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu). \tag{6.44}$$

□

The properties of MOMRS$^{M\text{-}D}$ which result from the ensemble averaging are given by:

Lemma 6.5 *Consider the extended MOMRS$^{M\text{-}D}$. For each M-tuple $(\boldsymbol{i}_\nu) \in \mathcal{X}^M_\infty$:*

1. *its expected value vector is $\mathcal{E}\{\mathbf{u}(\boldsymbol{i}_\nu)\} = [\mathcal{E}\{u_r(\boldsymbol{i}_\nu)\}]_{r=1,2,\dots,p}$, where:*

$$\mathcal{E}\{u_r(\boldsymbol{i}_\nu)\} = \begin{cases} A_{(\mathbf{o})} \sin\phi_{(\mathbf{o})} & \text{if } (\mathbf{o}) \in \mathcal{N}^M_{r,p} \\ 0 & \text{if } (\mathbf{o}) \notin \mathcal{N}^M_{r,p} \end{cases}. \tag{6.45}$$

2. *its correlation function matrix is $\mathcal{E}\{\mathbf{u}(\boldsymbol{i}_\nu)\mathbf{u}^T(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = [\mathcal{E}\{u_r(\boldsymbol{i}_\nu)u_s(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\}]_{r,s=1,2,\dots,p}$, where for $(\boldsymbol{\tau}_\nu) \in \mathcal{X}^M_\infty$:*

$$\mathcal{E}\{u_r(\boldsymbol{i}_\nu)u_s(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = \begin{cases} \mathcal{E}\{u_r(\boldsymbol{i}_\nu)u_r(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}. \tag{6.46}$$

$\mathcal{E}\{u_r(\boldsymbol{i}_\nu)u_r(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\}$ is the autocorrelation function of the rth MOMRS$^{M\text{-}D}$ element:

$$\mathcal{E}\{u_r(\boldsymbol{i}_\nu)u_r(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = \mathcal{E}\{u^0_r(\boldsymbol{i}_\nu)u^0_r(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\}$$

$$+\frac{1}{2} \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_{r,p} \setminus \mathcal{N}^M_S} A^2_{(\boldsymbol{n}_\nu)} \cos\left(\sum_{\nu=1}^M \Omega_\nu n_\nu \tau_\nu\right) + \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_S \cap \mathcal{N}^M_{r,p} \setminus \{(\mathbf{o})\}} (-1)^{\sum_{\nu=1}^M \tau_\nu \eta(\Omega_\nu n_\nu)} A^2_{(\boldsymbol{n}_\nu)} \sin^2\alpha, \tag{6.47}$$

where $\eta(\Omega_\nu n_\nu)$ is given by (6.30) and:

$$\mathcal{E}\{u^0_r(\boldsymbol{i}_\nu)u^0_r(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = \begin{cases} A^2_{(\mathbf{o})} \sin^2\phi_{(\mathbf{o})} & \text{if } (\mathbf{o}) \in \mathcal{N}^M_{r,p} \\ 0 & \text{if } (\mathbf{o}) \notin \mathcal{N}^M_{r,p} \end{cases}. \tag{6.48}$$

□

Proof of the above lemma proceeds similarly as for Lemma 2.2.

It follows from this lemma that the extended MOMRS$^{M\text{-}D}$ is a wide-sense stationary multidimensional multivariate random process. Similarly, as for the scalar case of multidimensional multisine random series, any change of the assumption about distributions of the random phase shifts $\phi_{(\boldsymbol{n}_\nu)}$ in the MOMRS$^{M\text{-}D}$ definition results in an extended MOMRS$^{M\text{-}D}$ for which elements of the expected value vector and autocorrelation function matrix are functions of the independent variable $(\boldsymbol{i}_\nu)$.

The independent variable domain averaging on any particular extended MOMRS$^{M\text{-}D}$ results in the following lemma:

Lemma 6.6 *Consider the extended MOMRS$^{M\text{-}D}$.*

1. *Its mean value vector is $\mathcal{M}\{\mathbf{u}(\boldsymbol{i}_\nu)\} = [\mathcal{M}\{u_r(\boldsymbol{i}_\nu)\}]_{r=1,2,\dots,p}$, where:*

$$\mathcal{M}\{u_r(\boldsymbol{i}_\nu)\} = \begin{cases} A_{(\mathbf{o})} \sin\phi_{(\mathbf{o})} & \text{if } (\mathbf{o}) \in \mathcal{N}^M_{r,p} \\ 0 & \text{if } (\mathbf{o}) \notin \mathcal{N}^M_{r,p} \end{cases}. \tag{6.49}$$

2. *Its correlation function matrix is $\mathbf{R}_{\mathbf{uu}}(\boldsymbol{\tau}_\nu) = [R_{u_r u_s}(\boldsymbol{\tau}_\nu)]_{r,s=1,2,\dots,p}$, where for $(\boldsymbol{\tau}_\nu) \in \mathcal{X}^M_\infty$:*

$$R_{u_r u_s}(\boldsymbol{\tau}_\nu) = \begin{cases} R_{u_r u_r}(\boldsymbol{\tau}_\nu) & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}. \tag{6.50}$$

$R_{u_r u_r}(\boldsymbol{\tau}_\nu)$ is the autocorrelation function of the rth MOMRS$^{M\text{-}D}$ element:

$$R_{u_r u_r}(\boldsymbol{\tau}_\nu) = R^0_{u_r u_r}(\boldsymbol{\tau}_\nu)$$

$$+\frac{1}{2} \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_{r,p} \setminus \mathcal{N}^M_S} A^2_{(\boldsymbol{n}_\nu)} \cos\left(\sum_{\nu=1}^M \Omega_\nu n_\nu \tau_\nu\right) + \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_S \cap \mathcal{N}^M_{r,p} \setminus \{(\mathbf{o})\}} (-1)^{\sum_{\nu=1}^M \tau_\nu \eta(\Omega_\nu n_\nu)} A^2_{(\boldsymbol{n}_\nu)} \sin^2\alpha, \tag{6.51}$$

where $\eta(\Omega_\nu n_\nu)$ is given by (6.30) and:

$$R^0_{u_r u_r}(\boldsymbol{\tau}_\nu) = \begin{cases} A^2_{(\mathbf{o})} \sin^2\phi_{(\mathbf{o})} & \text{if } (\mathbf{o}) \in \mathcal{N}^M_{r,p} \\ 0 & \text{if } (\mathbf{o}) \notin \mathcal{N}^M_{r,p} \end{cases}. \tag{6.52}$$

□

Proof of the above lemma proceeds similarly as for Lemma 6.3.

It follows from the above lemmas that similarly as MOMRS, the extended MOMRS$^{M\text{-}D}$ is a weakly ergodic multidimensional random process.

Frequency M-tuples distribution

The consecutive circular ordering of frequency M-tuples for multidimensional multisine random processes is done taking into account the Mth frequency axis. This ordering is denoted by the upper index c in symbols $\mathcal{N}^{c,M}_{r,p}$ $(r = 1, 2, \ldots, p)$. The frequency M-tuple $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M$ is a member of the set $\mathcal{N}^{c,M}_{r,p}$ $(r = 1, 2, \ldots, p)$ if:

$$r = n_M \bmod p + 1. \tag{6.53}$$

Thus defined circular ordering allows us to synthesise scalar ($p = 1$) and bivariate ($p = 2$) multidimensional white multisine random series for which whiteness holds for finite sample representations. It is a consequence of a constant bin spacing along all frequency axes.

For $p = 1$ ($\mathcal{N}^{c,M}_{1,1} = \mathcal{N}^M$) the constant bin spacing equal to $\Omega_1, \Omega_2, \ldots, \Omega_p$ is kept throughout each relative frequency axis in the range $[0, 2\pi)$, respectively.

When $p = 2$, a multidimensional bivariate orthogonal multisine random process (BOMRS$^{M\text{-}D}$) $\mathbf{u}(\boldsymbol{i}_\nu) = [u_r(\boldsymbol{i}_\nu)]_{r=1,2}$ is obtained, where:

- the $u^B_1(\boldsymbol{i}_\nu)$ element contains the M-D constant component and M-D sine components with frequency M-tuples from the set

$$\mathcal{N}^{c,M}_{1,2} = \Big\{\{0,\Omega_1,\ldots,\Omega_1(N_1-1)\} \times \cdots \times \{0,\Omega_{M-1},\ldots,\Omega_{M-1}(N_{M-1}-1)\}$$
$$\times\{2\Omega_M, 4\Omega_M, \ldots, \Omega_M(\frac{N_M}{2}-2)\} \cup \mathcal{N}^{M-1} \times \{0,\pi\}\Big\}. \tag{6.54}$$

The frequency bins along frequency axes $\omega_1 T_1, \ldots, \omega_{M-1} T_{M-1}, \omega_M T_M$ are equal to $\Omega_1, \ldots, \Omega_{M-1}, 2\Omega_M$, respectively. The Mth frequency axis is sampled at even harmonics of Ω_M. Its frequency bin is equal to $2\Omega_M$.

- the $u^B_2(\boldsymbol{i}_\nu)$ element contains M-D sine components with frequency M-tuples from the set:

$$\mathcal{N}^{c,M}_{2,2} = \Big\{\{0,\Omega_1,\ldots,\Omega_1(N_1-1)\} \times \cdots \times \{0,\Omega_{M-1},\ldots,\Omega_{M-1}(N_{M-1}-1)\}$$
$$\times\{\Omega_M, 3\Omega_M, \ldots, \Omega_M(\frac{N_M}{2}-1)\}\Big\}. \tag{6.55}$$

The frequency bins along frequency axes $\omega_1 T_1, \ldots, \omega_{M-1} T_{M-1}, \omega_M T_M$ are equal to $\Omega_1, \ldots, \Omega_{M-1}, 2\Omega_M$, respectively. The Mth frequency axis is sampled at odd harmonics of Ω_M. Its frequency bin is also equal to $2\Omega_M$.

6.3.2 Nonergodic Case

Definitions

Consider an M-D multivariate random series with the elements $u_r(\boldsymbol{i}_\nu)$ $(r = 1, 2, \ldots, p$ and $p > 1)$ being M-D scalar multisine random series for which the same relative frequency appears in all elements of the multivariate series. This condition implies nonergodicity of the multivariate series. The elements of the NMOMRS$^{M\text{-}D}$ have common frequencies but the independence of its M-D sine components random phase shifts implies orthogonality of its elements for ensemble averaging if constant components of NMOMRS elements are equal to 0. In the sequel, thus constructed multivariate series are called nonergodic M-D multivariate orthogonal multisine random series (NMOMRS$^{M\text{-}D}$). The NMOMRS$^{M\text{-}D}$ is defined as:

Definition 6.3 *The basic NMOMRS$^{M\text{-}D}$ is defined by the p-dimensional multivariate series $\mathbf{u}^B(\boldsymbol{i}_\nu) = \left[u^B_r(\boldsymbol{i}_\nu)\right]_{r=1,2,\ldots,p}$, where the rth NMOMRS$^{M\text{-}D}$ element is given by:*

$$u^B_r(\boldsymbol{i}_\nu) = \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M} A_{r,(\boldsymbol{n}_\nu)} \sin\Big(\sum_{\nu=1}^{M} \Omega_\nu n_\nu i_\nu + \phi_{r,(\boldsymbol{n}_\nu)}\Big), \tag{6.56}$$

where $(\boldsymbol{i}_\nu) \in \mathcal{X}^M$ and for $\nu = 1, 2, \ldots, M$:

- *$\Omega_\nu = \frac{2\pi}{N_\nu}$ denote fundamental relative frequencies,*
- *n_ν denote the consecutive harmonics of these frequencies,*

$A_{r,(\boldsymbol{n}_\nu)}$ are deterministic amplitudes of the M-D sine components ($A_{r,(\boldsymbol{n}_\nu)} \in \mathcal{R}$), $\phi_{r,(\boldsymbol{n}_\nu)}$ are phase shifts, of which $\phi_{r,(\mathbf{o})}$ are deterministic and the remaining phase shifts are random, independent and:

- *uniformly distributed on $[0, 2\pi)$ for $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M \setminus \mathcal{N}^M_S$,*
- *Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi + \alpha\}\right)$ for $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_S \setminus \{(\mathbf{o})\}$.*

□

The basic NMOMRS$^{M\text{-}D}$ is given in the frequency-domain for the (relative) frequency range $[0, 2\pi)^M$ by the p-dimensional vector $\mathbf{U}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \left[U^B_r(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\right]_{r=1,2,\ldots,p}$ of its M-D finite discrete Fourier transform with the rth element given by:

$$U^B_r(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \frac{\prod_{\nu=1}^{M} N_\nu}{2j} \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M} A_{r,(\boldsymbol{n}_\nu)} \left[e^{j\phi_{r,(\boldsymbol{n}_\nu)}} \prod_{\nu=1}^{M} \delta(m_\nu - n_\nu) - e^{-j\phi_{r,(\boldsymbol{n}_\nu)}} \prod_{\nu=1}^{M} \delta(m_\nu - (N_\nu - n_\nu))\right], \tag{6.57}$$

where $(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) \in \mathcal{N}^M_{2\pi}$.

Properties

The periodogram matrix of the basic NMOMRS$^{M\text{-}D}$ is given by:

Lemma 6.7 *Consider the basic NMOMRS$^{M\text{-}D}$. Its periodogram matrix is $\boldsymbol{\Phi}^B_{\mathbf{uu}}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \left[\Phi^B_{u_r u_s}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\right]_{r,s=1,2,\ldots,p}$, where for $(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) \in \mathcal{N}^M_{2\pi}$:*

$$\Phi^B_{u_r u_s}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \frac{\prod_{\nu=1}^{M} N_\nu T_\nu}{4} \Bigg\{ 4A_{r,(\mathbf{o})} A_{s,(\mathbf{o})} \sin\phi_{r,(\mathbf{o})} \sin\phi_{s,(\mathbf{o})} \prod_{\nu=1}^{M} \delta(m_\nu)$$
$$+ \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M \setminus \mathcal{N}^M_S} A_{r,(\boldsymbol{n}_\nu)} A_{s,(\boldsymbol{n}_\nu)} \Big[(\cos(\phi_{r,(\boldsymbol{n}_\nu)} - \phi_{s,(\boldsymbol{n}_\nu)}) - j\sin(\phi_{r,(\boldsymbol{n}_\nu)} - \phi_{s,(\boldsymbol{n}_\nu)})) \prod_{\nu=1}^{M} \delta(m_\nu - n_\nu)$$
$$+ (\cos(\phi_{r,(\boldsymbol{n}_\nu)} - \phi_{s,(\boldsymbol{n}_\nu)}) + j\sin(\phi_{r,(\boldsymbol{n}_\nu)} - \phi_{s,(\boldsymbol{n}_\nu)})) \prod_{\nu=1}^{M} \delta(m_\nu - (N_\nu - n_\nu)) \Big]$$
$$+4 \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_S \setminus \{(\mathbf{o})\}} A_{r,(\boldsymbol{n}_\nu)} A_{s,(\boldsymbol{n}_\nu)} \sin\phi_{r,(\frac{N}{2})} \sin\phi_{s,(\frac{N}{2})} \prod_{\nu=1}^{M} \delta(m_\nu - n_\nu) \Bigg\}. \tag{6.58}$$

□

Proof of the above lemma proceeds similarly as for Lemma 6.4.

It follows from the above lemma that expected value of $\boldsymbol{\Phi}^B_{\mathbf{uu}}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)$ is the matrix $\mathcal{E}\{\boldsymbol{\Phi}^B_{\mathbf{uu}}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\} = \left[\mathcal{E}\{\Phi^B_{u_r u_s}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\}\right]_{r,s=1,2,\ldots,p}$, where:

- its diagonal elements are:

$$\mathcal{E}\{\Phi^B_{u_r u_r}(\Omega_\nu m_\nu)\} = \frac{\prod_{\nu=1}^M N_\nu T_\nu}{4}\Bigg\{(4A^2_{r,(0)}\sin^2\phi_{r,(0)} + j0)\prod_{\nu=1}^M \delta(m_\nu)$$

$$+ \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M\setminus\mathcal{N}_S^M} (A^2_{r,(n_\nu)} + j0)\left[\prod_{\nu=1}^M \delta(m_\nu - n_\nu) + \prod_{\nu=1}^M \delta(m_\nu - (N_\nu - n_\nu))\right]$$

$$+4\sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_S^M\setminus\{(0)\}} (A^2_{r,(n_\nu)}\sin^2\alpha + j0)\prod_{\nu=1}^M \delta(m_\nu - n_\nu)\Bigg\}, \tag{6.59}$$

for $r = 1,2,\ldots,p$, $(\Omega_\nu m_\nu) \in \mathcal{N}^M_{2\pi}$.

- its off-diagonal elements are:

$$\mathcal{E}\{\Phi^B_{u_r u_s}(\Omega_\nu m_\nu)\} = \frac{\prod_{\nu=1}^M N_\nu T_\nu}{4}\Bigg\{(4A_{r,(0)}A_{s,(0)}\sin\phi_{r,(0)}\sin\phi_{s,(0)} + j0)\prod_{\nu=1}^M \delta(m_\nu)$$

$$+ \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M\setminus\mathcal{N}_S^M} (0 + j0)\left[\prod_{\nu=1}^M \delta(m_\nu - n_\nu) + \prod_{\nu=1}^M \delta(m_\nu - (N_\nu - n_\nu))\right]$$

$$+4\sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_S^M\setminus\{(0)\}} (0 + j0)\prod_{\nu=1}^M \delta(m_\nu - n_\nu)\Bigg\}, \tag{6.60}$$

for $r, s = 1,2,\ldots,p$, $r \neq s$, $(\Omega_\nu m_\nu) \in \mathcal{N}^M_{2\pi}$.

If all $A_{s,(0)} = 0$ or $\phi_{r,(0)} = 0$ $(r = 1,2,\ldots,p)$ then the $\mathcal{E}\{\Phi^B_{u_r u_r}(\Omega_\nu m_\nu)\}$ is a diagonal matrix.

The properties of NMOMRS$^{M\text{-}D}$ which result from the ensemble averaging are given by:

Lemma 6.8 *Consider the extended NMOMRS$^{M\text{-}D}$. For each M-tuple $(i_\nu) \in \mathcal{X}^M_\infty$:*

1. *its expected value vector is $\mathcal{E}\{\mathbf{u}(i_\nu)\} = [\mathcal{E}\{u_r(i_\nu)\}]_{r=1,2,\ldots,p}$, where:*

$$\mathcal{E}\{u_r(i_\nu)\} = A_{r,(0)}\sin\phi_{r,(0)}. \tag{6.61}$$

2. *its correlation function matrix is $\mathcal{E}\{\mathbf{u}(i_\nu)\mathbf{u}^T(i_\nu-\tau_\nu)\}=[\mathcal{E}\{u_r(i_\nu)u_s(i_\nu-\tau_\nu)\}]_{r,s=1,2,\ldots,p}$, where for $(\tau_\nu) \in \mathcal{X}^M_\infty$:*

$$\mathcal{E}\{u_r(i_\nu)u_s(i_\nu-\tau_\nu)\} = \begin{cases} \mathcal{E}\{u_r(i_\nu)u_r(i_\nu-\tau_\nu)\} & \text{if } r = s \\ A_{r,(0)}A_{s,(0)}\ \sin\phi_{r,(0)}\sin\phi_{s,(0)} & \text{if } r \neq s \end{cases} \tag{6.62}$$

$\mathcal{E}\{u_r(i_\nu)u_r(i_\nu-\tau_\nu)\}$ is the autocorrelation function of the rth MOMRS$^{M\text{-}D}$ element:

$$\mathcal{E}\{u(i_\nu)u(i_\nu-\tau_\nu)\} = A^2_{r,(0)}\sin^2\phi_{r,(0)}$$

$$+\frac{1}{2}\sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M\setminus\mathcal{N}_S^M} A^2_{r,(n_\nu)}\cos\Big(\sum_{\nu=1}^M \Omega_\nu n_\nu \tau_\nu\Big) + \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_S^M\setminus\{(0)\}} (-1)^{\sum_{\nu=1}^M \tau_\nu \eta(\Omega_\nu n_\nu)} A^2_{r,(n_\nu)}\sin^2\alpha, \tag{6.63}$$

where $\eta(\Omega_\nu n_\nu)$ is given by (6.30).

□

Proof of the above lemma proceeds similarly as for Lemma 2.2.

When the independent variable domain averaging on any particular extended NMOMRS$^{M\text{-}D}$ is analysed, the following lemma can be formulated:

Lemma 6.9 *Consider the extended NMOMRS$^{M\text{-}D}$.*

1. *Its mean value vector is $\mathcal{M}\{\mathbf{u}(i_\nu)\} = [\mathcal{M}\{u_r(i_\nu)\}]_{r=1,2,\ldots,p}$, where:*

$$\mathcal{M}\{u_r(i_\nu)\} = A_{r,(0)}\sin\phi_{r,(0)}. \tag{6.64}$$

2. *Its correlation function matrix is $\mathbf{R}_{\mathbf{uu}}(\tau_\nu) = [R_{u_r u_s}(\tau_\nu)]_{r,s=1,2,\ldots,p}$, where for $(\tau_\nu) \in \mathcal{X}^M_\infty$:*

$$R_{u_r u_s}(\tau_\nu) = A_{r,(0)}A_{s,(0)}\sin\phi_{r,(0)}\sin\phi_{s,(0)}$$

$$+\frac{1}{2}\sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M\setminus\mathcal{N}_S^M} A_{r,(n_\nu)}A_{s,(n_\nu)}\cos\Big(\sum_{\nu=1}^M \Omega_\nu n_\nu \tau_\nu + \phi_{r,(n_\nu)} - \phi_{s,(n_\nu)}\Big)$$

$$+\sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_S^M\setminus\{(0)\}} (-1)^{\sum_{\nu=1}^M \tau_\nu\eta(\Omega_\nu n_\nu)} A_{r,(n_\nu)}A_{s,(n_\nu)}\sin\phi_{r,(\frac{N}{2})}\sin\phi_{s,(\frac{N}{2})}, \tag{6.65}$$

where $\eta(\Omega_\nu n_\nu)$ is given by (6.30).

□

Proof of the above lemma proceeds similarly as for Lemma 6.3.

The expected value of $\mathbf{R}_{\mathbf{uu}}(\tau_\nu)$ is the matrix $\mathcal{E}\{\mathbf{R}_{\mathbf{uu}}(\tau_\nu)\} = [\mathcal{E}\{R_{u_r u_s}(\tau_\nu)\}]_{r,s=1,2,\ldots,p}$, where:

- its diagonal elements are:

$$R_{u_r u_r}(\tau_\nu) = A^2_{r,(0)}\sin^2\phi_{r,(0)}$$

$$+\frac{1}{2}\sum_{(\Omega_\nu n_\nu)\in\mathcal{N}^M\setminus\mathcal{N}_S^M} A^2_{r,(n_\nu)}\cos\Big(\sum_{\nu=1}^M \Omega_\nu n_\nu\tau_\nu\Big) + \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_S^M\setminus\{(0)\}} (-1)^{\sum_{\nu=1}^M \tau_\nu\eta(\Omega_\nu n_\nu)} A^2_{r,(n_\nu)}\sin^2\alpha \tag{6.66}$$

for $r = 1,2,\ldots,p$;

- its off-diagonal elements are:

$$R_{u_r u_s}(\tau_\nu) = A_{r,(0)}A_{s,(0)}\sin\phi_{r,(0)}\sin\phi_{s,(0)} \tag{6.67}$$

for $r, s = 1,2,\ldots,p$ and $r \neq s$.

It is worth to note that $\mathcal{E}\{\mathbf{R}_{\mathbf{uu}}(\tau_\nu)\} = \mathcal{E}\{\mathbf{u}(i_\nu)\mathbf{u}^T(i_\nu-\tau_\nu)\}$. If $A_{r,(0)} = 0$ or $\phi_{r,(0)} = 0$ $(r = 1,2,\ldots,p)$ then the $\mathcal{E}\{\mathbf{R}_{\mathbf{uu}}(\tau_\nu)\}$ and $\mathcal{E}\{\mathbf{u}(i_\nu)\mathbf{u}^T(i_\nu-\tau_\nu)\}$ are diagonal matrices.

6.4 *M*-D MULTIVARIATE NONORTHOGONAL MULTISINE RANDOM PROCESS

Definitions

The basic nonergodic multivariate nonorthogonal multisine random series (NMMRS$^{M\text{-}D}$) is defined by:

Definition 6.4 *The basic NMMRS$^{M\text{-}D}$ is defined by the p-dimensional multivariate series $\mathbf{u}^B(\boldsymbol{i}_\nu) = \left[u_r^B(\boldsymbol{i}_\nu)\right]_{r=1,2,\dots,p}$, where the rth ($r = 1,2,\dots,p$) NMMRS$^{M\text{-}D}$ element is given by:*

$$u_r^B(\boldsymbol{i}_\nu) = \sum_{t=1}^{p} \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu)\in\mathcal{N}^M} A_{r,t,(\boldsymbol{n}_\nu)} \sin\left(\sum_{\nu=1}^{M} \Omega_\nu n_\nu i_\nu + \phi_{t,(\boldsymbol{n}_\nu)} + \varphi_{r,t,(\boldsymbol{n}_\nu)}\right), \tag{6.68}$$

where $(\boldsymbol{i}_\nu) \in \mathcal{X}^M$ and for $\nu = 1,2,\dots,M$:

- $\Omega_\nu = \frac{2\pi}{N_\nu}$ *denote fundamental relative frequencies,*
- n_ν *denote the consecutive harmonics of these frequencies,*

and $A_{r,t,(\boldsymbol{n}_\nu)}$ are deterministic amplitudes of the M-D sine components ($A_{r,t,(\boldsymbol{n}_\nu)} \in \mathcal{R}$), $\phi_{t,(\boldsymbol{n}_\nu)}$ and $\varphi_{r,t,(\boldsymbol{n}_\nu)}$ are phase shifts, of which $\phi_{t,(\mathbf{o}_\nu)}$ and $\varphi_{r,t,(\boldsymbol{n}_\nu)}$ are deterministic and the remaining phase shifts $\phi_{t,(\boldsymbol{n}_\nu)}$ are random, independent and:

- *uniformly distributed on $[0,2\pi)$ for $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M$ and $t = 1,2,\dots,p$,*
- *Bernoulli distributed $\mathcal{B}\left(\frac{1}{2}, \{\alpha, \pi+\alpha\}\right)$ for $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}_S^M \setminus \{(\mathbf{o})\}$ and $t = 1,2,\dots,p$.*

□

The basic NMMRS$^{M\text{-}D}$ is defined in the frequency-domain by the p-dimensional vector $\mathbf{U}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \left[U_r^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\right]_{r=1,2,\dots,p}$ of its M-D finite discrete Fourier transforms with the rth element given by:

$$U_r^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \frac{\prod_{\nu=1}^{M} N_\nu}{2j} \sum_{t=1}^{p} \sum_{(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu)\in\mathcal{N}^M} A_{r,t,(\boldsymbol{n}_\nu)} \left[e^{j(\phi_{t,(\boldsymbol{n}_\nu)}+\varphi_{r,t,(\boldsymbol{n}_\nu)})} \prod_{\nu=1}^{M} \delta(m_\nu - n_\nu) \right.$$

$$\left. - e^{-j(\phi_{t,(\boldsymbol{n}_\nu)}+\varphi_{r,t,(\boldsymbol{n}_\nu)})} \prod_{\nu=1}^{M} \delta(m_\nu - (N_\nu - n_\nu)) \right], \tag{6.69}$$

where $(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) \in \mathcal{N}_{2\pi}^M$.

Properties

Similarly as for the 1-D NMMRS, the periodogram matrix $\boldsymbol{\Phi}_{\mathbf{uu}}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)$ of the basic NMMRS$^{M\text{-}D}$ can be written as:

$$\boldsymbol{\Phi}_{\mathbf{uu}}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \mathbf{K}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\boldsymbol{\Phi}_{\boldsymbol{\beta\beta}}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\mathbf{K}^*(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu), \tag{6.70}$$

where:

- elements of the matrix $\mathbf{K}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = [K_{u_r u_s}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)]_{r,s=1,2,\dots,p}$ are given by:

$$\frac{K_{u_r u_s}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)}{\sqrt{\prod_{\nu=1}^{M} N_\nu T_\nu}} = \begin{cases} A_{r,s,(\mathbf{o})} \sin\left(\phi_{t,(\mathbf{o})} + \varphi_{r,s,(\mathbf{o})}\right) + j0 & \text{if } (\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = (\mathbf{o}) \\ \frac{1}{2} A_{r,s,(\boldsymbol{m}_\nu)} e^{j\varphi_{r,s,(\boldsymbol{m}_\nu)}} & \text{if } (\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M \\ A_{r,s,(\boldsymbol{m}_\nu)} \sin\left(\alpha + \varphi_{r,s,(\boldsymbol{m}_\nu)}\right) + j0 & \text{if } (\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) \in \mathcal{N}_S^M \setminus \{(\mathbf{o})\} \end{cases}; \tag{6.71}$$

- the matrix $\boldsymbol{\Phi}_{\boldsymbol{\beta\beta}}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)$ is the periodogram matrix of a NMOMRS$^{M\text{-}D}$ with amplitudes of all sine components chosen so that $\mathcal{E}\{\boldsymbol{\Phi}_{\boldsymbol{\beta\beta}}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\} = \boldsymbol{I}$.

The above spectral factorisation of the NMMRS$^{M\text{-}D}$ periodogram matrix allows us to write the M-D finite discrete Fourier transform $\mathbf{U}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)$ of NMMRS$^{M\text{-}D}$ as:

$$\mathbf{U}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu) = \mathbf{K}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\boldsymbol{\beta}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu), \tag{6.72}$$

where $\boldsymbol{\beta}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)$ is the M-D finite discrete Fourier transform of a NMOMRS$^{M\text{-}D}$ with amplitudes of its M-D sine components chosen so that $\mathcal{E}\{\boldsymbol{\Phi}_{\boldsymbol{\beta\beta}}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\} = \boldsymbol{I}$. It is obvious that:

$$\mathcal{E}\{\boldsymbol{\Phi}_{\mathbf{uu}}^B(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\} = \mathbf{K}(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu)\mathbf{K}^*(j\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu). \tag{6.73}$$

The statistical properties of NMMRS$^{M\text{-}D}$ can be analysed similarly as it was done in Chapter 2 for the corresponding 1-D NMMRS.

6.5 SYNTHESIS AND SIMULATION

Synthesis and simulation of multidimensional multisine random processes follow the corresponding procedure for the 1-D multisine random processes. For a given deterministic amplitudes of M-D sine components of multidimensional multisine random processes, phase shifts for constant components, parameters of Bernoulli distributions, the corresponding spectrum is synthesised and a realisation of the basic multidimensional multisine random process approximation may be obtained using any M-D Fast Fourier Transform algorithm [5]. Additionally, in spite of random phase shifts, the periodogram and correlation function matrices for weakly ergodic or expected values of periodogram and correlation function matrices for nonergodic multidimensional multisine random processes are deterministic, real-valued functions. They are uniquely defined by the amplitudes of M-D sine components, phase shifts for constant components and parameters of Bernoulli distributions. It implies that like in the 1-D case, shapes of multidimensional multisine random process periodogram matrix elements can be fitted to shapes of any given power spectral density function matrix elements of M-D multivariate random process. This allows us to extend the proposed synthesis and simulation method of 1-D wide sense stationary random processes to the M-D case.

Let $\mathbf{v}(\boldsymbol{i}_\nu)$ be an M-D wide-sense stationary, real-valued multivariate random process with the power spectral density matrix $\boldsymbol{\Phi}_{\mathbf{vv}}(j\boldsymbol{\omega}_\nu \boldsymbol{T}_\nu) = [\Phi_{v_r v_s}(j\boldsymbol{\omega}_\nu \boldsymbol{T}_\nu)]_{r,s=1,2,\dots,p}$, which satisfies, for $(\boldsymbol{\omega}_\nu \boldsymbol{T}_\nu) \in [0,2\pi)^M$, the following conditions:

$$\boldsymbol{\Phi}_{\mathbf{vv}}(j\boldsymbol{\omega}_\nu \boldsymbol{T}_\nu) = \boldsymbol{\Phi}_{\mathbf{vv}}(j(\mathbf{2\pi} - \boldsymbol{\omega}_\nu \boldsymbol{T}_\nu)) \tag{6.74}$$

and:

$$\|\boldsymbol{\Phi}_{\mathbf{vv}}(j\boldsymbol{\omega}_\nu \boldsymbol{T}_\nu)\| < \infty, \tag{6.75}$$

where:

$$\|\boldsymbol{\Phi}_{\mathbf{vv}}(j\boldsymbol{\omega}_\nu \boldsymbol{T}_\nu)\| = \sqrt{\sum_{r=1}^{p}\sum_{s=1}^{p} \Phi_{v_r v_s}(j\boldsymbol{\omega}_\nu \boldsymbol{T}_\nu)}. \tag{6.76}$$

It is assumed that its autocorrelation function $\mathbf{R}_{\mathbf{vv}}(\boldsymbol{\tau}_\nu)$ for all M-tuple lags $(\boldsymbol{\tau}_\nu)$ with the elements $|\tau_\nu| > \tau_{\nu,0}$ ($\nu = 1,2,\dots,N-1$) satisfies:

$$\mathbf{R}_{\mathbf{vv}}(\boldsymbol{\tau}_\nu) = \mathbf{o}. \tag{6.77}$$

The power spectral density matrix $\boldsymbol{\Phi}_{\mathbf{vv}}(j\boldsymbol{\omega}_\nu \boldsymbol{T}_\nu)$ may be approximated by a periodogram matrix (or expected value of the periodogram matrix) of the corresponding multidimensional multisine random series with amplitudes of its M-D sine components chosen so as to make values of the periodogram matrix (or expected value of the periodogram matrix) equal to the corresponding values of power spectral density matrix of the original multidimensional random process for some equally spaced frequency M-tuples from the range $[0,2\pi)^M$. It can be interpreted as multidimensional sampling of the elements $\Phi_{v_r v_s}(\omega_\nu T_\nu)$ $(r, s = 1, 2, \ldots, p)$ in the frequency-domain. When $n_\nu^{'}$ sampling points (approximation nodes) are chosen for the νth frequency axis in the frequency range $[0, 2\pi)$, the multidimensional sampling does not produce aliasing [21] under spacings Δ_ν between the samples along this axis are such that:

$$\Delta_\nu \leq \Delta_{0,\nu} = \frac{2\pi}{2\tau_{\nu,0}}. \tag{6.78}$$

When maximum spacings $\Delta_{0,\nu}$ are chosen, the original power spectral densities $\Phi_{v_r v_s}(\omega_\nu T_\nu)$ may be recovered from their sampled values (periodograms of approximating multidimensional multisine random series) by using an M-D generalisation of the sinc function:

$$\Phi_{v_r v_s}(j\omega_\nu T_\nu) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_M=-\infty}^{\infty} \Phi_{v_r v_s}(j\Delta_\nu m_\nu) \prod_{\nu=1}^{M} \operatorname{sinc}\left(\frac{\pi(\omega_\nu T_\nu - \Delta_\nu m_\nu)}{\Omega_\nu}\right). \tag{6.79}$$

Asymptotic properties of the synthesised and simulated multidimensional multisine random process approximations of wide-sense stationary multidimensional random processes given by their power spectral density matrices are briefly summarised in the sequel. It should be emphasised that proofs of presented lemmas proceed similarly as for the power spectral density defined multisine random time-series (see Chapter 3).

6.5.1 Ergodic case

When the power spectral density matrix of an M-D causal, wide-sense stationary multivariate orthogonal random process is approximated by the periodogram matrix of a MOMRS^{M-D} with the consecutively circularly ordered frequency M-tuples and amplitudes chosen so as to make values of elements of the periodogram matrix equal to the corresponding values of the power spectral densities of the original random process for some frequency N-tuples from the range $[0, 2\pi)^N$, the obtained extended MOMRS^{M-D} turns asymptotically for $(N_\nu) \to (\infty)$ into an M-D Gaussian multivariate orthogonal multisine random process:

Lemma 6.10 *Assuming that:*

1. $\Phi_{\mathbf{vv}}(j\omega_\nu T_\nu) = diag[\Phi_{\mathbf{v_r v_r}}(\omega_\nu T_\nu) + j0]_{r=1,2,\ldots,p}$ *is the power spectral density matrix of an M-D wide-sense stationary, orthogonal, real-valued multivariate random process with zero mean vector and the variance matrix* $\boldsymbol{\sigma}^2_{\mathbf{vv}} = [\sigma^2_{v_r v_r}]_{r=1,2,\ldots,p}$, *where:*

$$\sigma^2_{v_r v_r} = \frac{1}{(2\pi)^M \prod_{\nu=1}^{M} T_\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_{v_r v_r}(\omega_\nu T_\nu) d(\omega_1 T_1) \cdots d(\omega_N T_N); \tag{6.80}$$

2. $A_{(\boldsymbol{n}_\nu)}$ *converges to* 0 *for* $(N_\nu) \to (\infty)$ *in such a way that for* $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^{c,M}_{r,p} \setminus \mathcal{N}^M_S$

$$\frac{A^2_{(\boldsymbol{n}_\nu)}}{4} \prod_{\nu=1}^{M} N_\nu T_\nu = \Phi_{v_r v_r}(\boldsymbol{\Omega}_\nu \boldsymbol{m}_\nu), \tag{6.81}$$

3. $A_{(\boldsymbol{n}_\nu)} = 0$ *for* $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_S$ *or* $\phi_{(\mathbf{o})} = \alpha = 0$,

then the extended MOMRS^{M-D} $\mathbf{u}(\boldsymbol{i}_\nu)$ *with the consecutively circularly ordered frequency M-tuples converges in distribution for $(N_\nu) \to (\infty)$ to an M-D Gaussian multivariate orthogonal multisine random process of type 1 (GMOMRS1^{M-D})* $\mathbf{g}(\boldsymbol{i}_\nu) = [g_r(\boldsymbol{i}_\nu)]_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\frac{1}{p}\boldsymbol{\sigma}^2_{\mathbf{vv}}$:

$$\mathbf{g}(\boldsymbol{i}_\nu) \in As\mathcal{N}(\mathbf{o}, \frac{1}{p}\boldsymbol{\sigma}^2_{\mathbf{vv}}). \tag{6.82}$$

Additionally the correlation function matrix $\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = \mathbf{R}_{\mathbf{gg}}(\boldsymbol{\tau}_\nu)$ *of the GMOMRS1^{M-D} converges to:*

$$\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = \mathbf{R}_{\mathbf{gg}}(\boldsymbol{\tau}_\nu)$$

$$= \frac{1}{(2\pi)^M p \prod_{\nu=1}^{M} T_\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_{\mathbf{vv}}(j\omega_\nu T_\nu) cos(\sum_{\nu=1}^{M} \omega_\nu T_\nu \tau_\nu) d(\omega_1 T_1) \cdots d(\omega_M T_M) = \frac{1}{p}\mathbf{R}_{\mathbf{vv}}(\boldsymbol{\tau}_\nu), \tag{6.83}$$

where $(\boldsymbol{\tau}_\nu) \in \mathcal{X}^M_\infty$.

□

6.5.2 Nonergodic case

Similarly as for the previous ergodic case, the power spectral density matrix of an M-D causal, wide-sense stationary multivariate orthogonal or nonorthogonal random process with finite powers of its elements may be approximated by the expected value of periodogram matrix of a multidimensional nonergodic multivariate multisine random series with amplitudes of M-D sine components chosen so as to make values of the expected value of its periodogram matrix equal to the corresponding values of power spectral density matrix of the original random process for $N_1 N_2 \cdots N_N$ frequency M-tuples from the range $[0, 2\pi)^M$. The extended multidimensional multivariate orthogonal or nonorthogonal multisine random series obtained from application of this approximation criterion turns asymptotically for $(N_\nu) \to (\infty)$ into an ergodic Gaussian multivariate orthogonal or nonorthogonal multisine random process. In the sequel, their properties are briefly summarised.

Multivariate orthogonal multidimensional multisine random processes

Lemma 6.11 *Assuming that:*

1. $\Phi_{\mathbf{vv}}(j\omega_\nu T_\nu) = diag[\Phi_{v_r v_r}(\omega_\nu T_\nu) + j0]_{r=1,2,\ldots,p}$ *is the power spectral density matrix of an M-D wide-sense stationary, orthogonal, real-valued multivariate random process with zero mean vector and the variance matrix* $\boldsymbol{\sigma}^2_{\mathbf{vv}} = [\sigma^2_{v_r v_r}]_{r=1,2,\ldots,p}$, *where:*

$$\sigma^2_{v_r v_r} = \frac{1}{(2\pi)^M \prod_{\nu=1}^{M} T_\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_{v_r v_r}(\omega_\nu T_\nu) d(\omega_1 T_1) \cdots d(\omega_N T_N); \tag{6.84}$$

2. $A_{r,(\boldsymbol{n}_\nu)}$ *converges to* 0 *for* $(N_\nu) \to (\infty)$ *in such a way that for* $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}_N \setminus \mathcal{N}^N_S$

$$\frac{A^2_{r,(\boldsymbol{n}_\nu)}}{4} \prod_{\nu=1}^{M} N_\nu T_\nu = \Phi_{v_r v_r}(j\omega_\nu T_\nu), \tag{6.85}$$

3. $A_{r,(\boldsymbol{n}_\nu)} = 0$ *for* $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M_S$ *or* $\phi_{r,(\mathbf{o})} = \alpha = 0$,

then the extended NMOMRS^{M-D} $\mathbf{u}(\boldsymbol{i}_\nu)$ *converges in distribution for $(N_\nu) \to (\infty)$ to an M-D ergodic Gaussian multivariate orthogonal multisine random process of type 2 (GMOMRS2^{M-D})* $\mathbf{g}(\boldsymbol{i}_\nu) = [g_r(\boldsymbol{i}_\nu)]_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\boldsymbol{\sigma}^2_{\mathbf{vv}}$:

$$\mathbf{g}(\boldsymbol{i}_\nu) \in As\mathcal{N}(\mathbf{o}, \boldsymbol{\sigma}^2_{\mathbf{vv}}). \tag{6.86}$$

Additionally the correlation function matrix $\mathcal{E}\{\mathbf{g}(\boldsymbol{i}_\nu)\mathbf{g}^T(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\}$ *of the GMOMRS^{M-D} converges to:*

$$\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = \mathbf{R}_{\mathbf{gg}}(\boldsymbol{\tau}_\nu)$$

$$= \frac{1}{(2\pi)^M \prod_{\nu=1}^{M} T_\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_{\mathbf{vv}}(j\omega_\nu T_\nu) \cos(\sum_{\nu=1}^{M} \omega_\nu T_\nu \tau_\nu) d(\omega_1 T_1) \cdots d(\omega_M T_M), \tag{6.87}$$

where $(\boldsymbol{\tau}_\nu) \in \mathcal{X}^M_\infty$.

□

Multivariate nonorthogonal multidimensional multisine random processes

Lemma 6.12 *Assuming that:*

1. $\mathbf{\Phi_{vv}}(j\omega_\nu T_\nu) = [\Phi_{v_r v_s}(j\omega_\nu T_\nu)]_{r,s=1,2,\ldots,p}$ *is the power spectral density matrix of a wide-sense stationary, real-valued multivariate nonorthogonal multidimensional random process with zero mean vector and the variance matrix* $\sigma^2_{\mathbf{vv}} = [\sigma^2_{v_r v_s}]_{r,s=1,2,\ldots,p}$, *where:*
$$\sigma^2_{v_r v_s} = \frac{1}{(2\pi)^M \prod_{\nu=1}^M T_\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_{v_r v_s}(\omega_\nu T_\nu) d(\omega_1 T_1) \cdots d(\omega_N T_N); \tag{6.88}$$
2. *for* $r, t = 1, 2, \ldots, p$ *values of* $A_{r,t,(n_\nu)}$ *converge to* 0 *for* $(N_\nu) \to (\infty)$ *in such a way that for* $(\Omega_\nu n_\nu) \in \mathcal{N}_N \setminus \mathcal{N}_S^N$:
$$\mathcal{E}\{\mathbf{\Phi}^B_{\mathbf{uu}}(j\Omega_\nu n_\nu)\} = \mathbf{\Phi_{vv}}(j\Omega_\nu n_\nu); \tag{6.89}$$
3. $A_{r,(n_\nu)} = 0$ *for* $(\Omega_\nu n_\nu) \in \mathcal{N}_S^M$ *or* $\phi_{r,(\mathbf{o})} = \alpha = 0$,

then the extended NMMRS^{M-D} $\mathbf{u}(\mathbf{i}_\nu)$ *converges in distribution for* $(N_\nu) \to (\infty)$ *to an ergodic Gaussian multivariate nonorthogonal multidimensional multisine random process (GNMMRS^{M-D})* $\mathbf{g}(\mathbf{i}_\nu) = [g_r(\mathbf{i}_\nu)]_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\sigma^2_{\mathbf{vv}}$:
$$\mathbf{g}(\mathbf{i}_\nu) \in As\mathcal{N}(\mathbf{o}, \sigma^2_{\mathbf{vv}}). \tag{6.90}$$
Additionally the correlation function matrix $\mathcal{E}\{\mathbf{g}(\mathbf{i}_\nu)\mathbf{g}^T(\mathbf{i}_\nu - \tau_\nu)\}$ *of the GNMMRS^{M-D} converges to:*
$$\mathcal{E}\{\mathbf{g}(\mathbf{i})\mathbf{g}^T(\mathbf{i}_\nu - \tau_\nu)\} = \mathbf{R_{gg}}(\tau_\nu)$$
$$= \frac{1}{(2\pi)^M \prod_{\nu=1}^M T_\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbf{\Phi_{vv}}(j\omega_\nu T_\nu) \cos(\sum_{\nu=1}^M \omega_\nu T_\nu \tau_\nu) d(\omega_1 T_1) \cdots d(\omega_M T_M), \tag{6.91}$$
where $(\tau_\nu) \in \mathcal{X}_\infty^M$.

□

Chapter 7

Multidimensional White Noise Approximation

This chapter addresses a direct extension of the 1-D white noises synthesis and simulation results presented in Chapter 4 to the multidimensional case. Multidimensional scalar, bivariate and multivariate white multisine random series are discussed.

7.1 SCALAR WHITE NOISE

When the power spectral density of the M-D white noise is approximated by the periodogram of SMRS^{M-D}, an extended multidimensional (N_ν)-lag white multisine random series (WSMRS^{M-D}) is obtained:

Definition 7.1 *An extended M-D scalar multisine random series* $x(\mathbf{i}_\nu)$ *is said to be* (N_ν)*-lag white if its autocorrelation function for lags* $(\tau_\nu) \in \mathcal{X}^M$ *is the same as the M-D white noise autocorrelation function, i. e.:*
$$\mathcal{E}\{x(\mathbf{i}_\nu)x(\mathbf{i}_\nu - \tau_\nu)\} = R_{xx}(\tau_\nu) = \begin{cases} \Gamma^2 & \text{if } (\tau_\nu) = (\mathbf{o}) \\ 0 & \text{if } (\tau_\nu) \in \mathcal{X}^M \setminus \{(\mathbf{o})\} \end{cases} \tag{7.1}$$

□

The statistical properties of WSMRS^{M-D} are given by the following lemma:

Lemma 7.1 *Assuming that:*

1. $\Phi_{vv}(\omega_\nu T_\nu) = \lambda^2$ $((\omega_\nu T_\nu) \in [0, 2\pi)^M)$ *is the power spectral density of a real-valued M-D white noise;*
2. $A_{(n_\nu)} = A$ *for* $(\Omega_\nu n_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M$ *and the value of* A *is chosen so that:*
$$\frac{A^2}{4} \prod_{\nu=1}^M N_\nu T_\nu = \lambda^2; \tag{7.2}$$
3. $A_{(n_\nu)} = \frac{A}{2}$ *for* $(\Omega_\nu n_\nu) \in \mathcal{N}_S^M$ *and* $\phi_{(\mathbf{o})} = \alpha = \frac{\pi}{2}$;

then the extended SMRS^{M-D} is an M-D white multisine random series (WSMRS^{M-D}) and:

1. *its periodogram is given by:*
$$\Phi^M_{uu}(\Omega_\nu m_\nu) = \lambda^2 \sum_{n_1=0}^{N_1-1} \cdots \sum_{n_M=0}^{N_M-1} \prod_{\nu=1}^M \delta(m_\nu - n_\nu), \tag{7.3}$$
where $(\Omega_\nu m_\nu) \in \mathcal{N}_{2\pi}^M$.

2. *its mean value is:*

$$\mathcal{M}\{u(i_\nu)\} = \mathcal{E}\{u(i_\nu)\} = \sqrt{\frac{1}{\prod_{\nu=1}^{M} N_\nu T_\nu}\lambda^2}. \tag{7.4}$$

3. *its autocorrelation function is:*

$$\mathcal{E}\{i_\nu)u(i_\nu - \tau_\nu)\} = R_{uu}(\tau_\nu) = \begin{cases} \frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu} & \text{if } (\tau_\nu) \in \mathcal{X}_0^M \\ 0 & \text{otherwise} \end{cases}. \tag{7.5}$$

□

Proof: *The proof of this this lemma proceeds similarly as for Lemma 4.1, i.e.:*

1. *It follows immediately from the assumptions 1, 2, 3 and from Lemma 6.1.*
2. *It follows Immediately from Definition 6.1.*
3. *Application of (6.34) to (7.3) results in:*

$$\mathcal{E}\{u(i_\nu)u(i_\nu - \tau_\nu)\} = R_{uu}(\tau_\nu) = \frac{A^2}{4}\sum_{m_1=0}^{N_1-1}\cdots\sum_{m_M=0}^{N_M-1}\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_M=0}^{N_M-1}\prod_{\nu=1}^{M}\delta(m_\nu - n_\nu)e^{j\sum_{\nu=1}^{M}\Omega_\nu m_\nu i_\nu}$$

$$= \frac{A^2}{4}\sum_{m_1=0}^{N_1-1}\cdots\sum_{m_M=0}^{N_M-1} e^{j\sum_{\nu=1}^{N}\Omega_\nu n_\nu \tau_\nu}. \tag{7.6}$$

This ends the proof when (6.7) is taken into account.

□

When $(N_\nu) \to (\infty)$, the WSMRS$^{M\text{-}D}$ converges to a Gaussian WSMRS$^{M\text{-}D}$ $g(i_\nu)$ with zero mean and the variance $\frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu}$:

$$g(i_\nu) \in As\mathcal{N}(0, \frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu}). \tag{7.7}$$

7.2 MULTIVARIATE WHITE NOISE

7.2.1 Ergodic Case

M-D bivariate $(N_1, \ldots, N_{M-1}, \frac{N_M}{2})$-lag white multisine random series

When the power spectral density matrix of an M-D bivariate white noise is approximated by the periodogram matrix of the extended BOMRS$^{M\text{-}D}$, an M-D bivariate orthogonal white multisine random series is obtained. It is characterised by the correlation function matrix which for a number of M-tuple lags behaves exactly like the correlation function matrix of M-D bivariate white noise. This series is called multidimensional bivariate $(N_1, \ldots, N_{M-1}, \frac{N_M}{2})$-lag white multisine random series (BOWMRS$^{M\text{-}D}$):

Definition 7.2 *An extended M-D bivariate orthogonal multisine random series $\mathbf{x}(i_\nu)$ is said to be $(N_1, \ldots, N_{M-1}, \frac{N_M}{2})$-lag white if its correlation function matrix $\mathbf{R_{xx}}(\tau_\nu) = \mathcal{E}\{\mathbf{x}(i_\nu)\mathbf{x}^T(i_\nu - \tau_\nu)\}$ for lags $(\tau_\nu) \in \mathcal{X}^{M-1} \times \{0, 1, \ldots, \frac{N_M}{2} - 1\}$ is the same as for the M-D bivariate white noise correlation function matrix - its elements satisfy the conditions:*

$$R_{x_1x_1}(\tau_\nu) = R_{x_2x_2}(\tau_\nu) = \begin{cases} \Gamma^2 & \text{if } (\tau_\nu) = (\mathrm{o}) \\ 0 & \text{if } (\tau_\nu) \in \mathcal{X}^{M-1} \times \{0, 1, \ldots, \frac{N_M}{2} - 1\} \setminus \{(\mathrm{o})\} \end{cases}, \tag{7.8}$$

$$R_{x_1x_2}(\tau_\nu) = R_{x_2x_1}(\tau_\nu) = 0. \tag{7.9}$$

□

The spectral and correlation properties of the BOWMRS$^{M\text{-}D}$ are the following:

Lemma 7.2 *Assuming that:*

1. *$\Phi_{\mathbf{vv}}(j\omega_\nu T_\nu) = \lambda^2 I$ $((\omega_\nu T_\nu) \in [0, 2\pi)^M)$) is the power spectral density matrix of an M-D real-valued bivariate white noise;*
2. *$A_{(n_\nu)} = A$ for $(\Omega_\nu n_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M$ and the value of A is chosen so that:*

$$\frac{A^2}{4}\prod_{\nu=1}^{M} N_\nu T_\nu = \lambda^2, \tag{7.10}$$

3. *$A_{(n_\nu)} = \frac{A}{2}$ for $(\Omega_\nu n_\nu) \in \mathcal{N}_S^M$ and $\phi_{(\mathrm{o})} = \alpha = \frac{\pi}{2}$,*

then the extended BOMRS$^{M\text{-}D}$ is a bivariate $(N_1, \ldots, N_{M-1}, \frac{N_M}{2})$-lag white multisine random series (BOWMRS$^{M\text{-}D}$) and:

1. *its periodogram matrix is $\Phi^B_{\mathbf{uu}}(j\Omega_\nu m_\nu) = diag\left[\Phi^B_{u_ru_r}(\Omega_\nu m_\nu) + j0\right]_{r=1,2}$, where for $(\Omega_\nu m_\nu) \in \mathcal{N}_{2\pi}^M$:*

$$\Phi^B_{u_1u_1}(\Omega_\nu m_\nu) = \lambda^2 \left\{ \prod_{\nu=1}^{M}\delta(m_\nu) + \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_{1,2}^{c,M}\setminus\mathcal{N}_S^M} \left[\prod_{\nu=1}^{M}\delta(m_\nu - n_\nu) + \prod_{\nu=1}^{M}\delta(m_\nu - (N_\nu - n_\nu))\right] + \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_S^M\setminus\{(\mathrm{o})\}}\prod_{\nu=1}^{M}\delta(m_\nu - n_\nu)\right\}, \tag{7.11}$$

$$\Phi^B_{u_2u_2}(\Omega_\nu m_\nu) = \lambda^2 \sum_{(\Omega_\nu n_\nu)\in\mathcal{N}_{2,2}^{c,M}} \left[\prod_{\nu=1}^{M}\delta(m_\nu - n_\nu) + \prod_{\nu=1}^{M}\delta(m_\nu - (N_\nu - n_\nu))\right], \tag{7.12}$$

2. *its mean value vector is*

$$\mathcal{M}\{\mathbf{u}(i_\nu)\} = \mathcal{E}\{\mathbf{u}(i_\nu)\} = \begin{bmatrix} \sqrt{\frac{1}{\prod_{\nu=1}^{M} N_\nu T_\nu}\lambda^2} \\ 0 \end{bmatrix}. \tag{7.13}$$

3. *its correlation function matrix is $\mathcal{E}\{\mathbf{u}(i_\nu)\mathbf{u}(i_\nu - \tau_\nu)\} = \mathbf{R_{uu}}(\tau_\nu) = diag[R_{u_ru_r}(\tau_\nu)]_{r=1,2}$, where for $(\tau_\nu) \in \mathcal{X}_\infty^M$:*

$$R_{u_1u_1}(\tau_\nu) = \begin{cases} \frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu} & \text{if } (\tau_\nu) \in \mathcal{X}_0^{M-1} \times \{0, \frac{N_M}{2}, \ldots\} \\ 0 & \text{otherwise} \end{cases}, \tag{7.14}$$

$$R_{u_2u_2}(\tau_\nu) = \begin{cases} \frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu} & \text{if } (\tau_\nu) \in \mathcal{X}_0^M \\ -\frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu} & \text{if } (\tau_\nu) \in \mathcal{X}_0^{M-1} \times \{\frac{N_M}{2}, \frac{3N_M}{2}, \ldots\} \\ 0 & \text{otherwise} \end{cases}. \tag{7.15}$$

□

Proof: *This lemma can be proven similarly as Lemma 4.4. It should be noticed that for all M-tuples $(n_1, n_2, \ldots, n_{M-1}, n_M^e)$ with n_M^e zero or even it holds that:*

$$\sum_{n_1=0}^{N_1-1} \cdots \sum_{n_{M-1}=0}^{N_{M-1}-1} \sum_{n_M^e=0}^{N_M-2} e^{j(\sum_{\nu=1}^{M-1} \Omega n_\nu \tau_\nu + \Omega n_M^e \tau_M)}$$

$$= \begin{cases} \frac{\prod_{\nu=1}^{M} N_\nu}{2} & \text{if}(\tau_\nu) \in \mathcal{X}_0^{M-1} \times \left\{0, \frac{N_M}{2}, \ldots\right\} \\ 0 & \text{otherwise} \end{cases}, \tag{7.16}$$

and for $(n_1, n_2, \ldots, n_{M-1}, n_M^o)$, where n_M^o is odd it holds that:

$$\sum_{n_1=0}^{N_1-1} \cdots \sum_{n_{M-1}=0}^{N_{M-1}-1} \sum_{n_M^o=1}^{N_M-1} e^{j(\sum_{\nu=1}^{M-1} \Omega n_\nu \tau_\nu + \Omega n_M^o \tau_M)}$$

$$= \begin{cases} \frac{\prod_{\nu=1}^{M} N_\nu}{2} & \text{if}(\tau_\nu) \in \mathcal{X}_0^{M} \\ -\frac{\prod_{\nu=1}^{M} N_\nu}{2} & \text{if}(\tau_\nu) \in \mathcal{X}_0^{M-1} \times \left\{\frac{N_M}{2}, \frac{3N_M}{2}, \ldots\right\} \\ 0 & \text{otherwise} \end{cases} \tag{7.17}$$

so that:

$$\sum_{n_1=0}^{N_1-1} \cdots \sum_{n_M=0}^{N_M-1} e^{j\sum_{\nu=1}^{M} \Omega n_\nu \tau_\nu} = \sum_{n_1=0}^{N_1-1} \cdots \sum_{n_{M-1}=0}^{N_{M-1}-1} \sum_{n_M^e=0}^{N_M-2} e^{j(\sum_{\nu=1}^{M-1} \Omega n_\nu \tau_\nu + \Omega n_M^e \tau_M)}$$

$$+ \sum_{n_1=0}^{N_1-1} \cdots \sum_{n_{M-1}=0}^{N_{M-1}-1} \sum_{n_M^o=1}^{N_M-1} e^{j(\sum_{\nu=1}^{M-1} \Omega n_\nu \tau_\nu + \Omega n_M^o \tau_M)} \tag{7.18}$$

□

Elements of the basic $N_1 \cdot N_2 \cdots N_M$-sample BOWMRS$^{M\text{-}D}$ have the mean values $\sqrt{\frac{1}{\prod_{\nu=1}^{M} N_\nu T_\nu}\lambda^2}$ and 0, respectively. The corresponding variances are equal to $\frac{(\prod_{\nu=1}^{M} N_\nu - 2)}{\prod_{\nu=1}^{M} N_\nu}\frac{\lambda^2}{2T}$ and $\frac{\lambda^2}{2\prod_{\nu=1}^{M} T_\nu}$. When $(N_\nu) \to (\infty)$, the variance matrix of the BOWMRS$^{M\text{-}D}$ converges to $\frac{\lambda^2}{2\prod_{\nu=1}^{M} T_\nu} I$, and its mean value vector tends to a zero vector.

Gaussian Multivariate White Noise

When the power spectral density matrix of an M-D multivariate white noise is approximated by the periodogram matrix of the extended MOMRS$^{M\text{-}D}$, an extended M-D white multisine random series (MOWMRS$^{M\text{-}D}$) is obtained. For $p = 1, 2$ (WSMRS$^{M\text{-}D}$ and BOWMRS$^{M\text{-}D}$) whiteness holds for finite $N_1 \cdot N_2 \cdots N_M$-sample series. Correlation matrices of MOWMRS$^{M\text{-}D}$ with the number of elements $p > 2$ coincide only asymptotically for $(N_\nu) \to (\infty)$ with correlation matrices of an M-D p-variate white noise and asymptotically the MOWMRS$^{M\text{-}D}$ is an M-D Gaussian multivariate random series. Its spectral and correlation properties are given by the following lemma:

Lemma 7.3 *Assuming that:*

1. $\Phi_{\mathbf{vv}}(j\omega_\nu T_\nu) = \lambda^2 I$ $((\omega_\nu T_\nu) \in [0, 2\pi)^M)$ *is the power spectral density matrix of an M-D real-valued bivariate white noise;*

2. $A_{(\boldsymbol{n}_\nu)} = A$ *for* $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}^M \backslash \mathcal{N}_S^M$ *and the value of A converges to 0 for $(N_\nu) \to (\infty)$ in such a way that:*

$$\frac{A^2}{4} \prod_{\nu=1}^{M} N_\nu T_\nu = \lambda^2, \tag{7.19}$$

3. $A_{(\boldsymbol{n}_\nu)} = \frac{A}{2}$ *for* $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}_S^M$ *and* $\phi_{(\mathbf{o})} = \alpha = \frac{\pi}{2}$,

then the extended MOMRS$^{M\text{-}D}$ $\mathbf{u}(\boldsymbol{i}_\nu)$ with the consecutively circularly ordered frequency M-tuples converges in distribution for $(N_\nu) \to (\infty)$ to an M-D Gaussian multivariate white multisine random series of type 1 (GMOWMRS1$^{M\text{-}D}$) $\mathbf{g}(\boldsymbol{i}_\nu) = [g_r(\boldsymbol{i}_\nu)]_{r=1,2,\ldots,p}$ with zero mean vector and the variance matrix $\frac{\lambda^2}{p\prod_{\nu=1}^{M} T_\nu} I$:

$$\mathbf{g}(\boldsymbol{i}_\nu) \in As\mathcal{N}(\mathbf{o}, \frac{\lambda^2}{p\prod_{\nu=1}^{M} T_\nu} I). \tag{7.20}$$

Its correlation function matrix is $\mathcal{E}\{\mathbf{g}(\boldsymbol{i}_\nu)\mathbf{g}^T(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = \mathbf{R}_{\mathbf{gg}}(\boldsymbol{\tau}_\nu) = [R_{g_r g_s}(\boldsymbol{\tau}_\nu)]_{r,s=1,2,\ldots,p}$, where for $(\tau_\nu) \in \mathcal{X}_\infty^M$:

$$R_{g_r g_s}(\tau_\nu) = \begin{cases} R_{g_r g_r}(\tau_\nu) & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}. \tag{7.21}$$

The autocorrelation function $R_{g_r g_r}(\tau_\nu)$ of the rth GWMOMRS$^{M\text{-}D}$ element converges to:

$$R_{g_r g_r}(\tau_\nu) = \begin{cases} \frac{\lambda^2}{p\prod_{\nu=1}^{M} T_\nu} & \text{if } (\tau_\nu) = (\mathbf{o}) \\ 0 & \text{if } (\tau_\nu) \neq (\mathbf{o}) \end{cases}. \tag{7.22}$$

□

Proof: *The proof of this lemma follows immediately from Lemma 6.10 when it is noticed that the mean value vector of $\mathbf{u}(\boldsymbol{i}_\nu)$ is:*

$$\mathcal{M}\{\mathbf{u}(\boldsymbol{i}_\nu)\} = \left[\sqrt{\frac{1}{\prod_{\nu=1}^{N}(N_\nu T_\nu)}\lambda^2}, 0, \ldots, 0\right]. \tag{7.23}$$

The corresponding variance matrix is:

$$\mathbf{R}_{\mathbf{uu}}(\mathbf{o}) = \frac{2\lambda^2}{\prod_{\nu=1}^{N}(N_\nu T_\nu)} \operatorname{diag}\left[n_r'\right]_{r=1,2,\ldots,p}, \tag{7.24}$$

where n_r' $(r = 1, 2, \ldots, p)$ is the number of elements of the set $\mathcal{N}_{r,p}^{c,M} \setminus (\mathbf{o})$. When $(N_\nu) \to (\infty)$, the mean value vector tends to a zero vector and n_r' approaches $\frac{\prod_{\nu=1}^{N} N_\nu}{2p}$. It implies that $\mathbf{R}_{\mathbf{uu}}(\mathbf{o})$ tends to $\frac{\lambda^2}{p\prod_{\nu=1}^{N} T_\nu} I$ vector.

□

7.2.2 Nonergodic Case

M-D(N_ν)-lag pseudo-white multisine random process

When the power spectral density matrix of an M-D multivariate white noise is approximated by the expected value of the periodogram matrix of extended NMOMRS$^{M\text{-}D}$, a nonergodic multivariate orthogonal pseudo-white multisine random series (NMOPWMRS$^{M\text{-}D}$) is obtained.

Definition 7.3 *An extended M-D nonergodic multivariate orthogonal multisine random series $\mathbf{x}(i_\nu)$ is said to be (N_ν)-lag pseudo-white if elements of its correlation function matrix $\mathcal{E}\{\mathbf{x}(i_\nu)\mathbf{x}^T(i_\nu - \tau_\nu)\} = [\mathcal{E}\{x_r(i_\nu)x_s(i_\nu - \tau_\nu)\}]_{r,s=1,2,\ldots,p}$ for lags $(\tau_\nu) \in \mathcal{X}^M$ satisfy the following conditions:*

-
$$\mathcal{E}\{x_r(i_\nu)x_r(i_\nu - \tau_\nu)\} = \begin{cases} \Gamma^2 & \text{if } (\tau_\nu) = (\mathbf{o}) \\ \gamma_r(\tau_\nu)\Gamma^2 & \text{if } (\tau_\nu) \in \mathcal{X}^M \setminus \{(\mathbf{o})\} \end{cases} \tag{7.25}$$
for $r = 1, 2, \ldots, p$ and $|\gamma_r(\tau_\nu)| \ll 1$;

-
$$\mathcal{E}\{x_r(i_\nu)x_s(i_\nu - \tau_\nu)\} = 0 \tag{7.26}$$
for $r, s = 1, 2, \ldots, p$ and $r \neq s$.

□

The statistical properties of the NMOPWMRS$^{M\text{-}D}$ are given by the following lemma:

Lemma 7.4 *Assuming that:*

1. *$\boldsymbol{\Phi}_{\mathbf{vv}}(j\Omega_\nu m_\nu) = \lambda^2 I$ $((\omega_\nu T_\nu) \in [0, 2\pi)^M)$ is the power spectral density matrix of a real-valued M-D multivariate white noise;*
2. *$A_{r,(\mathbf{n}_\nu)} = A$ for $(\Omega_\nu \mathbf{n}_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M$ and the value of A is chosen so that:*
$$\frac{A^2}{4}\prod_{\nu=1}^{M} N_\nu T_\nu = \lambda^2, \tag{7.27}$$
3. *$A_{r,(\mathbf{o})} = 0$ or $\phi_{r,(\mathbf{o})} = 0$ for $r = 1, 2, \ldots, p$,*
4. *$A_{r,(\mathbf{n}_\nu)} = \frac{A}{2}$ for $(\Omega_\nu \mathbf{n}_\nu) \in \mathcal{N}_S^M \setminus \{\mathbf{o}\}$ and $\alpha = \frac{\pi}{2}$,*

then the extended NMOMRS$^{M\text{-}D}$ $\mathbf{u}(i_\nu)$ is an M-D nonergodic multivariate (N_ν)-lag pseudo-white multisine random series (NMOPWMRS$^{M\text{-}D}$) and:

1. *its expected value vector is $\mathcal{E}\{\mathbf{u}(i_\nu)\} = \mathbf{o}$.*
2. *its correlation function matrix is $\mathcal{E}\{\mathbf{u}(i_\nu)\mathbf{u}^T(i_\nu - \tau_\nu)\} = [\mathcal{E}\{u_r(i_\nu)u_s(i_\nu - \tau_\nu)\}]_{r,s=1,2,\ldots,p}$, where for $(\tau_\nu) \in \mathcal{X}_\infty^M$:*
$$\mathcal{E}\{u_r(i_\nu)u_s(i_\nu - \tau_\nu)\} = \begin{cases} \mathcal{E}\{u_r(i_\nu)u_r(i_\nu - \tau_\nu)\} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \tag{7.28}$$
$\mathcal{E}\{u_r(i_\nu)u_r(i_\nu - \tau_\nu)\}$ is the autocorrelation function of the rth NMOMRS$^{M\text{-}D}$ element:
$$\mathcal{E}\{u_r(i_\nu)u_r(i_\nu - \tau_\nu)\} = \begin{cases} \dfrac{\prod_{\nu=1}^M N_\nu - 1}{\prod_{\nu=1}^M N_\nu}\dfrac{\lambda^2}{\prod_{\nu=1}^M T_\nu} & \text{if } (\tau_\nu) \in \mathcal{X}_0^M \\ 0 & \text{otherwise} \end{cases} \tag{7.29}$$

□

Proof:

1. *It follows immediately from the assumption 3 and from Lemma 6.8.*

2. *It follows from the assumption 3 and Lemma 6.8 that the NMOPWMRS$^{M\text{-}D}$ correlation function is $\mathcal{E}\{\mathbf{u}(i_\nu)\mathbf{u}^T(i_\nu - \tau_\nu)\} = diag[\mathcal{E}\{u_r(i_\nu)u_r(i_\nu - \tau_\nu)\}]_{r=1,2,\ldots,p}$. This ends proof when it is noticed that the rth element $u_r(i_\nu)$ of NMOPWMRS$^{M\text{-}D}$ $\mathbf{u}(i_\nu)$ is a WSMRS$^{M\text{-}D}$ (see Lemma 7.1) with the removed expected value $\mathcal{E}\{u_r(i_\nu)\} = \sqrt{\frac{1}{\prod_{\nu=1}^M N_\nu T_\nu}\lambda^2}$.*

□

The independent variable domain averaging on any particular extended NMOPWMRS$^{M\text{-}D}$ results in:

Lemma 7.5 *Consider the extended NMOPWMRS$^{M\text{-}D}$.*

1. *Its periodogram matrix is $\boldsymbol{\Phi}_{\mathbf{uu}}^B(j\Omega_\nu m_\nu) = \left[\Phi_{u_r u_s}^B(j\Omega_\nu m_\nu)\right]_{r,s=1,2,\ldots,p}$, where for $(\Omega_\nu m_\nu) \in \mathcal{N}_{2\pi}^M$:*
$$\begin{aligned} \Phi_{u_r u_s}^B(j\Omega_\nu m_\nu) &= (0 + j0)\prod_{\nu=1}^M \delta(m_\nu) \\ &+\lambda^2 \sum_{(\Omega_\nu \mathbf{n}_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M} \Big[(\cos(\phi_{r,(\mathbf{n}_\nu)} - \phi_{s,(\mathbf{n}_\nu)}) - j\sin(\phi_{r,(\mathbf{n}_\nu)} - \phi_{s,(\mathbf{n}_\nu)}))\prod_{\nu=1}^M \delta(m_\nu - n_\nu) \\ &+(\cos(\phi_{r,(\mathbf{n}_\nu)} - \phi_{s,(\mathbf{n}_\nu)}) + j\sin(\phi_{r,(\mathbf{n}_\nu)} - \phi_{s,(\mathbf{n}_\nu)}))\prod_{\nu=1}^M \delta(m_\nu - (N_\nu - n_\nu))\Big] \\ &+ \sum_{(\Omega_\nu \mathbf{n}_\nu) \in \mathcal{N}_S^M \setminus \{(\mathbf{o})\}} (\lambda^2 \sin\phi_{r,(\frac{N}{2})}\sin\phi_{s,(\frac{N}{2})} + j0)\prod_{\nu=1}^M \delta(m_\nu - n_\nu). \end{aligned} \tag{7.30}$$
2. *Its mean value vector is: $\mathcal{M}\{\mathbf{u}(\mathbf{i}_\nu)\} = \mathbf{o}$.*
3. *Its correlation function matrix is $\mathbf{R}_{\mathbf{uu}}(\tau_\nu) = [R_{u_r u_s}(\tau_\nu)]_{r,s=1,2,\ldots,p}$, where for $(\tau_\nu) \in \mathcal{X}_\infty^M$:*
$$\begin{aligned} R_{u_r u_s}(\tau_\nu) = \frac{\lambda^2}{\prod_{\nu=1}^M N_\nu T_\nu}\Big[&\sum_{(\Omega_\nu \mathbf{n}_\nu) \in \mathcal{N}^M \setminus \mathcal{N}_S^M} \cos\Big(\sum_{\nu=1}^M \Omega_\nu n_\nu \tau_\nu + \phi_{r,(\mathbf{n}_\nu)} - \phi_{s,(\mathbf{n}_\nu)}\Big) \\ &+ \sum_{(\Omega_\nu \mathbf{n}_\nu) \in \mathcal{N}_S^M \setminus \{(\mathbf{o})\}} (-1)^{\sum_{\nu=1}^M \tau_\nu \eta(\Omega_\nu n_\nu)} \sin\phi_{r,(\frac{N}{2})}\sin\phi_{s,(\frac{N}{2})}\Big], \end{aligned} \tag{7.31}$$
where $\eta(\Omega_\nu n_\nu)$ is given by (6.30).

□

Proof: *The proof follows from Lemma 6.9 when $A_{r,(\mathbf{n}_\nu)}$ defined by the assumptions 2,3,4 of Lemma 7.4 are used.*

□

The expected value of $\mathbf{R}_{\mathbf{uu}}(\tau_\nu)$ is the diagonal matrix:

$$\mathcal{E}\{\mathbf{R}_{\mathbf{uu}}(\tau_\nu)\} = \begin{cases} \dfrac{\prod_{\nu=1}^M N_\nu - 1}{\prod_{\nu=1}^M N_\nu}\dfrac{\lambda^2}{\prod_{\nu=1}^M T_\nu} & \text{if } (\tau_\nu) \in \mathcal{X}_0^M \\ 0 & \text{otherwise} \end{cases} \tag{7.32}$$

M-D Gaussian multivariate white multisine random series

The extended NMOPWMRS$^{M\text{-}D}$ turns asymptotically for $(N_\nu) \to (\infty)$ into an M-D ergodic Gaussian multivariate white multisine random series:

Lemma 7.6 *Assuming that:*

1. $\mathbf{\Phi}_{\mathbf{vv}}(j\omega_\nu T_\nu) = \lambda^2 I$ $((\omega_\nu T_\nu) \in [0, 2\pi)^M)$ *is the power spectral density matrix of a real-valued M-D multivariate white noise;*

2. $A_{r,(\boldsymbol{n}_\nu)} = A$ *for* $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}_M \setminus \mathcal{N}_S^M$ *and the value of A is chosen so that:*

$$\frac{A^2}{4} \prod_{\nu=1}^{M} N_\nu T_\nu = \lambda^2, \tag{7.33}$$

3. $A_{r,(\boldsymbol{n}_\nu)} = 0$ *or* $\phi_{r,(\mathbf{o})} = \alpha = 0$ *for* $(\boldsymbol{\Omega}_\nu \boldsymbol{n}_\nu) \in \mathcal{N}_S^M$,

then the extended NMOMRS$^{M\text{-}D}$ $\mathbf{u}(\boldsymbol{i}_\nu)$ *converges in distribution for* $(N_\nu) \to (\infty)$ *to an M-D ergodic Gaussian multivariate white multisine random series of type 2 (GMOWMRS2$^{M\text{-}D}$)* $\mathbf{g}(\boldsymbol{i}_\nu) = [g_r(\boldsymbol{i}_\nu)]_{r=1,2,\ldots,p}$ *with zero mean vector and the variance matrix* $\frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu} \boldsymbol{I}$*:*

$$\mathbf{g}(\boldsymbol{i}_\nu) \in As\mathcal{N}(\mathbf{o}, \frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu} I). \tag{7.34}$$

Additionally the correlation function matrix $\mathcal{E}\{\mathbf{g}(\boldsymbol{i}_\nu)\mathbf{g}^T(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\}$ *of GMOWMRS2$^{M\text{-}D}$ converges to:*

$$\mathcal{E}\{\mathbf{g}(\boldsymbol{i}_\nu)\mathbf{g}^T(\boldsymbol{i}_\nu - \boldsymbol{\tau}_\nu)\} = \mathbf{R}_{\mathbf{gg}}(\boldsymbol{\tau}_\nu) = \begin{cases} \frac{\lambda^2}{\prod_{\nu=1}^{M} T_\nu} I & \text{if } (\boldsymbol{\tau}_\nu) = (\mathbf{o}) \\ O & \text{if } (\boldsymbol{\tau}_\nu) \neq (\mathbf{o}) \end{cases}. \tag{7.35}$$

□

Proof of this lemma follows immediately from Lemma 6.11.

Chapter 8

Conclusions

An efficient synthesis and simulation method of wide-sense stationary scalar and multivariate one- and multi-dimensional random processes, given by diagrams of their power spectral densities, is presented. The method is based on approximating the power spectral densities by periodograms (or expected values of periodograms) of multisine random time-series or multidimensional multisine random processes with deterministic amplitudes and random phase shifts. The periodograms are used to synthesise spectra of the corresponding multisine random processes. Transforming the synthesised spectra by the inverse finite discrete Fourier transform gives the simulated multisine random process approximations. It was shown that multisine random process approximations thus obtained have spectral and correlation properties very close to those of the original wide-sense stationary random processes. Asymptotically, they turn into Gaussian random processes.

The proposed approach is applicable if only the power spectral density diagrams of random processes to be simulated are given. There is no necessity to solve the spectral factorisation problem in order to calculate the corresponding parametric approximation. It is especially important for random processes which have nonrational power spectral densities [78] or (and) which are multidimensional [15], [59], [20] because accuracy of the parametric approximation is crucial in reconstructing of the properties of original random processes.

The proposed method, when applied to power spectral densities of white noises, allows simulate different types of interesting scalar and multivariate orthogonal, white, pseudo-white or asymptotically white, ergodic and nonergodic, time-series and multidimensional random processes.

An extension of the proposed approach to the generation of wide-sense stationary continuous-time band-limited random processes, defined also by their power spectral densities, has been presented.

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SYNTHESIS AND SIMULATION OF RANDOM PROCESSES

Summary

This monograph presents an approach to the synthesis and simulation of wide-sense stationary random processes given by diagrams of their power spectral densities. The approach is based on multisine random time-series, which are sums of discrete-time sines with deterministic amplitudes and random phase shifts.

The essence of the presented approach is to approximate the power spectral density by the periodogram of a multisine time-series with amplitudes chosen so that for a given number of equally spaced frequencies from the range $[0, 2\pi)$, the periodogram is equal to the original power spectral density. The periodogram may be used in turn to construct the corresponding spectrum provided the phase shifts for each sine component are chosen. It is well known, that any periodogram corresponds to infinitely many different multisine time-series with different phase shifts. In the proposed approach, the phase shifts are used to define properties of the multisine random time-series in the time-domain. This concludes the synthesis part of the procedure. To simulate the synthesised time-series with predefined spectral properties, the spectrum with the chosen phase shifts is transformed into the time-domain using the inverse finite discrete Fourier transform.

In Chapter 2, time- and frequency- domain definitions of scalar as well as different multivariate multisine random time-series are introduced. Their statistical properties, resulting from ensemble and time-domain averaging, are discussed. The weak ergodicity of multisine random time-series is examined. It is shown that periodograms of weakly ergodic multisine random time-series as well as expected values of periodograms for nonergodic multisine random time-series are uniquely defined by amplitudes of their sine components. This idea is behind the proposed random process synthesis and simulation method.

Chapter 3 is devoted to the synthesis and simulation of multisine random time-series defined by their power spectral densities. In the presented approach, the power spectral density matrix of a multivariate wide-sense stationary random process to be simulated is approximated by the periodogram of a multivariate multisine random time-series with sine component amplitudes chosen so as to make values of the periodogram matrix (or expected value of the periodogram matrix) equal to the corresponding values of power spectral density matrix of the original random process for some equally spaced frequencies (being approximation nodes) from the range $[0, 2\pi)$. This approximation criterion can be interpreted as sampling of the power spectral density matrix in the frequency domain. A lower bound on the number of approximation nodes (samples of multisine random time-series to be simulated) follows from the reconstruction criterion. Statistical properties of the synthesised multisine random process approximations are determined. It was shown that such multisine random process approximations converge to ergodic Gaussian multisine random processes for the number of approximation nodes tending to infinity. The proposed approach is illustrated by examples of simulating and identifying scalar and multivariate random processes given by rational and nonrational power spectral densities. An extension of the proposed random process synthesis and simulation method to the generation of wide-sense stationary continuous-time band-limited random signals, given also by their power spectral densities, is included.

Multisine white noise approximations obtained by using the proposed random process synthesis and simulation method are presented in Chapter 4. The following cases are discussed: weakly ergodic scalar and bivariate white and pseudo-white multisine random time-series which are asymptotically Gaussian, weakly ergodic multivariate orthogonal

asymptotically Gaussian and white multisine random time-series, nonergodic multivariate orthogonal white and pseudo-white multisine random time-series which are asymptotically ergodic and Gaussian. Their whiteness is compared for finite and infinite periods of multisine random time-series. Asymptotic Gaussianess of the synthesised multisine random process approximations is discussed. Simulation examples are included.

Simulation of Gaussian random processes is the subject of Chapter 5. Simulation schemes based on the proposed approach and rules of computer simulation are established, including a proposition of simulation time-scale contraction. The proposed schemes are illustrated by simulation examples.

In Chapter 6, an extension of multisine random time-series ideas given in Chapter 2 to a multidimensional (M-D) case is presented. Scalar and multivariate M-D multisine random processes are defined and their time- and frequency- domain properties are established. Similarly as for the 1-D case, definitions of M-D multisine random processes are closely related to the M-D finite discrete Fourier transform. A set of the frequency M-tuples of the M-D sine series such that a sum of the M-D sine series has its spectrum lines defined for all frequency M-tuples present in definition of the finite discrete Fourier transform is a key to define M-D multisine random processes. It is shown that the multidimensional multisine random processes inherit properties of the 1-D multisine random time-series. The defined M-D multisine random processes are used to synthesise and simulate wide-sense stationary scalar and multivariate M-D random processes given by their power spectral densities. The synthesis and simulation follow the corresponding procedure for 1-D multisine random processes. Asymptotic properties of synthesised M-D multisine random process approximations are discussed.

The problem of synthesising and simulating various types of scalar, bivariate and multivariate ergodic and nonergodic multidimensional white multisine random processes is summarised in Chapter 7.

SYNTEZA I SYMULACJA PROCESÓW LOSOWYCH

Streszczenie

W monografii przedstawiono podstawy teoretyczne oraz zastosowanie nowej metody syntezy i symulacji stacjonarnych w szerszym sensie procesów losowych na podstawie wykresu ich gęstości widmowej mocy. Zaproponowana metoda korzysta z wielosinusoidalnych sygnałów losowych, które są sumą harmonicznych składowych sinusoidalnych o deterministycznych amplitudach i losowych fazach.

Punktem wyjścia proponowanej metody jest definiowanie procesu losowego w dziedzinie częstotliwości za pomocą periodogramu wielosinusoidalnego sygnału losowego. Amplitudy poszczególnych składowych sinusoidalnych sygnału wielosinusoidalnego dobierane są tak, by jego periodogram był równy gęstości widmowej mocy procesu losowego dla pewnej liczby równoodległych częstotliwości z zakresu $[0, 2\pi)$. Na podstawie tak zdefiniowanego periodogramu dokonuje się syntezy widma amplitudowego i fazowego sygnału wielosinusoidalnego. Przedstawienie periodogramu za pomocą widma amplitudowego i fazowego jest jednoznaczne w odniesieniu do widma amplitudowego i niejednoznaczne w odniesieniu do widma fazowego: ten sam periodogram można uzyskać dla jednego określonego widma amplitudowego i nieskończenie wielu różnych widm fazowych. Ową niejednoznaczność wykorzystano do kształtowania własności wielosinusoidalnych sygnałów losowych w dziedzinie czasu. W wyniku odwrotnego przekształcenia Fouriera widma zespolonego (z widmem amplitudowym determinującym periodogram i widmem fazowym determinującym własności losowe w dziedzinie czasu) otrzymuje się proces losowy o założonych właściwościach widmowych.

W rozdziale 2 zdefiniowano skalarne i wektorowe (ortogonalne i nieortogonalne) wielosinusoidalne sygnały losowe w dziedzinie czasu i częstotliwości. Porównano ich własności statystyczne, analizując wyniki uśredniania po zbiorze realizacji i uśredniania w dziedzinie czasu. Na tej podstawie dokonano podziału wektorowych wielosinusoidalnych sygnałów losowych na ergodyczne i nieergodyczne. Wykazano, że periodogram dla ergodycznych wielosinusoidalnych sygnałów losowych oraz wartość oczekiwana periodogramu dla nieergodycznych sygnałów wielosinusoidalnych przyjmują wartości deterministyczne jednoznacznie określone poprzez amplitudy składowych sinusoidalnych. Konsekwencją tej własności jest możliwość dowolnego kształtowania periodogramu (lub jego wartości oczekiwanej) poprzez wybór amplitud składowych sinusoidalnych.

W kolejnym rozdziale przedstawiono algorytmy syntezy wielosinusoidalnych sygnałów losowych zdefiniowanych za pomocą macierzy gęstości widmowych mocy oraz ich symulację z wykorzystaniem odwrotnego przekształcenia Fouriera. W prezentowanym podejściu, macierz gęstości widmowych mocy stacjonarnego w szerszym sensie procesu losowego, który ma być symulowany jest aproksymowana przez macierz periodogramu wielosinusoidalnego sygnału losowego o amplitudach składowych sinusoidalnych dobieranych tak, by jego macierz periodogramu (lub jej wartość oczekiwana) była równa macierzy gęstości widmowej dla pewnej liczby równoodległych częstotliwości z zakresu $[0, 2\pi)$. Takie kryterium aproksymacji interpretowane jest jako próbkowanie gęstości widmowej mocy w dziedzinie częstotliwości. Z warunku rekonstrukcji gęstości widmowej mocy na podstawie periodogramu wynika dolne ograniczenie na liczbę wezłów aproksymacji (okres wielosinusoidalnego sygnału losowego). Wykazano, że gdy liczba wezłów aproksymacji wzrasta do nieskończoności, wielosinusoidalne sygnały losowe stają się asymptotycznie gausowskie, a te które były nieergodyczne stają się asymptotycznie ergodyczne. Przedstawiona metoda zilustrowana jest przykładami symulacji i identyfikacji skalarnych i wektorowych procesów losowych zadanych

w postaci gęstości widmowych mocy będących wymiernymi i niewymiernymi funkcjami częstotliwości. Załączono również propozycję rozszerzenia powyższego podejścia na przypadek generacji ciagłych procesów losowych zdefiniowanych również poprzez gęstość widmową mocy.

Rozdzial 4 poświęcony jest wielosinusoidalnym sygnałom losowym otrzymanym w wyniku aproksymacji gęstości widmowej mocy białego szumu. Analizowane są przypadki skalarnych i wektorowych, ergodycznych i nieergodycznych sygnałów wielosinusoidalnych o własnościach białego szumu. Porównywana jest ich „białość" dla skończonego i nieskończonego okresu sygnału wielosinusoidalnego. Dyskutowana jest również ich asymptotyczna gausowskość. Rozważania są zilustrowane przykładami.

Tematem kolejnego rozdziału jest zastosowanie wielosinusoidalnych sygnałów losowych do syntezy i symulacji procesów gausowskich. Na podstawie analizy warunków symulacji komputerowej oraz własności wielosinusoidalnych sygnałów losowych zaproponowano schematy symulacji procesów gausowskich. Zaproponowane schematy zilustrowano przykładami.

W rozdziale 6 uogólniono definicje jednowymiarowych (1-D) wielosinusoidalnych procesów losowych analizowanych w rozdziale 2 na przypadek wielowymiarowy (M-D). Zdefiniowano M-wymiarowe skalarne i wektorowe wielosinusoidalne procesy losowe. Podobnie jak poprzednio, ich definicje są ściśle powiązane z M-wymiarową skończoną dyskretną transformatą Fouriera. Kluczem do zdefiniowania M-wymiarowych wielosinusoidalnych procesów losowych okazało się skonstruowanie takiego zbioru M-tek częstotliwości, by odpowiednia suma M-wymiarowych sinusoid posiadała linie widma dla wszystkich M-tek częstotliwości występujących w definicji transformaty. Analizując własności statystyczne M-wymiarowych wielosinusoidalnych procesów losowych stwierdzono, że dziedziczą one własności ich jednowymiarowych odpowiedników. Konsekwencją tego jest możliwość przeprowadzenia syntezy i symulacji stacjonarnych M-wymiarowych procesów losowych zadanych poprzez gęstość widmową mocy tak samo jak dla sygnałów 1-D. Podsumowano własności asymptotyczne tak otrzymanych M-wymiarowych wielosinusoiadalnych sygnałów losowych.

W kolejnym rozdziale przedstawiono wyniki syntezy M-wymiarowych wielosinusoidalnych aproksymacji M-wymiarowego białego szumu. Analizowano przypadki skalarne i wektorowe.