Silesian J. Pure Appl. Math.
vol. 6, is. 1 (2016), 23–26

Waldemar HOŁUBOWSKI, Sebastian ŻUREK

Institute of Mathematics, Silesian University of Technology, Gliwice, Poland

NOTE ON SIMPLE LIE ALGEBRAS OF INFINITE MATRICES

Abstract. We introduce Lie algebras of infinite $\mathbb{N} \times \mathbb{N}$ matrices, with coefficients in a commutative rings, which have nonzero entries only in finite number of rows and study its properties. We show that algebra of matrices with trace 0 is uncountably dimensional simple Lie algebra for any ground field.

1. Introduction

Let R be a commutative ring and let $M_n(R)$ denote an $R-$algebra of $n \times n$ matrices over R. It becomes a Lie algebra under Lie product $[A, B] = AB - BA$. We denote it by $\mathfrak{gl}_n(R)$. By $\mathfrak{sl}_n(R)$ we denote a Lie subalgebra of $\mathfrak{gl}_n(R)$ consisting of all matrices A with $\mathrm{tr}(A) = 0$.

The direct limit $\mathfrak{gl}_\infty(R)$ of algebras $\mathfrak{gl}_n(R)$ under natural embeddings $\mathfrak{gl}_n(R) \to \mathfrak{gl}_{n+1}(R)$, given by:

$$A \to \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$$

is a Lie algebra of countable dimension. It can be viewed as a Lie algebra of infinite $\mathbb{N} \times \mathbb{N}$ matrices A, which have only finite number of nonzero entries. Similarly, we obtain a Lie subalgebra $\mathfrak{sl}_\infty(R)$ of matrices A having $\mathrm{tr}(A) = 0$. Note that

2010 Mathematics Subject Classification: 17B20, 17B65.
Keywords: simple Lie algebra, infinite dimensional Lie algebra, infinite matrix.
Corresponding author: W. Hołubowski (w.holubowski@polsl.pl).
Received: 29.07.2016.

the trace "tr" is a well defined function in this case since there is finitely many nonzero entries on the main diagonal.

Let K be a field of characteristic 0. It is known that $\mathfrak{sl}_n(K)$ is a simple Lie algebra [4] (it has no nontrivial ideals) of dimension n^2-1. The direct limit $\mathfrak{sl}_\infty(K)$ of algebras $\mathfrak{sl}_n(K)$ under natural embeddings is a simple Lie algebra of countable dimension [1].

In [3], the first author introduced a new Lie algebra of infinite matrices of uncountable dimension over a field of characteristic 0 and proved its simplicity. In this note, we generalize this example to Lie algebra of infinite matrices over a commutative ring and prove its simplicity over any ground field K.

Denote by $M_{fr}(\infty,\mathbf{R})$ a set of infinite $\mathbb{N}\times\mathbb{N}$ matrices over $\mathbf{R}$ having only finite number of nonzero rows. We note that a matrix in $M_{fr}(\infty,\mathbf{R})$ can have infinitely many nonzero coefficients in a nonzero row. A standard matrix multiplication of two matrices $C=A\cdot B$, given by the formula $c_{ij}=\sum_{k=1}^{\infty} a_{ik}b_{kj}$ is well defined, because in this infinite sum there is only a finite number of nonzero summands $a_{ik}b_{kj}$. So, $M_{fr}(\infty,\mathbf{R})$ is an associative $\mathbf{R}-$algebra.

Thus $M_{fr}(\infty,\mathbf{R})$, with respect to Lie product $[A,B]=AB-BA$, forms a Lie algebra denoted by $\mathfrak{gl}_{fr}(\infty,\mathbf{R})$. By $\mathfrak{sl}_{fr}(\infty,\mathbf{R})$ we denote a Lie subalgebra of $\mathfrak{gl}_{fr}(\infty,\mathbf{R})$ consisting of matrices U such that $\mathrm{tr}(U)=0$ (the notion of trace is well defined in this case).

We define the subset $\mathfrak{gl}(n,\infty,\mathbf{R})$ ($\mathfrak{sl}(n,\infty,\mathbf{R})$) of $\mathfrak{gl}_{fr}(\infty,\mathbf{R})$ ($\mathfrak{sl}_{fr}(\infty,\mathbf{R})$ respectively), consisting of all matrices which have nonzero entries only in first n rows. So $\mathfrak{gl}(n,\infty,\mathbf{R})$ and $\mathfrak{sl}(n,\infty,\mathbf{R})$ are Lie subalgebras.

By I_n we denote the subsets of $\mathfrak{gl}(n,\infty,\mathbf{R})$ consisting of matrices which have a zero $n\times n$ matrix in left upper corner.

Theorem 1. *The center of* $\mathfrak{gl}(n,\infty,\mathbf{R})$ ($\mathfrak{sl}(n,\infty,\mathbf{R})$) *is trivial. The set* I_n *is an abelian ideal of* $\mathfrak{gl}(n,\infty,\mathbf{R})$ ($\mathfrak{sl}(n,\infty,\mathbf{R})$ *respectively*) *and the factor algebra* $\mathfrak{gl}(n,\infty,\mathbf{R})/I_n$ ($\mathfrak{sl}(n,\infty,\mathbf{R})/I_n$) *is isomorphic to* $\mathfrak{gl}_n(R)$ ($\mathfrak{sl}_n(R)$ *respectively*).

It is clear that $(\mathfrak{gl}(n,\infty,\mathbf{R}))_{n>0}$ and $(\mathfrak{sl}(n,\infty,\mathbf{R})_{n>0}$ form ascending sequences of Lie algebras and

$$\mathfrak{gl}_{fr}(\infty,\mathbf{R})=\bigcup_{n>0}\mathfrak{gl}(n,\infty,\mathbf{R}),$$
$$\mathfrak{sl}_{fr}(\infty,\mathbf{R})=\bigcup_{n>0}\mathfrak{sl}(n,\infty,\mathbf{R}).$$

In other words, $\mathfrak{gl}_{fr}(\infty, \mathbf{R})$ is a direct limit of nonsimple Lie algebras $\mathfrak{gl}(n, \infty, \mathbf{R})$, which have uncoutable dimension in case $\mathbf{R} = K$- a field.

Our main result is the following

Theorem 2. *For every field K the Lie algebra $\mathfrak{sl}_{rf}(\infty, \mathbf{K})$ is simple and has uncountable dimension.*

2. Notations and proofs

For any $i, j \in \mathbb{N}$ denote by E_{ij}, the matrix unit, the infinite matrix whose only nonzero entry is 1 in the (i, j) position. Sometimes, if there is no ambiguity we denote by E_{ij} its finite $n \times n$ analogue. The product of any matrix units E_{ij} and E_{kl} is given by

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj},$$

where δ_{ij}- Kronecker's symbol. Moreover

$$[E_{ik}, E_{kj}] = E_{ij}.$$

for pairwise distinct i, j, k. The set $\{E_{ij} \mid i, j \in \mathbb{N}\}$ form a basis of $\mathfrak{gl}_\infty(\mathbf{K})$, and the set $\{E_{ij}, E_{rr} - E_{ss} \mid i, j, r, s \in \mathbb{N}, i \neq j, r \neq s\}$ form a generating set for $\mathfrak{sl}_\infty(\mathbf{K})$. So, $\mathfrak{gl}_\infty(\mathbf{K})$ and $\mathfrak{sl}_\infty(\mathbf{K})$ are countably dimensional. We see that $\mathfrak{gl}_{rf}(\infty, \mathbf{K})$ and $\mathfrak{sl}_{rf}(\infty, \mathbf{K})$ are uncountably dimensional. The symbol 0 may stand for the element of a field as well as for the matrix consisting only of zeros.

Proof of Theorem 1. If $U = \left(\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right)$ is from the center, then commuting U with E_{ij} we conclude that $B = 0$ and A belongs to the center of finite dimensional Lie algebra. So A must be a scalar matrix $\alpha\big(\sum\limits_{i=1}^{n} E_{ii}\big)$ and the equation $[X, \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)] = 0$ implies $\alpha = 0$.

The straightforward computation shows that I_n is an ideal. Abelianity follows from the fact that the nonzero coefficients V_{ij} of $V \in I_n$ satisfy the inequalities $i \leq n$ and $j > n$. The last claim is obvious. □

Proof of Theorem 2. The proof is the same as in [3]. The simplicity of $\mathfrak{sl}_{fr}(\infty, \mathbf{K})$ was proved there in three steps. If I is a nonzero ideal of $\mathfrak{sl}_{fr}(\infty, \mathbf{K})$, then:

1) I contains at least one matrix unit E_{ij},

2) I contains $\mathfrak{sl}(n, \mathbf{K})$ for any $n \geq 1$,

3) the ideal I coincides with the Lie algebra $\mathfrak{sl}_{rf}(\infty, \mathbf{K})$.

It is clear that all steps do not depend on characteristic of K. Moreover, if $\mathbf{K}$ is not a field the first step is false. □

Remark 3. Naturally arise a question about generalization of Theorem 2 to orthogonal and symplectic Lie algebras. One can define $\mathfrak{so}(n,\infty, \mathbf{K})$ and $\mathfrak{sp}(2n,\infty, \mathbf{K})$ and prove an analogue of Theorem 1 for these Lie algebras. However, we have not a natural embedding as for $\mathfrak{gl}(n, \infty, \mathbf{K})$ and $\mathfrak{sl}(n, \infty, \mathbf{K})$ in these cases. Since

$$\mathfrak{so}(n, \infty, \mathbf{K}) = \{U = \left(\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right) : \quad A + A^t = 0 \},$$

any embedding $\mathfrak{so}(n, \infty, \mathbf{K}) \to \mathfrak{so}(n+k, \infty, \mathbf{K})$ which put rows $n+1, n+2, \ldots, n+k$ entirely of zeros, implies from the condition $A + A^t = 0$ that in U the columns $n + 1, n + 2, \ldots, n + k$ are entirely of zeros. So we cannot obtain uncountably dimensional orthogonal (and similarly symplectic) analogue of $\mathfrak{sl}_{fr}(\infty, \mathbf{K})$. The only possibility of generalization gives a natural embedding which get a stable (countably dimensional) Lie algebra $\mathfrak{so}_\infty(\mathbf{K})$ or $\mathfrak{so}_\infty(\mathbf{K})$. This remark agrees with the results of Baranov [1] and Baranov, Strade [2] on direct limits of classical simple Lie algebras.

References

1. Baranov A.A.: *Finitary simple Lie algebras.* J. Algebra **219** (1999), 299–329.
2. Baranov A.A., Strade H.: *Finitary Lie algebras.* J. Algebra **254** (2002), 173–211.
3. Hołubowski W.: *A new simple Lie algebra of uncountable dimension.* Linear Algebra Appl. **492** (2016), 9–12.
4. Jacobson N.: *Lie Algebras.* Wiley-Interscience, New York 1962.