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Zdeněk Sobotka

Mathematical Institute
Czechoslovak Academy of Sciences

MODELLING OF SPECIAL RHEOLOGICAL PROCESSES

Summary. The paper deals with special rhe logical processes concerning the deformation of linear viscoelastic bodies. The rheological relations yield the equations for the evaluation of test results and determining the rheological parameters.

1. INTRODUCTION

The author derives the relations for special rheological processes connected with deformations of linear viscoelastic bodies. These relations can be used for the evaluation of the tests of viscoelstic bodies at the constant stress and stress rate. The starting point is represented by the rheological equation of the Kelvin-Voigt body:

$$\sigma = E\sigma + \lambda \frac{d\varepsilon}{dt} , \qquad (1)$$

where E is the modulus of elasticity and λ denotes the coefficient of viscosity.

Integrating the foregoing first-order differential equation, we obtain the expression for the time-varying strain [1]:

$$\varepsilon = e^{-Et/\lambda} \left[\int_{t_0}^{t} \sigma(\tau)\, e^{E\tau/\lambda}\, d\tau + \varepsilon_0\, e^{Et_0/\lambda} \right] , \qquad (2)$$

where ε_0 is the initial strain at the time t_0 .

For the constant stress σ_0 and zero-initial strain ε_0 , we have

$$\varepsilon = \frac{\sigma_0}{E} (1 - e^{-Et/\lambda}) . \qquad (3)$$

Introducing into Eq. (2) the constant stress rate $s = \sigma/t$ and integrating, we obtain for $\varepsilon_0 = 0$ [1] :

$$\varepsilon = \frac{\sigma}{E} \left[1 - \frac{\lambda s}{E\sigma} (1 - e^{-E\sigma/\lambda s})\right] . \qquad (4)$$

2. VISCOELASTIC DEFORMATION AT THE CONSTANT STRESS

If a viscoelastic material corresponding to the Kelvin-Voigt rheological model is tested at the constant stress σ_0 , its strain is defined by Eq. (3). For determining two unknown

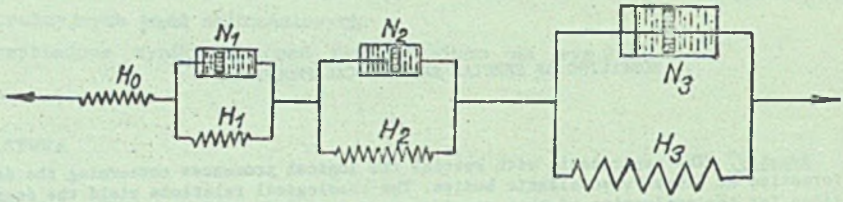

Fig. 1. Rheological model consisting of one Hookean elastic element and three Kelvin-Voigt groups.

Rys. 1. Model reologiczny składający sie z elementu sprężystego Hookea i trzech elementow lepkosprężystych Kelvina-Voigta

rheological characteristics E and λ , two experimental data are needed which can be expressed by the ralations

$$\varepsilon_1 = \frac{\sigma_0}{E}\left(1 - e^{-\alpha t_1}\right), \quad \varepsilon_2 = \frac{\sigma_0}{E}\left(1 - e^{-\alpha t_2}\right), \tag{5}$$

where $\alpha = E/\lambda$ is the reciprocal time of retardation.

Dividing the first relation (5), by the second equation , we can eliminate the modulus E and obtain the transcendental equation for determining the unknown α :

$$1 - k = e^{-\alpha t_1} - k\, e^{-\alpha t_2}, \tag{6}$$

where $k = \varepsilon_1/\varepsilon_2$.

The modulus of elasticity then follows from Eqs. (5).

The strain at the constant stress σ_0 , corresponding to the rheological model in Fig. 1, is expressed by

$$\varepsilon = \sigma_0 \left[\frac{1}{E_0} + \frac{1}{E_1}\left(1 - e^{-\alpha_1 t}\right) + \frac{1}{E_2}\left(1 - e^{-\alpha_2 t}\right) + \frac{1}{E_3}\left(1 - e^{-\alpha_3 t}\right)\right]. \tag{7}$$

This relation contains seven rheological characteristics which should be determined from test results. Therefore, seven values of strains ε_K obtained at seven times t_K are needed. Introducing the notations

$$R_0 = \frac{1}{E_0}, \quad Q_1 = \frac{1}{E_1}, \quad Q_2 = \frac{1}{E_3}, \quad Q_3 = \frac{1}{E_4}, \quad R_1 = e^{-\alpha_1}, \quad R_2 = e^{-\alpha_2}, \quad R_3 = e^{-\alpha_3}, \tag{8}$$

we obtain from the relation (7) for the evaluation of test data measured in seven times t_K , the non-linear equations of the following type

$$Q_0 + Q_1 - Q_1R_1^{t_K} + Q_2 - Q_2R_2^{t_K} + Q_3 - Q_3R_3^{t_K} = \frac{\varepsilon_K}{\sigma_0} \tag{9}$$

In order to make the solution of this problem less complex, the author has introduced an approximate solution of this problem on expanding the exponential functions as follows

$$e^{-\alpha t} = 1 - \alpha t + \frac{1}{2}\alpha^2t^2 - \frac{1}{3!}\alpha^3t^3 + \frac{1}{4!}\alpha^4t^4 - \frac{1}{5!}\alpha^5t^5 + \frac{1}{6!}\alpha^6t^6 - \dots . \tag{10}$$

Introducing seven resolving functions

$$S_0 = \frac{1}{E_0}\,, \quad S_1 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\,, \quad S_2 = \frac{E_1}{\lambda_1^2} + \frac{E_2}{\lambda_2^2} + \frac{E_3}{\lambda_3^2}\,, \quad S_3 = \frac{E_1^2}{\lambda_1^3} + \frac{E_2^2}{\lambda_2^3} + \frac{E_3^2}{\lambda_3^3}\,,$$

$$S_4 = \frac{E_1^3}{\lambda_1^4} + \frac{E_2^3}{\lambda_2^4} + \frac{E_3^3}{\lambda_3^4}\,, \quad S_5 = \frac{E_1^4}{\lambda_1^5} + \frac{E_2^4}{\lambda_2^5} + \frac{E_3^4}{\lambda_3^5}\,, \quad S_6 = \frac{E_1^5}{\lambda_1^6} + \frac{E_2^5}{\lambda_2^6} + \frac{E_3^5}{\lambda_3^6}\,, \tag{11}$$

we obtain seven linear equations with seven unknowns S_K :

$$S_0 + S_1t_1 - \frac{1}{2}S_2t_1^2 + \frac{1}{3!}S_3t_1^3 - \frac{1}{4!}S_4t_1^4 + \frac{1}{5!}S_5t_1^5 - \frac{1}{6!}S_6t_1^6 = \frac{\varepsilon_1}{\sigma_0}\,, \tag{12}$$

$$S_0 + S_1t_2 - \frac{1}{2}S_2t_2^2 + \frac{1}{3!}S_3t_2^3 - \frac{1}{4!}S_4t_2^4 + \frac{1}{5!}S_5t_2^5 - \frac{1}{6!}S_6t_2^6 = \frac{\varepsilon_2}{\sigma_0} \tag{13}$$

. . . .

For determining the moduli of elasticity E_K and coefficients of viscosity λ_K, we then have the set of non-linear equations (11) with seven unknowns. However, for determining the strain at the constant stress σ_0, the parameters are not needed since we have for it the relation with resolving functions S_K :

$$\varepsilon = \sigma_0\left(S_0 + S_1t - \frac{1}{2}S_2t^2 + \frac{1}{3!}S_3t^3 - \frac{1}{4!}S_4t^4 + \frac{1}{5!}S_5t^5 - \frac{1}{6!}S_6t^6\right) . \tag{14}$$

3. STRAIN AT THE CONSTANT STRESS RATE

The strain of a viscoelastic body at the constant stress rate represented by the rheological model in Fig. 1 follows from Eq. (4):

$$\varepsilon = \sigma\left\{\frac{1}{E_0} + \frac{1}{E_1}\left[1 - \frac{\lambda_1 s}{E_1\sigma}\left(1 - e^{-E_1\sigma/\lambda_1 s}\right)\right] + \frac{1}{E_2}\left[1 - \frac{\lambda_2 s}{E_2\sigma}\left(1 - e^{-E_2\sigma/\lambda_2 s}\right)\right] + \right.$$
$$\left. + \frac{1}{E_3}\left[1 - \frac{\lambda_3 s}{E_3\sigma}\left(1 - e^{-E_3\sigma/\lambda_3 s}\right)\right]\right\} . \tag{15}$$

Expanding the exponential functions and introducing the resolving functions given by Eqs. (11), we have

$$\varepsilon = \frac{\sigma}{E_0} + \frac{\sigma^2}{2s}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right) - \frac{\sigma^3}{3!s^2}\left(\frac{E_1}{\lambda_1^2} + \frac{E_2}{\lambda_2^2} + \frac{E_3}{\lambda_3^2}\right) + \frac{\sigma^4}{4!s^3}\left(\frac{E_1^2}{\lambda_1^3} + \frac{E_2^2}{\lambda_2^3} + \frac{E_3^2}{\lambda_3^3}\right) - \dots , \tag{16}$$

$$\varepsilon = S_0\sigma + \frac{1}{2s}S_1\sigma^2 - \frac{1}{3!s^2}S_2\sigma^3 + \frac{1}{4!s^3}S_3\sigma^4 - \frac{1}{5!s^4}S_4\sigma^5 + \frac{1}{6!s^5}S_5\sigma^6 - \dots . \tag{17}$$

For determining the unknown resolving functions S_K, we can measure the stresses at seven different stress rates s_K and at a given strain ε_R. We then have seven linear equations of the following type:

$$S_0\sigma_K + \frac{1}{2s_K} S_1\sigma_K^2 - \frac{1}{3!s_K^2} S_2\sigma_K^3 + \frac{1}{4!s_K^3} S_3\sigma_K^4 - \frac{1}{5!s_K^4} S_4\sigma_K^5 + \frac{1}{6!s_K^5} S_5\sigma_K^6 - \frac{1}{7!s_K^6} S_6\sigma_K^7 = \varepsilon_R . \quad (18)$$

REFERENCE

[1] Sobotka Z.: Rheology of Materials and Engineering Structures. Elsevier, Amsterdam - Oxford - New York - Tokyo 1984.

MODELOWANIE SPECJALNYCH PROCESÓW REOLOGICZNYCH

Streszczenie

W pracy przedstawiono równania dla specjalnych procesów reologicznych liniowych ciał lepkosprężystych przy działaniu naprężenia stałego i stałej prędkości naprężeń. Autor wprowadza układy równań dla określenia parametrów reologicznych na podstawie doświadczeń.

МОДЕЛИРОВАНИЕ СПЕЦИАЛЬНЫХ РЕОЛОГИЧЕСКИХ ПРОЦЕССОВ

Резюме

В статье предлагаются уравнения для специальных реологических процессов линейных вязкоупругих тел при действии постоянного напряжения и постоянной скорости напряжения. Автор получает системы уравнений для определения реологических параметров.