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Zdeněk SOBOTKA
Czech Academy of Sciences

ZWIĄZKI RÓŻNICZKOWE I CAŁKOWE W LEPKOSPRĘŻYSTOŚCI

Streszczenie. W pracy rozpatruje się różne struktury różniczkowych równań zwyczajnych ze współczynnikami zmiennymi, które zostały wyrowadzone na podstawie modeli reologicznych ciał lepkosprężystych. Dla przedstawienia odpowiednich wyrazów całkowych wprowadzono specjalne funkcje rozwiązujące. Badano struktury związków różniczkowych i całkowych oraz pokazano, że metody reologiczne mogą być pożyteczne dla racjonalnych rozwiązań różniczkowych równań zwyczajnych.

DIFFERENTIAL AND INTEGRAL RELATIONS IN VISCOELASTICITY

Summary. The paper is concenred with various structures of ordinary differential equations with variable coefficients which are derived on the basis of rheological models for viscoelastic bodies. For the formulations of corresponding integrals, the special resolving functions are introduced. The analytical structures of the compled differential and integral relations are investigated in detail. It is shown that the rheological procedures may contribute to the rational solutions of ordinary differential equations.

DIFFERENTIAL - UND INTEGRAL - BEZIEHUNGEN IN DER VISKOELASTIZITÄL

Zusammenfassung. In der Arbeit werden verschiedene Strukturen der gemeinen Differentialgleichungen mit veränderlichen Koeffizienten, die auf Grund der rheologischen Modelle von viskoelastischen Körpern abgeleitet wurden, erörtet. Um die entsprechenden Integralausdrücke zu formulieren, hat man spezielle Lösungfunktionen eingeleitet. Man hat die Strukturen von Differential - und - Integral - Beziehungen untersucht und aufgezeigt, daß rheologische Methoden für rationale Lösungen der gemeinen Differentialgleichungen verwendbar sein können.

1. INTRODUCTORY CONSIDERATIONS

The paper contains some results of the search for analogies between the rheological and mathematical structures. It can be shown that every differential equation can be represented by a rheological model. Various configurations of the ordinary differential equations and their integrals are derived on the basis of viscoelastic relations and rheological models. For instance, the strain ε of a viscoelastic body represented by the rheological model with one isolated Hookean elastic element and three Kelvin - Voigt viscoelastic groups connected in series and each consistinng of the elastic and viscous element connected in parallel depends on the time - varying stress $\sigma(t)$ in the following manner

$$\varepsilon = \frac{\sigma(t)}{E_0(t)} + \sum_{k=1}^{3} e^{-\int_{t_0}^{t} d_k(\tau)d\tau} \left| \int_{t_0}^{t} \frac{\sigma(\tau)}{\lambda_k(\tau)} e^{\int_{t_0}^{t} d_k(s)ds} d\tau + C_k \right| , \tag{1.1}$$

where $\alpha_k(t)=E_k(t)/\lambda_k(t)$ are the reciprocal times of retardation defined by the ratios of moduli of elasticity $E_k(t)$ and coefficients of viscosity $\lambda_k(t)$.

Eliminating the constans oof integration C_k by successive multiplication by the exponential functions and differentiation of Eq.(1) yields the third - order differential equation:

$$\begin{aligned} &\varepsilon''' + (\alpha_1 + \alpha_2 + \alpha_3)\varepsilon'' + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + 2\alpha_1' + \alpha_2')\varepsilon' + \\ &+ \left[\alpha_1\alpha_2\alpha_3 + \alpha_1'\alpha_3 + (\alpha_1\alpha_2)' + \alpha_1'' \right] \varepsilon = f(t) , \end{aligned} \tag{1.2}$$

where the time - varying function f(t) depends on the stress, its time derivatives, moduli of elasticity, coefficients of viscosity and their time derivatives.

The actual solution of differential equation (1.2) can consist in determining the functions α_k and substituting them into Eq.(1.1). As can be seen, the order of differential equation to be solved is then reduced.

From Eq.(1.2), the author has obtained the following alternative expression for the strain and simultaneously the solution of this non - homogeneous ordinary differential equation

$$\varepsilon = e^{-\int_{t_0}^{t}\alpha_1(s)ds}\left\| \int_{t_0}^{t} e^{\int_{t_0}^{\tau_1}[\alpha_1(s)-\alpha_2(s)]ds} d\tau_1 \left\{ \int_{t_0}^{\tau_1} e^{\int_{t_0}^{\tau_2}[\alpha_2(s)-\alpha_3(s)]ds} d\tau_2 \cdot \right.\right.$$
$$\left.\left. \cdot \left[\int_{t_0}^{\tau_2} f(\tau_3) e^{\int_{t_0}^{\tau_3}\alpha_3(s)ds} d\tau_3 + C_3 \right] + C_2 \right\} + C_1 \right\| . \tag{1.3}$$

The proof of the foregoing solution can be carried out by multiplying it by the exponential functions and differentiating it, successively.

Therefore, both expressions (1.1) and (1.3) can be taken for the solutions of the differential equation (1.2). The solution of the n-th order ordinary differential equation is then expressed either by the sum of n simple integrals or by the n-fold integral.

2. GENERAL FORM OF DIFFERENTIAL EQUATIONS AND RELATED INTEGRALS

The exponential functions in te foregoing introductory considerations can be replaced by the general integrable functions. The author has called them the resolving functions or briefly the R - functions.

In this manner, a more general integral representation is expressed by

$$\varepsilon = \frac{1}{R_1(t)}\left\| \int_{t_0}^{t} \frac{R_1(\tau_1)}{R_2(\tau_1)} d\tau_1 \left\{ \int_{t_0}^{\tau_1} \frac{R_2(\tau_2)}{R_3(\tau_2)} d\tau_2 \left[\int_{t_0}^{\tau_2} f(\tau_3) R_3(\tau_3) d\tau_3 + C_3 \right] + C_2 \right\} + C_1 \right\| , \tag{2.1}$$

where $R_k(t)$ are the general continuous integrable functions with continuous derivatives.

Eliminating the integrals and constants of integration, we obtain the corresponding differential equation:

$$\varepsilon'''' + \left(\frac{R_1'}{R_1} + \frac{R_2'}{R_2} + \frac{R_3'}{R_3} \right)\varepsilon''' + \left[2\left(\frac{R_1'}{R_1}\right)' + \left(\frac{R_2'}{R_2}\right)' + \frac{R_1'R_2'}{R_1R_2} + \frac{R_1'R_3'}{R_1R_3} + \frac{R_2'R_3'}{R_2R_3} \right]\varepsilon' +$$
$$+ \left[\frac{R_1'R_2'R_3'}{R_1R_2R_3} + \left(\frac{R_1'}{R_1}\right)'\frac{R_3'}{R_3} + \left(\frac{R_1'R_2'}{R_1R_2}\right) + \left(\frac{R_1'}{R_1}\right)'' \right]\varepsilon = f(t) . \tag{2.2}$$

As can be seen, the deriveratives of R - functions in the foregoing equation are logarithmic ones so that we can write:

$$\varepsilon''' + \left(\frac{d\ln R_1}{dt} + \frac{d\ln R_2}{dt} + \frac{d\ln R_3}{dt} \right)\varepsilon'' + \left[\left(2\frac{d^2\ln R_1}{dt^2} + \frac{d^2\ln R_2}{dt^2} + \right.\right.$$

$$\left. + \frac{d\ln R_1}{dt}\frac{d\ln R_2}{dt} + \frac{d\ln R_1}{dt}\frac{d\ln R_3}{dt} + \frac{d\ln R_2}{dt}\frac{d\ln R_3}{dt} \right)\varepsilon' + \tag{2.3}$$

$$\left. + \frac{d\ln R_1}{dt}\frac{d\ln R_2}{dt}\frac{d\ln R_3}{dt} + \frac{d^2\ln R_1}{dt^2}\frac{d\ln R_3}{dt} + \frac{d}{dt}\left(\frac{d\ln R_1}{dt}\frac{d\ln R_2}{dt} \right) + \frac{d^3\ln R_1}{dt^3} \right]\varepsilon = f(t) \ .$$

The differential and integral relations can be formulated in an alternative manner. Introducing other resolving functions $S_k(t)$, the integral relation may be expressed in the following form

$$\varepsilon = \frac{1}{S_1(t)}\left[\!\left[\int_{t_0}^{t} \frac{d\tau_1}{S_1(\tau_1)} \left\{ \int_{t_0}^{\tau_1} \frac{d\tau_2}{S_3(\tau_2)} \left[\int_{t_0}^{\tau_2} f(\tau_3)d\tau_3 + C_3 \right] + C_2 \right\} + C_1 \right]\!\right] . \tag{2.4}$$

Eliminating again the integrals and constants of integration yields:

$$\varepsilon''' + \left[\frac{S_1'}{S_1} + \frac{(S_1S_2)'}{S_1S_2} + \frac{(S_1S_2S_3)'}{S_1S_2S_3} \right]\varepsilon'' +$$

$$+ \left[2\frac{S_1''}{S_1} + 2\frac{S_1'S_2'}{S_1S_2} + \frac{S_1'S_3'}{S_1S_3} + \frac{(S_1S_2)''}{S_1S_2} + \frac{(S_1S_2)'}{S_1S_2}\frac{S_3'}{S_3} \right]\varepsilon' + \tag{2.5}$$

$$+ \left[\frac{(S_1'S_2')'}{S_1S_2} + \frac{(S_1''S_2)'}{S_1S_2} + \frac{S_1''S_3'}{S_1S_3} + \frac{S_1'S_2'S_3'}{S_1S_2S_3} \right]\varepsilon = f(t) \ .$$

3. SECONND-ORDER DIFFERENTIAL EQUATION WITH VARIOUS R-FUNCTIONS

Starting from the integral relation:

$$\varepsilon = \frac{1}{R_1(t)}\left\{ \int_{t_0}^{t} \frac{R_1(\tau_1)}{R_2(\tau_1)} d\tau_1 \left[\int_{t_0}^{\tau_1} f(\tau_2)R_2(\tau_2)d\tau_2 + C_2 \right] + C_1 \right\} , \tag{3.1}$$

we obtain second - order differential equation:

$$\varepsilon'' + \left(\frac{R_1'}{R_1} + \frac{R_2'}{R_2} \right) \varepsilon' + \left[\left(\frac{R_1'}{R_1} \right)' + \frac{R_1' R_2'}{R_1 R_2} \right] \varepsilon = f(t) \ . \tag{3.2}$$

Introducing into the foregoing equation the functions:

$$R_1 = k_1 \sin at \ , \qquad R_2 = k_2 \sin bt \ , \tag{3.3}$$

we obtain the differential equation with trigonometric coefficients:

$$\varepsilon'' + (a \cos at + b \cos bt)\, \varepsilon' + \left(ab \cot at \cot bt - \frac{a^2}{\sin^2 at} \right) \varepsilon = f(t) \ . \tag{3.4}$$

The related integral is expressed by

$$\varepsilon = \frac{1}{\sin at} \left\{ \int_{t_0}^{t} \frac{\sin a\tau_1}{\sin b\tau_1} d\tau_1 \left[\int_{t_0}^{\tau_1} f(\tau_2) \sin b\tau_2 \, d\tau_2 + C_2 \right] + C_1 \right\} , \tag{3.5}$$

Introducing the resolving functions:

$$R_1 = \frac{k_1}{t^m} \ , \qquad R_2 = \frac{k_2}{t^n} \tag{3.6}$$

yields the differential equation:

$$\varepsilon'' - \frac{1}{t}(m + n)\, \varepsilon' + \frac{1}{t^2}(mn + n)\, \varepsilon = f(t) \ . \tag{3.7}$$

The corresponding integral has the following form

$$\varepsilon = t^m \left\{ \int_{t_0}^{t} \tau_1^{n-m} d\tau_1 \left[\int_{t_0}^{\tau_1} \frac{f(\tau_2)}{\tau_2^n} d\tau_2 + C_2 \right] + C_1 \right\} . \tag{3.8}$$

Proceeding in this manner and choosing various R - functions, we can obtain a very extensive set of differential relations with coupled integrals which represent a useful clue for the solutions of non - homogeneous ordinary differential equations with variable coefficients.

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Recenzent: Dr hab. inż. Jerzy Świder

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Streszczenie

Przedmiotem pracy są niektóre wyniki badań analogii pomiędzy strukturami reologicznymi i matematycznymi. Można wykazać, że każde równanie różniczkowe zwyczajne może być przedstawione przez model reologiczny ciała lepkosprężystego, które się składa z elementów sprężystych i lepkich. Rząd takiego równania odpowiada liczbie elementów lepkich. Niektóre równania różniczkowe są przedstawione przez modele zawierające elementy z charakterystykami urojonymi albo zespolonymi. W pracy umieszczono różniczkowe równania zwyczajne ze współczynikami zmiennymi i odpowiednie całki wielokrotne ze stałymi całkowania, które otrzymuje się na podstawie modeli reologicznych. Równania różniczkowe i odpowiednie wyrazy całkowe związane są za pomocą specjalnych funkcji rozwiązujących. Wybierając różne funkcje rozwiązujące, możemy otrzymać szeroki zbiór różniczkowych równań zwyczajnych wraz z ich całkami. Taki zbiór może doprowadzić do wielu racjonalnych rozwiązań różniczkowych równań zwyczajnych.