

*transport elements, nonstationary arrivals,
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Július REBO¹
Ondrej BARTL²

TIME CONTROL OF A NONSTATIONARY DISCRETE ACCUMULATION PROCESS

An accumulation process of transport elements with nonstationarity in arrival patterns is studied in the paper. Arrival rates vary periodically in time within a cycle. A method to divide the cycle into accumulation periods is suggested with respect to some optimisation criteria.

KONTROLA CZASU W NIESTACJONARNYCH DYSKRETNYCH PROCESACH AKUMULACJI

Referat rozpatruje proces akumulacji elementów transporty z brakiem stacjonarności wzorców przyjazdów. Przyjazdy okresowo ulegają zmianie w sposób cykliczny. Zaproponowano metodę podzielenia cyklu na okresy akumulacji w odniesieniu do pewnych kryteriów optymalizacji.

1. INTRODUCTION

When transporting elements such as containers, a naturally convenient way is to gather them first and then transport in groups. Rules to control accumulation process of transport elements are expected to economise transportation. The rules usually form a compromise between transporting frequently with a poorly utilised capacity of transport means and letting elements waiting relatively long to increase capacity utilisation. As elements arrive at an accumulation site at random, the accumulation process is a stochastic process. Supposing an arrival process with independent increments, the evolution of the accumulation process can be represented by a discrete-time Markov chain, where the process state is a discrete random variable. A time control problem for the discrete accumulation process is analysed in the paper. If a stationary Poisson process represents arrivals of transport elements to the place of their accumulation, an optimal duration of the accumulation period can be determined. Each time when the period of accumulation is over, a group of elements not exceeding

¹ Faculty of Management Science and Informatics, University of Žilina, Campus Prievidza, Bakalárska 2, 971 01 Prievidza, Slovakia, rebo@utcpd.sk

² Faculty of Management Science and Informatics, University of Žilina, Univerzitná 8215/1, 010 26 Žilina, Slovakia, bartl@kst.fri.utc.sk

transportation capacity is sent to be transported. In the case of a nonstationary Poisson arrival stream where few distinct arrival patterns are periodically repeated in a cycle, a method to obtain an approximate solution to the optimal time control problem is designed.

2. DISCRETE ACCUMULATION PROCESS

The state of the accumulation process at any time is the number of elements waiting on the accumulation site. State values important for decision-making are those at the ends of accumulation periods. Let t_0, t_1, t_2, \dots denote the ends of accumulation periods, where time $t_0 \equiv 0$ is the beginning of the accumulation process. Then the duration of the i th accumulation period is given by $T_i = t_i - t_{i-1}$, $i = 1, 2, \dots$. Let M_i be the transportation capacity of transport means available at the end of the i th period to carry elements away. Denoting by S_i the state of the accumulation process at the end of the i th period just before starting the transport of elements, it is clear that the quantity $A_i = \min\{S_i, M_i\}$ represents the portion of elements transported. Then the number of elements remaining to wait on the accumulation site is represented by $Z_i = S_i - A_i = S_i - \min\{S_i, M_i\} = \max\{0, S_i - M_i\}$. Let X_i be a random variable representing the number of elements that arrive during the i th accumulation period. The behaviour of the discrete accumulation process can be described, for $i = 0, 1, 2, \dots$, by the following transition equation

$$S_{i+1} = S_i - A_i + X_{i+1} = S_i - \min\{S_i, M_i\} + X_{i+1} = Z_i + X_{i+1} = \max\{0, S_i - M_i\} + X_{i+1} \quad (1)$$

3. STATIONARY ARRIVALS

If a stationary Poisson process $\{X(t), t \geq 0\}$, where $X(t)$ is the number of arrivals up to time t , represents the arrival stream of elements and the transportation capacity available at the end of any accumulation period is $M > 0$ elements, then the lengths T_i of accumulation periods are equal and each of them is therefore the duration T of the accumulation cycle, i.e. $T_i = T$, $i = 1, 2, \dots$. The number of elements that arrive during the accumulation period T is Poisson distributed. Hence, $\pi_k = P\{X(T) = k\} = e^{-\lambda T} (\lambda T)^k / k!$, $T \geq 0$, $k = 0, 1, 2, \dots$, where the parameter $\lambda > 0$ is the arrival rate of the process. In case the period length $T > 0$ satisfies the stabilisation condition

$$\rho = \frac{\lambda T}{M} < 1 \quad (2)$$

the accumulation process eventually reaches the steady-state mode of operation when the limits $p_k = \lim_{i \rightarrow \infty} P\{S_i = k\}$, $k = 0, 1, 2, \dots$, exist such that $p_k \geq 0, \forall k$, and $\sum_{k=0}^{\infty} p_k = 1$ (the proof based on the use of embedded discrete-time Markov chains is given in [3]). The limiting probability distribution $\{p_k, k = 0, 1, 2, \dots\}$ can be obtained as the solution to the set of equations

$$\begin{aligned} p_0 &= (p_0 + p_1 + \dots + p_M) \pi_0 \\ p_k &= (p_0 + p_1 + \dots + p_M) \pi_k + p_{M+1} \pi_{k-1} + \dots + p_{M+k} \pi_0, \quad k = 1, 2, \dots \end{aligned} \quad (3)$$

together with the condition $\sum_{k=0}^{\infty} p_k = 1$. The probability generating functions $F(z)$ for process state probabilities and $\Phi(z)$ for arrival probabilities will be used to solve the system (3).

Multiplying the k th equation ($k = 0, 1, \dots$) in (3) by the corresponding power z^k and summing the modified equations on k , we get (see e.g. [3])

$$F(z) = \frac{\Phi(z) \sum_{r=0}^{M-1} (z^M - z^r) p_r}{z^M - \Phi(z)} = \frac{\sum_{r=0}^{M-1} (z^M - z^r) p_r}{z^M e^{\rho M(1-z)} - 1} \quad (4)$$

with the notation $F(z) = \sum_{r=0}^{\infty} p_r z^r$ and $\Phi(z) = \sum_{r=0}^{\infty} \pi_r z^r = e^{-\rho M(1-z)}$, where $|z| \leq 1$. Since the series $\sum_{r=0}^{\infty} p_r z^r$ has to be convergent for $|z| \leq 1$, the roots of the polynomial in the numerator of the fraction in (4) must coincide with the zeros of denominator, z_k , in the circle $|z| \leq 1 + \delta$, where $\delta > 0$ is an arbitrary small real number. After some manipulations (like those in [3, 4]) we get from (4) the probability generating function $F(z)$ in the form

$$F(z) = \frac{(M - \rho M)(z-1) \prod_{r=1}^{M-1} \frac{z - z_r}{1 - z_r}}{z^M e^{\rho M(1-z)} - 1}. \quad (5)$$

Computing the first derivative of the function (5) at point $z = 1$ enables us to obtain the mean value of the process state at the end of any accumulation period in the steady-state mode of operation. Thus,

$$E(S) = \begin{cases} \frac{\rho(2 - \rho)}{2(1 - \rho)}, & M = 1, \\ \frac{M - (M - \lambda T)^2}{2(M - \lambda T)} + \sum_{r=1}^{M-1} (1 - z_r)^{-1}, & M \geq 2. \end{cases} \quad (6)$$

As in the steady-state operation mode the transition equation (1) can be written in the form $S = Z + X(T)$, the mean value of the number of elements remaining on the accumulation site immediately after transport at the end of any accumulation period is given by

$$E(Z) = E(S) - E(X(T)) = E(S) - \lambda T. \quad (7)$$

The expected consumption of time, ω , spent by elements waiting on the accumulation site during an accumulation period of length T can be expressed in specified time units according to [3] as follows

$$\omega = E(Z)T + \frac{\lambda T}{2} T = \left(E(S) - \frac{\lambda T}{2} \right) T. \quad (8)$$

For $M \geq 2$, we need to know the roots $z_r, r = 0, 1, \dots, M-1$, of the so-called characteristic equation $z^M - e^{-\rho M(1-z)} = 0$ representing the relation, in which the denominator of the fraction on the right side of (5) equals zero. The root $z_0 = 1$ obviously satisfies the characteristic equation. A method to find out the other roots is described in [3, 4]. The method is based on the numerical technique from [1]. After computing the roots for various combinations of the values of the accumulation process parameters M and $\rho = \lambda T/M$, we can approximate the expression $\sum_{r=1}^{M-1} (1-z_r)^{-1}$ by an appropriate function $f(M, \rho)$. A linear approximation turns out to be accurate enough and rather simple to be used for optimisation purposes. The resulting form of the approximation is given, for $M \geq 2$ and $\rho \in (0, 1)$, by

$$\sum_{r=1}^{M-1} (1-z_r)^{-1} \approx f(M, \rho) = a_1(M)\rho + a_0(M) = (0.4045M - 0.6609)\rho + 0.525M - 0.5114 \quad (9)$$

Using this formula, the values of expectations (6), (7), and (8) explicitly depend on the parameters λ, T, M of the accumulation process even if $M \geq 2$.

3.1. OPTIMISATION OF ACCUMULATION PERIOD

Optimisation criteria for determining an optimal length T^* of the accumulation period can reflect the costs associated with the accumulation process or the waiting time consumption. The cost criterion uses the transportation cost per element, $c_1 > 0$, the fixed cost of the accumulation process per period, $c_2 > 0$, and the waiting cost per element per unit time, $c_3 > 0$. Since the equilibrium equation $\lambda T = \rho M$ with the transportation capacity utilisation $\rho \in (0, 1)$ holds in the steady-state mode of operation, the expected cost per period incurred to carry out the accumulation process, when the accumulation period is long T time units, is given by $C(T) = c_1 \lambda T + c_2 + c_3 \omega$. The corresponding expected cost per element is stated by

$$N(\rho) = \frac{C(T)}{\lambda T} \approx c_1 + \frac{c_2}{\rho M} + \frac{c_3}{\lambda} \left[\frac{M - (M - \rho M)^2}{2(M - \rho M)} - \frac{\rho M}{2} + a_1(M)\rho + a_0(M) \right], M \geq 2. \quad (10)$$

The formulae (8), (6), and (9) have been used to obtain (10) for the case with the disposable transportation capacity greater than one element, which is more likely to occur in practice.

Differentiating function $N(\rho)$ and letting the third derivative equal zero yields the equation $\rho^4 - 2\rho^3 + [(Mc_3 - 2\lambda c_2)/(2Ma_1c_3) + 1]\rho^2 + [(2\lambda c_2)/(Ma_1c_3)]\rho - (\lambda c_2)/(Ma_1c_3) = 0$ with a_1 written instead of $a_1(M)$ for short. A feasible solution ρ^* , lying in the interval $(0, 1)$, can be calculated using a proper numerical technique. Then the optimal length of the accumulation period is given by $T^* = \rho^* M / \lambda$.

The waiting time criterion takes into account the expected time that an element, remaining on the accumulation site after the end of the accumulation period, spends waiting to be transported later. Using Little's formula, it follows that the expected waiting time per element is stated by the function

$$W(\rho) = \frac{E(Z)}{\lambda} \approx \frac{1}{2\lambda(1-\rho)} - \frac{M(1-\rho)}{2\lambda} - \frac{\rho M}{\lambda} + \frac{1}{\lambda} [a_1(M)\rho + a_0(M)], \quad M \geq 2. \quad (11)$$

The formulae (7), (6), and (9) have been used to obtain (11) for the transportation capacity greater than one element. Letting the first derivative of the function $W(\rho)$ equal zero leads to the equation $1/[2\lambda(1-\rho)^2] - M/(2\lambda) + a_1(M)/\lambda = 0$, where solution ρ^* satisfying the constraint $0 < \rho < 1$ is given by $\rho^* = 1 - 1/\sqrt{M - 2a_1(M)}$. The expression within the radical sign is positive for $M \geq 2$ since $a_1(M) = 0.4045M - 0.6609$. The optimal length of the accumulation period is then prescribed by the formula $T^* = (1 - 1/\sqrt{M - 2a_1(M)})M/\lambda$ as it follows from the relation $T^* = \rho^*M/\lambda$.

4. NONSTATIONARY ARRIVALS

If the arrival stream of elements is represented by a nonstationary Poisson arrival process $\{X(t), t \geq 0\}$, then the arrival rate, i.e. the expected number of arrivals per unit time, is not a constant. An intensity function $\lambda(t), t \geq 0$, describes arrival rates for particular time instants t . As random variable $X(t)$ is the number of arrivals up to time t , the expected number of arrivals up to time t is given by the quantity $M(t) = E[X(t)] = \int_0^t \lambda(u) du, t \geq 0$. The number of arrivals in the time interval $(t, t+s]$ is Poisson distributed. The probability of k arrivals between times t and $t+s$ is stated by $p_k(t, s) = P\{X(t+s) - X(t) = k\} = e^{-L(t, s)} [L(t, s)]^k / k!$ for any $t, s \geq 0$ and $k = 0, 1, 2, \dots$, where $L(t, s) = \int_t^{t+s} \lambda(u) du$, see e.g. [2]. We will suppose that the intensity function is a periodic function with period $T > 0$. Then $\lambda(t+lT) = \lambda(t)$ for $l = 0, 1, 2, \dots$ and $t \in [0, T]$. The disposable transportation capacity is also supposed to vary periodically with the same cycle duration T . Periodicity in arrival rates and in transportation capacities justifies periodicity in rules for time control of the accumulation process. Thus, we need to divide the accumulation cycle T somehow into a proper number of accumulation periods with respect to a criterion of interest chosen for optimisation purposes. The way we suggest provides an approximate solution to the problem.

The starting step is to partition the cycle length T into disjoint subintervals that correspond to particular arrival patterns. Let $m > 0$ denote the number of subintervals $(b_{j-1}, b_j]$ splitting the interval $(0, T]$ to match arrival patterns. The border points of subintervals are such that $0 \equiv b_0 < b_1 < \dots < b_{m-1} < b_m \equiv T$. A constant arrival rate $\lambda_j > 0$ relates to the j th subinterval, $j \in J = \{1, 2, \dots, m\}$, being determined as the average arrival intensity on the j th subinterval according to $\lambda_j = \int_{b_{j-1}}^{b_j} \lambda(t) dt / \tau_j$, with $\tau_j = b_j - b_{j-1}$, $j \in J$.

Let M_j be the disposable transportation capacity for every transport of elements within the j th subinterval. As an approximation to the original nonstationary optimal time control

problem, we can determine an optimal length T_j^* of the accumulation period autonomously for each pair of parameters $\lambda_j, M_j, j \in J$, using the techniques for stationary accumulation processes described before. Then we find a proper number k_j of accumulation periods T_j^* to match the length τ_j of the j th subinterval for each $j \in J$ so that $k_j T_j^* \approx \tau_j$. The resulting total number of accumulation periods per accumulation cycle is $n = k_1 + k_2 + \dots + k_m$. The corresponding duration of the accumulation cycle turns out to be $T^0 = \sum_{j=1}^m k_j T_j^*$. We compare this calculated length of the accumulation cycle with the required length T associated with the periodicity in arrival rates, and after finding any difference, we have to modulate the lengths of n accumulation periods to hold the required duration T of the accumulation cycle.

4.1. COORDINATION OF ACCUMULATION PERIODS WITH CYCLE LENGTH

Let us suppose that we have n accumulation periods with lengths $T_1^0, T_2^0, \dots, T_n^0$ that form an accumulation cycle, whose calculated duration is $T^0 = \sum_{i=1}^n T_i^0$. Let $\Delta T^0 = T - T^0$ denote the difference between a required cycle duration T and the calculated duration T^0 . To refer to accumulation periods $T_i^0, i \in I = \{1, 2, \dots, n\}$, we use the term stage for short. The function $a: I \rightarrow J$ defined by $a(i) = \min\{j \in J: \sum_{l=1}^j k_l \geq i\}, i \in I$, can be used to specify the arrival pattern related to each stage within the accumulation cycle. Recall that $J = \{1, 2, \dots, m\}$ is the set of subscripts denoting arrival patterns and k_l is the number of accumulation periods for the l th arrival pattern, $l \in J$, in each accumulation cycle. The duration of the i th stage is then given by the equality $T_i^0 = T_{a(i)}^*$, $i \in I$. The equalities $\lambda_i^0 = \lambda_{a(i)}, M_i^0 = M_{a(i)}, \rho_i^0 = \rho_{a(i)}^*$ specify the respective arrival rate, transportation capacity and utilisation of the transportation capacity for any stage $i \in I$ in the accumulation cycle. If the difference ΔT^0 equals zero, then no recalculation of the stage lengths $T_i^0, i \in I$, is made implying that the change ΔT_i^0 of the stage length is equal to zero for each stage $i \in I$. In the case of a positive difference we have to enlarge the lengths of stages, whilst we must shorten the lengths of stages in the case of a negative difference.

Anyway, we need to determine proper values of the stage length changes ΔT_i^0 , so that the new lengths of stages, $\tilde{T}_i^0 = T_i^0 + \Delta T_i^0, i \in I$, will satisfy the required duration T of the accumulation cycle, i.e. $\sum_{i=1}^n \tilde{T}_i^0 = T$. Each stage can have changed its duration in such extent only, that the new stage length will not violate the stabilisation condition for the respective autonomous stationary accumulation process, i.e. $\tilde{\rho}_i^0 \equiv \lambda_i^0 \tilde{T}_i^0 / M_i^0 < 1, i \in I$. Then the expected number of elements arriving per accumulation cycle is less than the disposable transportation capacity per cycle, i.e. $\int_0^T \lambda(t) dt < \sum_{i=1}^n M_i^0$, which ensures the stabilisation of the original nonstationary accumulation process into the steady-state mode of operation provided that there exists a periodicity in the process parameters represented by arrival rates, transportation capacities and lengths of accumulation periods.

If it is not possible to find such changes $\Delta T_i^0, i \in I$, in the lengths of stage: that all stabilisation conditions are satisfied, a new partition of the accumulation cycle T into its subintervals $(b_{j-1}, b_j]$ with a modified number m of subintervals should be tried. This return to the starting step of the procedure for the time control optimisation of a nonstationary (but periodic) accumulation process brings a new setting of the process parameters. It may enable one to obtain feasible lengths \tilde{T}_i^0 of accumulation periods within the accumulation cycle T with a modified number n of periods.

In general, there are several feasible solutions when proper changes ΔT_i^0 of the lengths of accumulation periods are looked for. To select the only one, an optimal coordination problem can be formulated using a suitable optimisation criterion. Having new, recalculated lengths \tilde{T}_i^0 of accumulation periods as close to the original, calculated lengths T_i^0 as possible seems to be a natural requirement. It means that we need to minimise the mean square deviation between the new and old stage lengths for a given number n of stages in the accumulation cycle. The corresponding optimisation problem is stated as follows:

$$\text{minimise } F(\Delta T_1^0, \Delta T_2^0, \dots, \Delta T_n^0) = \sum_{i=1}^n (\Delta T_i^0)^2 \tag{12}$$

subject to

$$\sum_{i=1}^n \Delta T_i^0 = \Delta T^0 \tag{13}$$

$$\lambda_i^0 (T_i^0 + \Delta T_i^0) < M_i^0, \quad i \in I = \{1, 2, \dots, n\} \tag{14}$$

Recall that $\Delta T^0 = T - T^0$ denotes the difference between the required cycle duration T and the calculated duration T^0 . The sum of new stage lengths $\tilde{T}_i^0 = T_i^0 + \Delta T_i^0$ will therefore form the required cycle duration as prescribed by the constraint (13). Each new stage length must satisfy the stabilisation condition for the related arrival rate and transportation capacity as required by the set of constraints (14). Since the number of stages is a fixed finite number n , the objective function (12) representing the total square deviation attains its minimum at the same points as does the mean square deviation $\frac{1}{n} \sum_{i=1}^n (\Delta T_i^0)^2$. Note that the optimisation problem (12), (13) without the stabilisation constraints (14) has the solution in the form $\Delta T_i^0 = \Delta T^0/n, i \in I$, as follows solving the corresponding unconstrained minimisation problem with the objective function $\sum_{i=1}^n (\Delta T_i^0)^2 + y [\sum_{i=1}^n \Delta T_i^0 - \Delta T^0]$ where y is a Lagrange multiplier. Thus, splitting the difference ΔT^0 uniformly among the changes $\Delta T_i^0, i \in I$, is optimal in such a case. This fact can encourage one to try a bit different objective function so that the constraints (13), (14) are accompanied by the objective

$$\text{minimise } G(\Delta T_1^0, \Delta T_2^0, \dots, \Delta T_n^0) = \sum_{i=1}^{n-1} (w_i \Delta T_i^0 - w_{i+1} \Delta T_{i+1}^0)^2 + (w_n \Delta T_n^0 - w_1 \Delta T_1^0)^2 \tag{15}$$

with weights $w_i \in (0, 1], i \in I$, used to assess the importance of each stage length change. The function G , where all weights equal 1, represents in some sense the variance of the changes. The problem (15) with unit weights and no restrictions leads to equal changes, which means

that if the restriction (13) is imposed on the problem then $\Delta T_i^0 = \Delta T^0/n, i \in I$. The function F with weights, in the form $F_1 = \sum_{i=1}^n w_i (\Delta T_i)^2$ or $F_2 = \sum_{i=1}^n (w_i \Delta T_i)^2$, and the function G might alternate for each other in the optimisation problem (15), (13), (14).

Should operational conditions prefer having accumulation intervals as equal as possible, the objective function representing in some sense the variance of stage lengths can be employed. The corresponding optimisation problem with weights would consist of the constraints (13), (14) and the objective

$$\min H(\Delta T_1^0, \dots, \Delta T_n^0) = \sum_{i=1}^{n-1} (T_i^0 + w_i \Delta T_i^0 - T_{i+1}^0 - w_{i+1} \Delta T_{i+1}^0)^2 + (T_n^0 + w_n \Delta T_n^0 - T_1^0 - w_1 \Delta T_1^0)^2 \quad (16)$$

A solution to the problem (16), (13), (14) would approach regular transports of elements with regularity modified by an impact of weights.

There are various ways how to select the values of coefficients $w_i, i \in I$, if we want them to differ from 1 (except for few values). For instance, we can let $w_i = T_i^0 / T_{\max}^0, i \in I$, or $w_i = \rho_i^0 / \rho_{\max}^0, i \in I$, where $T_{\max}^0 = \max\{T_i^0, i \in I\}$ and $\rho_{\max}^0 = \max\{\rho_i^0, i \in I\}$. In case the problem (12), (13), (14) is considered, with the objective function F replaced by its weighted version F_1 or F_2 , it holds that the higher the stage weight, the higher the tendency of the stage length to stay unchanged. In general, we can experiment with the coefficients w_i not only having their values in the interval $(0, 1]$. Then we can, e.g., let $w_i = c_{3,a(i)}, i \in I$, where $c_{3,a(i)}$ is the waiting cost per element per unit time associated with the $a(i)$ th arrival pattern.

The optimisation problem with the objective (12) or (15) or (16) and the constraints (13) and (14) is a mathematical programming problem. To solve it, some available techniques can be used as, for example, Rosen's gradient projection methods with linear constraints or Wolfe's quadratic method.

A solution to the optimal coordination problem mentioned above prescribes the changes ΔT_i^0 of the original stage lengths T_i^0 to obtain such new lengths $\tilde{T}_i^0 = T_i^0 + \Delta T_i^0, i \in I$, that the sum of these recalculated lengths of stages satisfies the required duration T of the accumulation cycle. If the change ΔT_i^0 is positive, then the duration of the i th accumulation period within the accumulation cycle will be prolonged. If the change ΔT_i^0 is negative, then the duration of the i th accumulation period will be shortened. In case no feasible solution exists, a new iteration of the whole procedure for determining proper lengths of accumulation periods in a nonstationary accumulation process with periodic operational conditions must be made.

BIBLIOGRAPHY

- [1] ČERNÝ J. and KLUVÁNEK P., Základy matematickej teórie dopravy (Fundamentals of the Mathematical Theory of Transport), Veda, Bratislava 1991
- [2] GROSS D. and HARRIS C.M., Fundamentals of Queueing Theory, Wiley, New York 1974
- [3] REBO J., Diskrétné modely zhromažďovania (Discrete Accumulation Models), Faculty of Management Science and Informatics, University of Žilina, Žilina 2001, (Dissertation)
- [4] REBO J., The discrete process of storage, in: Proceedings of the 16th International Conference on Mathematical Methods in Economics, Vydavatelství ZČU, Cheb 1998, 171–178

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