

**APPLICATION OF SHAPE SENSITIVITY ANALYSIS  
IN NUMERICAL MODELLING OF SOLIDIFICATION PROCESS**E. MAJCHRZAK<sup>1</sup>, M. JASIŃSKI<sup>2</sup>, G. KAŁUŻA<sup>3</sup><sup>1,2,3</sup> Department for Strength of Materials and Computational Mechanics

Silesian University of Technology, 44-100 Gliwice, Konarskiego 18a, Poland

<sup>1</sup> Czestochowa University of Technology, 42-200 Czestochowa, Dabrowskiego 73

## SUMMARY

In the paper the influence of the casting geometrical parameters on the course of solidification process is discussed. In order to solve the problem, the methods of sensitivity analysis are applied. Here the direct approach of shape sensitivity analysis is used and then the special methods of the governing equations and boundary initial conditions differentiation should be introduced. The basic problem and additional one connected with the shape sensitivity analysis are solved using the boundary element method. The BEM algorithm for parabolic equations is adapted to the solidification process modelling by the introduction of numerical procedure called the artificial heat source method. In the final part of the paper the results of computations are presented.

*Key words: solidification process, shape sensitivity analysis*

**1. GOVERNING EQUATIONS**

Heat transfer processes proceeding in domain of casting are described by the energy equation (2D problem is analyzed)

$$(x, y) \in \Omega: C(T) \frac{\partial T(x, y, t)}{\partial t} = \lambda \nabla^2 T(x, y, t) \quad (1)$$

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<sup>1</sup> *prof. dr hab. inż., ewa.majchrzak@polsl.pl*

<sup>2</sup> *dr inż., marek.jasinski@polsl.pl*

<sup>3</sup> *mgr inż.*

where  $C(T)$  [J/(m<sup>3</sup> K)] is the substitute thermal capacity [1],  $\lambda$  [W/(mK)] is the thermal conductivity,  $T, x, y, t$  denote temperature, spatial co-ordinates and time. The substitute thermal capacity of the casting material is defined as follows

$$C(T) = \begin{cases} c_s & T < T_s \\ c_p + \frac{L}{T_L - T_s} & T_s \leq T < T_L \\ c_L & T \geq T_L \end{cases} \quad (2)$$

where  $c_s, c_p, c_L$  are the volumetric specific heats of liquid, mushy zone and solid state,  $L$  is the latent heat,  $T_s$  and  $T_L$  correspond to the border temperatures, respectively [1].

The influence of mould sub-domain on the course of solidification process is substituted by the condition

$$(x, y) \in \Gamma: T(x, y, t) = T_b \quad (3)$$

where  $T_b$  is the boundary temperature resulting from the well known Schwarz solution [1]. Mathematical model of the process considered is supplemented by initial condition

$$t = 0: T(x, y, t) = T_p \quad (4)$$

where  $T_p$  is the pouring temperature.

## 2. SHAPE SENSITIVITY ANALYSIS

Among the different sensitivity problems, especially important are the shape sensitivity ones [2, 3]. They consist in finding the sensitivity of structural response to variations in the initial shape of the body. The major question is how the temperature, its gradient etc. is modified due to the transformation of structure. We assume that  $b$  is the shape design parameter. Using the concept of material derivative we can write [2, 3]

$$\frac{DT}{Db} = \frac{\partial T}{\partial b} + \frac{\partial T}{\partial x} v_x + \frac{\partial T}{\partial y} v_y \quad (5)$$

where  $v = v(x, y, b)$  is the velocity associated with design parameter  $b$ .

Taking into account the definition (5) it can be proved that

$$\frac{D}{Db} \left( \frac{\partial^2 T}{\partial x^2} \right) = \frac{\partial^2}{\partial x^2} \left( \frac{DT}{Db} \right) - 2 \frac{\partial^2 T}{\partial x^2} \frac{\partial v_x}{\partial x} - \frac{\partial T}{\partial x} \frac{\partial^2 v_x}{\partial x^2} - 2 \frac{\partial^2 T}{\partial x \partial y} \frac{\partial v_y}{\partial x} - \frac{\partial T}{\partial y} \frac{\partial^2 v_y}{\partial x^2} \quad (6)$$

and

$$\frac{D}{Db} \left( \frac{\partial^2 T}{\partial y^2} \right) = \frac{\partial^2}{\partial y^2} \left( \frac{DT}{Db} \right) - 2 \frac{\partial^2 T}{\partial y^2} \frac{\partial v_y}{\partial y} - \frac{\partial T}{\partial x} \frac{\partial^2 v_x}{\partial y^2} - 2 \frac{\partial^2 T}{\partial x \partial y} \frac{\partial v_x}{\partial y} - \frac{\partial T}{\partial y} \frac{\partial^2 v_y}{\partial y^2} \quad (7)$$

at the same time

$$\frac{D}{Db} \left( \frac{\partial T}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{DT}{Db} \right) \quad (8)$$

If the direct approach of sensitivity analysis is applied, then differentiation of equation (1) and conditions (3), (4) with respect to the parameter  $b$  for 2D domain oriented in Cartesian co-ordinate system leads to the following boundary initial problem

$$\begin{aligned} (x, y) \in \Omega : \quad C(T) \frac{\partial U}{\partial t} &= \lambda \frac{\partial^2 U}{\partial x^2} + \lambda \frac{\partial^2 U}{\partial y^2} - 2\lambda \left( \frac{\partial^2 T}{\partial x^2} \frac{\partial v_x}{\partial x} + \frac{\partial^2 T}{\partial y^2} \frac{\partial v_y}{\partial y} \right) - \\ &\quad \lambda \frac{\partial T}{\partial x} \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) - 2\lambda \frac{\partial^2 T}{\partial x \partial y} \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) - \\ &\quad \lambda \frac{\partial T}{\partial y} \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) \quad (9) \\ (x, y) \in \Gamma : \quad U &= 0 \\ t = 0 : \quad U &= 0 \end{aligned}$$

where  $U = DT/Db$ . It should be pointed out that in order to solve the problem (9), the values of  $v_x, v_y$  should be known.

### 3. RESULTS OF COMPUTATIONS

The cross-section of casting in the form of square of dimensions  $0.04 \times 0.04$  [m] has been considered. It is assumed that the shape parameter  $b$  corresponds to the half of square diagonal and  $v_x = x/b, v_y = y/b$ .

The following input data have been introduced:  $\lambda = 35$  [W/mK],  $c_S = 5.175$  [MJ/m<sup>3</sup> K],  $c_P + L/(T_L - T_S) = 61.4$  [MJ/m<sup>3</sup> K],  $c_L = 5.74$  [MJ/m<sup>3</sup> K], pouring temperature  $T_p = 1550$  °C, liquidus temperature  $T_L = 1505$  °C, solidus temperature  $T_S = 1470$  °C and the boundary temperature  $T_b = 1465$  °C.

For the assumed shape parameter  $b$  and the values of  $v_x, v_y$ , the additional problem (9) takes a form

$$\begin{aligned}
 (x, y) \in \Omega: \quad C(T) \frac{\partial U(x, y, t)}{\partial t} &= \lambda \nabla^2 U(x, y, t) - \frac{2}{b} C(T) \frac{\partial T(x, y, t)}{\partial t} \\
 (x, y) \in \Gamma: \quad U(x, y, t) &= 0 \\
 t = 0: \quad U(x, y, t) &= 0
 \end{aligned} \tag{10}$$

The basic and additional problems have been solved using the first scheme of the boundary element method [4] supplemented by artificial heat source procedure [5]. The boundary  $\Gamma$  of the domain considered has been divided into 84 constant boundary elements, while the interior has been divided into 256 constant internal cells. Time step:  $\Delta t = 1$  [s]. In Figures 1 and 2 the distribution of temperature and function  $U$  for time 10 and 20 [s] is shown.

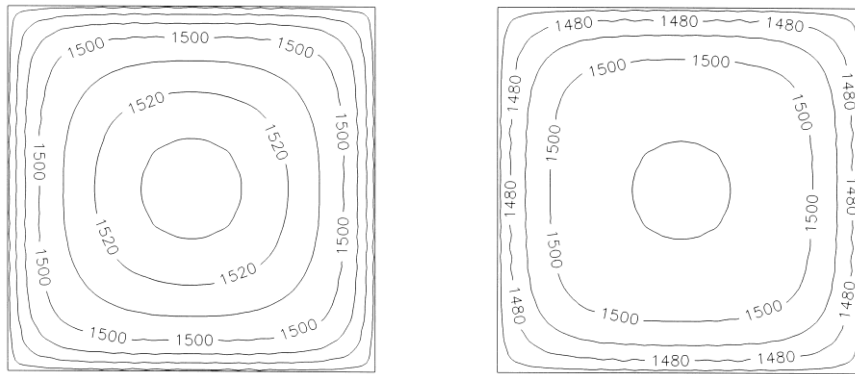


Fig. 1. Temperature distribution for time 10 and 20 [s]  
Rys. 1. Rozkład temperatury po czasie 10 i 20 [s]

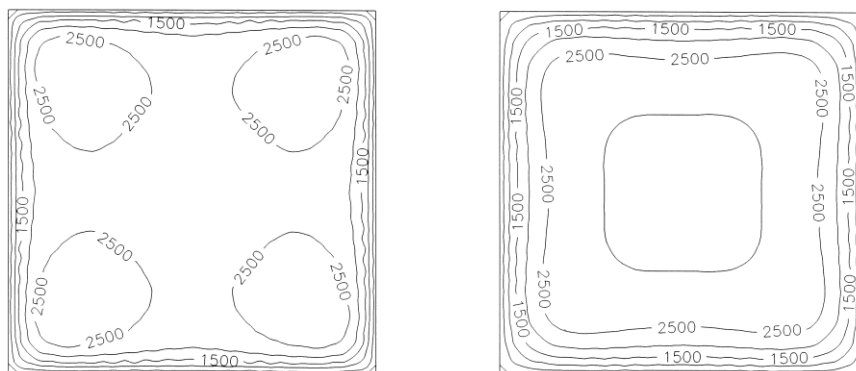


Fig. 2. Distribution of function  $U$  for time 10 and 20 [s]  
Rys. 2. Rozkład funkcji wrażliwości  $U$  po czasie 10 i 20 [s]

Figure 3 illustrates the cooling curves and the courses of function  $U$  at the points 1 - (0.02125, 0.02125), 2 - (0.02625, 0.02625) and 3 - (0.03125, 0.03125).

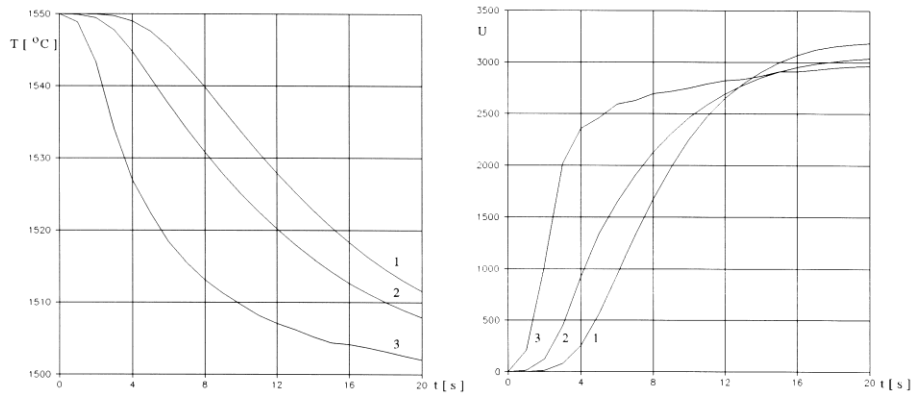


Fig. 3. Courses of function  $T$  and  $U$  at the points 1, 2, 3

Rys. 3. Przebieg funkcji  $T$  i  $U$  w punktach 1, 2, 3

On the basis of the knowledge of temperature  $T$  and sensitivity function  $U$  for time  $t$  and shape parameter  $b$ , the temperature in the domain for  $b+\Delta b$  can be obtained using the Taylor formula

$$T(x, y, b + \Delta b, t) = T(x, y, b, t) + U(x, y, b, t) \Delta b \quad (11)$$

In Figure 4 the temperature distribution for shape parameter  $b+\Delta b$  ( $\Delta b = 0.05b$ ) is shown.

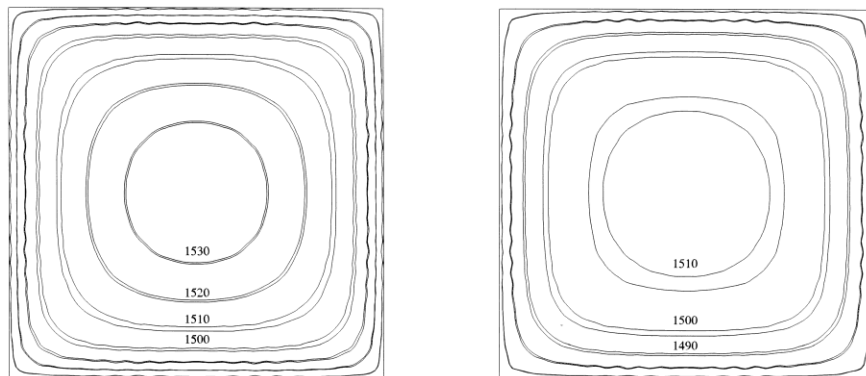


Fig. 4. Comparison of temperature distribution for 10 and 20 [s]

Rys. 4. Porównanie rozkładu temperatury po czasie 10 i 20 [s]

The solution has been obtained in two ways. The first has been found on the basis of temperature field corresponding to the basic parameter  $b$  and the knowledge of sensitivity function distribution (the Taylor formula (11) has been used), while the second solution has been found directly for shape parameter  $b+\Delta b$  without the application of sensitivity approach. It is visible, that both solutions are very close and this fact confirms the correctness of the considerations presented in this paper.

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*This paper has been sponsored by KBN (Grant No 3 T08B 004 28).*

#### ZASTOSOWANIE ANALIZY WRAŻLIWOŚCI KSZTAŁTU W NUMERYCZNYM MODELOWANIU PROCESU KRZEPNIĘCIA

W artykule analizowano wpływ parametrów geometrycznych odlewu na przebieg procesu krzepnięcia. W celu rozwiązania tak sformułowanego zadania zastosowano metodę analizy wrażliwości. Wykorzystano tzw. podejście bezpośrednie bazujące na definicji pochodnej materialnej, które polega na różniczkowaniu równań tworzących opis matematyczny procesu względem parametru kształtu. Zadanie podstawowe i dodatkowe związane z analizą wrażliwości rozwiązano stosując metodę elementów brzegowych uzupełnioną procedurą sztucznego źródła ciepła. W końcowej części artykułu przedstawiono wyniki obliczeń numerycznych.

Recenzował Prof. Wojciech Kapturkiewicz