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Tatiana OLEJNIKOVA
Department of Applied Mathematics, Faculty of Civil Engineering,
Technical University in Košice, Slovakia

THREE-AXIAL CYCLICAL SURFACES OF REVOLUTION

Summary. In the paper, there are presented special class of surfaces, three-axial cyclical surfaces of revolution created by the Euclidean metric transformation of a simultaneous revolutiosn about three different axes. Four specific subclasses of surfaces are classified with respect to the superposition of the three axes of revolution. Special positions of axes are chosen in the coordinate axes or in lines parallel to the coordinate axes, or in the general position. For every case transformation matrices of composed revolutions are derived. Some trajectories of the point movement and also cyclical surfaces created by translating of a circle along of these trajectories are visualised from their vector equations.

TRZYOSIOWE, CYKLICZNE POWIERZCHNIE OBROTOWE

Streszczenie. Artykuł przedstawia specjalne klasy powierzchni – trzyosiowe, cykliczne powierzchnie obrotowe. Cztery specjalne podklasy powierzchni są określone z uwagi na superpozycje trzech osi obrotu. Specjalne pozycje osi są wybrane w osiach współrzędnych lub liniach równoległych do osi współrzędnych, lub w pozycji ogólnej. Dla każdej transformacji tworzone są macierze obrotu. Niektóre trajektorie ruchu punktu, a także powierzchnie cykliczne tworzone przez przełożenie okręgu wzdłuż tych trajektorii są wizualizowane z ich równań wektorów.

1. Introduction

Composition of three revolutions about three different axes in the space determines a metric transformation denoted as the three-axial revolution and we can analytically represented it by a regular square matrix of rank 4, with entries as a real functions of one real variable. Matrix can be derived as the product of multiplication of matrices representing the consecutive revolutions.

Trajectory of the three-axial revolutionary movement is a space curve which represented the class of general revolutionary movements in the space. Let ${}^1o$, ${}^2o$, ${}^3o$ be the three different axes of three revolutions. There can be characterised four subgroups of movements with respect to the superposition of three axes of revolutions:

I. movement with three parallel axes ${}^1o \parallel {}^2o \parallel {}^3o$
   1. all axes are located in one plane
   2. axes are not located in one plane

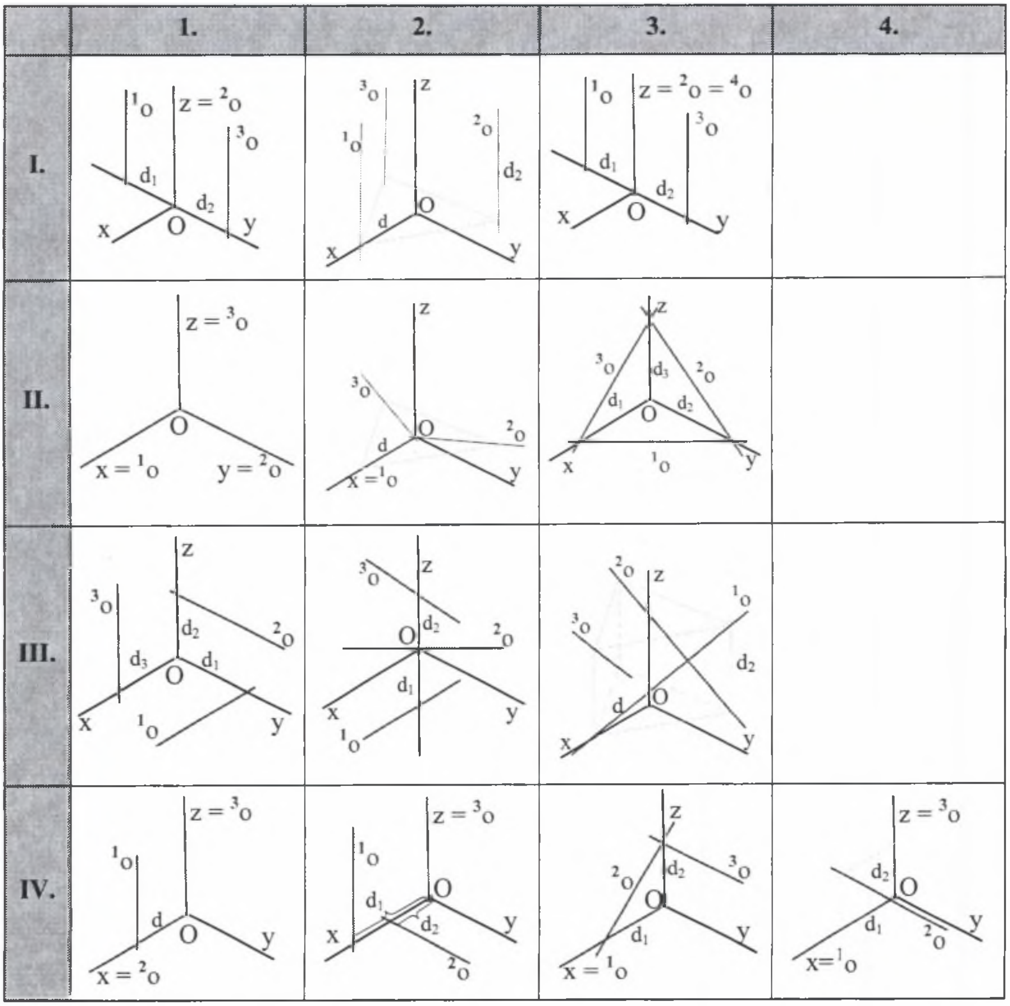
II. movement with three intersect axes ${}^1o \times {}^2o \times {}^3o$
   1. all axes intersect in one point and they are perpendicular to each other ${}^1o \cap {}^2o \cap {}^3o = \mathrm{O}$ and ${}^1o \perp {}^2o \perp {}^3o$
   2. all axes intersect in one common point and they are situated in one plane ${}^1o \cap {}^2o \cap {}^3o = \mathrm{O}$ and ${}^1o, {}^2o, {}^3o \subset xy$
   3. axes intersect in three different points created triangular Δ XYZ ${}^1o \cap {}^2o = \mathrm{X}, {}^2o \cap {}^3o = \mathrm{Y}, {}^1o \cap {}^3o = \mathrm{Z}$

III. movement with three skew axes ${}^1o / {}^2o / {}^3o$
   1. axes of skew lines have one common point ${}^1o \parallel x, {}^2o \parallel y, {}^3o \parallel z$
   2. axis of skew lines is one common line
   3. axes of skew lines are three different lines

IV. movement with combinations of superpositions
   1. two axes are parallel and third is intersect with them ${}^1o \parallel {}^2o, {}^3o \times {}^1o, {}^3o \times {}^2o$
   2. two axes are parallel and third is skew with them ${}^1o \parallel {}^3o, {}^2o / {}^1o, {}^2o / {}^3o$
   3. two axes are skew and third is intersect with them ${}^1o \times {}^2o, {}^2o \times {}^3o, {}^1o / {}^3o$
   4. two axes are intersect and third is skew with them ${}^1o \times {}^2o, {}^1o / {}^3o, {}^2o / {}^3o$

All possibilities of relativ superpositions of three axes of revolutions are presented in Tab. 1. Special positions of axes are chosen in the coordinate axes x, y, z or in lines parallel to them, or in the general position defined by distances or angles. All triangles in Tab. 1 are equiangular excepting case IV. 3.

Table 1



2. Vector equation of the three-axial cyclical surface of revolution

Let curve k be a trajectory of the point $\mathbf{A}=(x_0,y_0,z_0,1)$, which revolves concurrently about three different axes ${}^1o$, ${}^2o$, ${}^3o$ with angular velocities $w_i = m_i v$ in the directions (right-handed or left-handed) determined by parameters $q_i = \pm 1$ for $i=1,2,3$.

Let us denoted the transformation matrices of particular revolutions about three different axes as $\mathbf{T}_i(w_i,q_i)$, then general revolution composite from these three revolutions will be represent by matrix $\mathbf{T}_{123}(v)=\mathbf{T}_1(w_1,q_1).\mathbf{T}_2(w_2,q_2).\mathbf{T}_1(w_3,q_3)$. The trajectory of the point A which is amenable of this three-axial revolution is the curve k defined analytically by a vector function

$$\mathbf{r}(v)=\mathbf{A}\,.\,\mathbf{T}_{123}(v)=(x_0,y_0,z_0,1).\mathbf{T}_1(w_1,q_1).\mathbf{T}_2(w_2,q_2).\mathbf{T}_3(w_3,q_3),\ v\in\langle 0,2\pi\rangle.$$

Three-axial cyclical surface of revolution will be created by translation of the circle $c=(\mathrm{R},\mathrm{r})$ along the curve k so that the circle is located always in the normal plane of the curve k determined by principal normal and by binormal of k in every point $\mathrm{R}\in k$, $c\subset nb$. The circle c is determined by vector function $\mathbf{c}(u)=(0,\cos u,\sin u,1)$, for $u\in\langle 0,2\pi\rangle$ in coordinate system (O,x,y,z) and after then is transformed into coordinate system (R,t,n,b). Vector equation of the three-axial cyclical surface of revolution is

$$\mathbf{P}(u,v)=\mathbf{r}(v)+\mathbf{c}(u).\mathbf{M}(v),\ u\in\langle 0,2\pi\rangle,\ v\in\langle 0,2\pi\rangle,$$

where $\mathbf{M}(v)$ is a regular square matrix of rank 4, with entries as a real functions of one real vaqriable and it represented the transformation of the coordinate system (O,x,y,z) into coordinate system (R,t,n,b) identical with trihedron determined by tangent, principal normal and binormal of the curve k with unit vectors $\mathbf{t}(v)=(t_1,t_2,t_3)$, $\mathbf{n}(v)=(n_1,n_2,n_3)$, $\mathbf{b}(v)=(b_1,b_2,b_3)$:

$$\mathbf{M}(v)=\begin{pmatrix} t_1 & t_2 & t_3 & 0\\ n_1 & n_2 & n_3 & 0\\ b_1 & b_2 & b_3 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

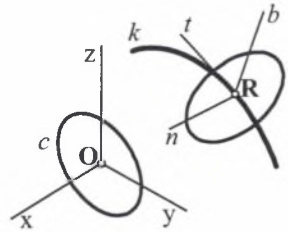


Fig. 1.

$$\mathbf{t}(v)=(t_1,t_2,t_3)=\frac{\frac{d\mathbf{r}}{dv}}{\left|\frac{d\mathbf{r}}{dv}\right|},\ \mathbf{n}(v)=(n_1,n_2,n_3)=\frac{\frac{d^2\mathbf{r}}{dv^2}}{\left|\frac{d^2\mathbf{r}}{dv^2}\right|},$$

$$\mathbf{b}(v)=(b_1,b_2,b_3)=\frac{\mathbf{t}\times\mathbf{n}}{|\mathbf{t}\times\mathbf{n}|},\ v\in\langle 0,2\pi\rangle.$$

The revolutions about the coordinate axes *x*, *y*, *z* are represented by regular square matrices of rank 4 $\mathbf{T}_j(w_i,q_i), j=x,y,z$ for $i=1,2,3$, the transformation matrix of the translation with vector $(a_1,a_2,a_3,1)$ is represented by matrix $\mathbf{T}(a_1,a_2,a_3)$:

$$\mathbf{T}_x(w_i,q_i)=\begin{pmatrix}1&0&0&0\\0&\cos w_i&q_i\sin w_i&0\\0&-q_i\sin w_i&\cos w_i&0\\0&0&0&1\end{pmatrix},\quad \mathbf{T}_y(w_i,q_i)=\begin{pmatrix}\cos w_i&0&q_i\sin w_i&0\\0&1&0&0\\-q_i\sin w_i&0&\cos w_i&0\\0&0&0&1\end{pmatrix},$$

$$\mathbf{T}_z(w_i,q_i)=\begin{pmatrix}\cos w_i&q_i\sin w_i&0&0\\-q_i\sin w_i&\cos w_i&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad \mathbf{T}(a_1,a_2,a_3)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\a_1&a_2&a_3&1\end{pmatrix}$$

If the axis of any revolution is parallel with any of coordinate axes x, y, z, then the revolution is represented by a product of three matrices represented translation and revolution about corresponding coordinate axis. For axample if io || x in the distance $d_i=|{}^io\,x|$ then matrix represented revolution about this axis io is

$$\mathbf{T}_i(v)=\mathbf{T}(0,-d_i,0).\mathbf{T}_x(w_i,q_i).\ \mathbf{T}(0,d_i,0)\ .$$

If the axis of any revolution intersect any coordinate axis angle-wise α, then revolution is represented by product of matrices represented turn by angle α and revolution about corresponding coordinate axis. For example if angle between io and x is α, then matrix represented revolution about this axis io is

$$\mathbf{T}_i(v)=\mathbf{T}_z(\alpha,-1).\mathbf{T}_x(w_i,q_i).\ \mathbf{T}_z(\alpha,+1)\ .$$

3. Transformation matrices represented three revolutions

Transformation matrices represented three revolutions about axes with superpositions e presented in Tab. 1

I. 1. $T_1(w_1,q_1)=T(0,d_1,0).T_z(w_1,q_1).T(0,-d_1,0)$, $T_2(w_2,q_2)=T_z(w_2,q_2)$,

$T_3(w_3,q_3)=T(0,-d_2,0).T_z(w_3,q_3).T(0,d_1,0)$

I. 2. $T_1(w_1,q_1)=T(0,-d,0).T_z(w_1,q_1).T(0,d,0)$,

$T_2(w_2,q_2)=T_z(\alpha,-1).T(0,-d,0).T_z(w_2,q_2).T(0,d,0).T_z(\alpha,1)$,

$T_3(w_3,q_3)=T_z(2\alpha,-1).T(0,-d,0).T_z(w_3,q_3).T(0,d,0).T_z(2\alpha,1)$, $\alpha=\frac{2\pi}{3}$

II. 1. $T_1(w_1,q_1)=T_x(w_1,q_1)$, $T_2(w_2,q_2)=T_y(w_2,q_2)$, $T_3(w_3,q_3)=T_z(w_3,q_3)$

II. 2. $T_1(w_1,q_1)=T_x(w_1,q_1)$, $T_2(w_2,q_2)=T_z(\alpha,-1).T_x(w_2,q_2).T_z(\alpha,1)$,

$T_3(w_3,q_3)=T_z(2\alpha,-1).T_x(w_3,q_3).T_z(2\alpha,1)$, $\alpha=\frac{2\pi}{3}$

II. 3. $T_1(w_1,q_1)=T(-d_1,0,0).T_z(\alpha,+1).T_x(w_1,q_1).T_z(\alpha,-1).T(d_1,0,0)$,

$T_2(w_2,q_2)=T(0,-d_2,0_2).T_x(\beta,+1).T_y(w_2,q_2).T_x(\beta,-1).T(0,d_2,0)$,

$T_3(w_3,q_3)=T(0,0,-d_3).T_y(\gamma,+1).T_x(w_3,q_3).T_y(\gamma,-1).T(0,0,d_3)$,

$\alpha=\arctan\frac{d_2}{d_1}$, $\beta=\arctan\frac{d_3}{d_2}$, $\gamma=\arctan\frac{d_1}{d_3}$.

III. 1. $T_1(w_1,q_1)=T(0,d_1,0).T_x(w_1,q_1).T(0,-d_1,0)$,

$T_2(w_2,q_2)=T(0,0,-d_2).T_y(w_2,q_2).T(0,0,-d_2)$

$T_3(w_3,q_3)=T(-d_3,0,0).T_z(w_3,q_3).T(-d_3,0,0)$

III. 2. $T_1(w_1,q_1)=T(0,0,d_1).T_x(w_1,q_1).T(0,0,-d_1)$,

$T_2(w_2,q_2)=T_z(\alpha,-1).T_x(w_2,q_2).T_z(\alpha,1)$,

$T_3(w_3,q_3)=T_z(2\alpha,-1).T(0,0,-d_2).T_z(w_3,q_3).T(0,0,d_2).T_z(2\alpha,1)$, $\alpha=\frac{2\pi}{3}$

III. 3. $T_i(w_i,q_i)=T_z((i-1)\gamma,-1).T(-d_1,0,0).T_z(\alpha,+1).T_y(\beta,+1).T_x(w_i,q_i).$
$.T_y(\beta,-1).T_z(\alpha,-1).T(d_1,0,0).T_z((i-1)\gamma,+1),\ i=1,2,3$,

$\alpha=\frac{\pi}{6}$, $\beta=\mathrm{arctg}\frac{d_2}{2d_1\cos\alpha}$, $\gamma=\frac{2\pi}{3}$

IV. 1. $T_1(w_1,q_1)=T(0,d_1,0).T_z(w_1,q_1).T(0,-d_1,0)$, $T_2(w_2,q_2)=T_y(w_2,q_2)$,

$T_3(w_3,q_3)=T_z(w_3,q_3)$

IV. 2. $T_1(w_1,q_1)=T(0,d_1,0).T_z(w_1,q_1).T(0,-d_1,0)$,

$T_2(w_2,q_2)=T(-d_2,0,0).T_y(w_2,q_2).T(d_2,0,0)$, $T_3(w_3,q_3)=T_z(w_3,q_3)$

IV. 3. $T_1(w_1,q_1)=T_x(w_1,q_1)$,

$T_2(w_2,q_2)=T(-d_1,0,0).T_y(\alpha,-1).T_x(w_2,q_2).T_y(\alpha,+1).T(d_1,0,0)$,

$T_3(w_3,q_3)=T(0,0,-d_2).T_y(w_3,q_3).T(0,0,d_2)$

IV. 4. $\mathbf{T}_1(w_1,q_1)=\mathbf{T}(0,d_1,0).\mathbf{T}_z(w_1,q_1).\mathbf{T}(0,-d_1,0)$,

$\mathbf{T}_2(w_2,q_2)=\mathbf{T}(-d_1,0,-d_2).\mathbf{T}_y(w_2,q_2).\mathbf{T}(d_1,0,d_2)$, $\mathbf{T}_3(w_3,q_3)=\mathbf{T}_z(w_3,q_3)$

4. Display of the three-axial cyclical surfaces of revolution

Design of the cyclical three-axial surface of revolution changes in dependence on superposition of axes of revolutions, in dependence on position of point A with respect to these axes of revolutions and also in dependence on parameters m_i, q_i.

We will describe the creation of the cyclical surface with axes in superposition III. 3. and its forming by modification of its parameters. Axes of revolutions are skew each other, whereas they are diagonals of side faces of right triangular prism with base face in the plane *xy*, with center of base face in origin and one vertex is in the point (d_1, 0, 0, 1), d_2 is hight of the prism (Fig. 2). The revolutions of one point about axes ${}^1o$, ${}^2o$, ${}^3o$ are displayed in figure 3. In figure 4 there is displayed the curve *k* created by revolutionary movement of the point A composit from these three revolutions, where all parameters m_i, q_i = 1.

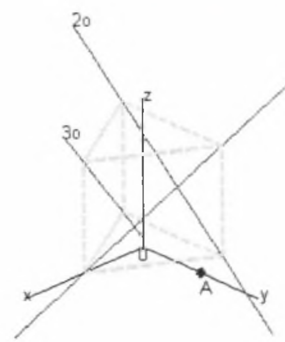


Fig. 2

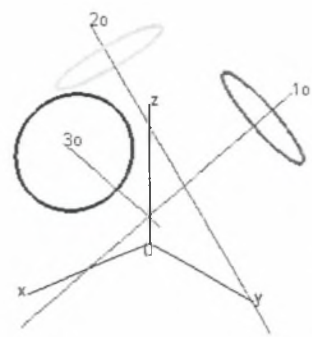


Fig. 3

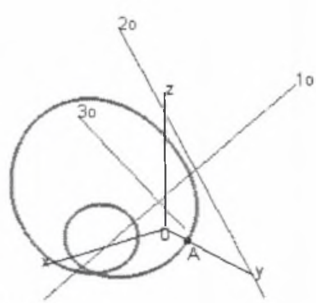


Fig. 4

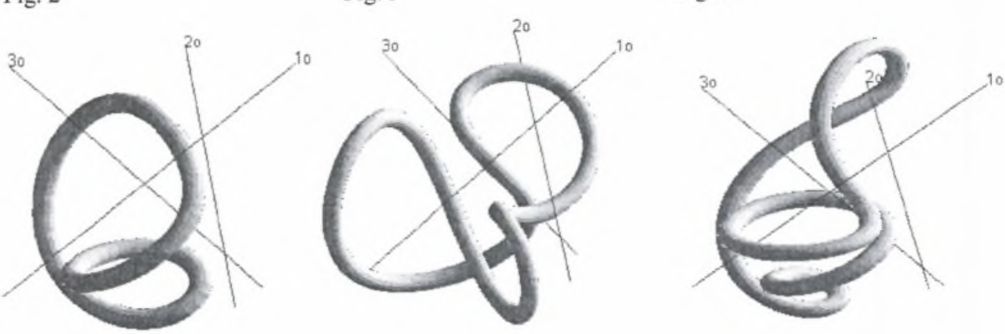


Fig. 5

In figure 5 there are displayed three cyclical surfaces with parameters m_1, m_2, m_3 equal subsequently from left (1,1,1), (1,3,1), (3,1,1) and q_1, q_2, q_3 are all equal to +1 in addition to second surface, where $q_2 = -1$. If all parameters m_i are identical, surface has identical form. These surfaces have none nodal point.

In figure 6 there are displayed two cyclical surfaces with axes in superposition III. 1. Axes of revolutions are skew each other and they are parallel with coordinate axes. Parameters m_i are subsequently from left (1,1,3) and (1,3,1). Also there is displayed the surface with axes in superposition III. 2 whih have common axis of skew axes of revolutions and its parameters m_i are (2,1,2) and parameters q_i are (–1,1,–1).

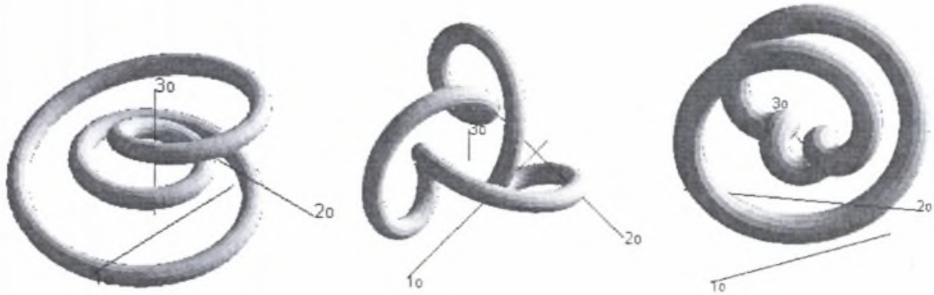


Fig. 6

In the case if axes of revolutions are intersect with one common point (II. 1., II. 2.), the curve *k* is situated always on the sphere with centre in this common point and radius is equal to the distance the point A from it.

In figure 7 there are displayed the curve *k*, point A and axes of revolutions situated in one plane *xy* with common point in origin O, where parameters m_i are consequently equal (1,1,1) and (4,2,1), parameters q_i are all equal +1. The curve *k* is situated on the sphere with centre in origin O and with radius $r' = |AO|$. There is displayed the cyclical surface created by translation of the circle along the curve *k*, situated always in normal plane of curve *k*.

Fig. 7

In figure 8 there is displayed the curve k, point A and axes of revolutions identical with coordinate axes with common point in origin O, where parameters m_i consequently equal (2,1,2), parameters q_i are +1. There are displayed also two cyclical surfaces, where the first has parameter $q_2 = +1$ and the second one $q_2 = -1$.

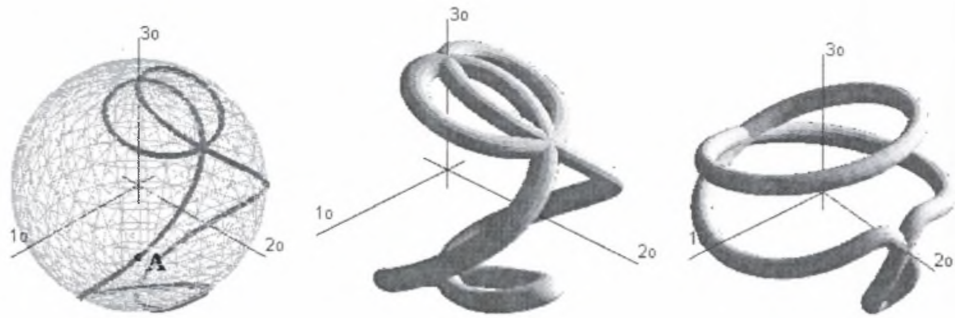


Fig. 8

In figure 9 there are displayed four cyclical surfaces with axes in superposition II. 3., where intersection axes has three common points create equiangular triangel. The parameters of the first surface are (1,1,3), of the second are (3,–1,1), the third are (–1,4,–1), the fourth are (2,1,2), where sign minus represent corresponding parameter $q_i = -1$.

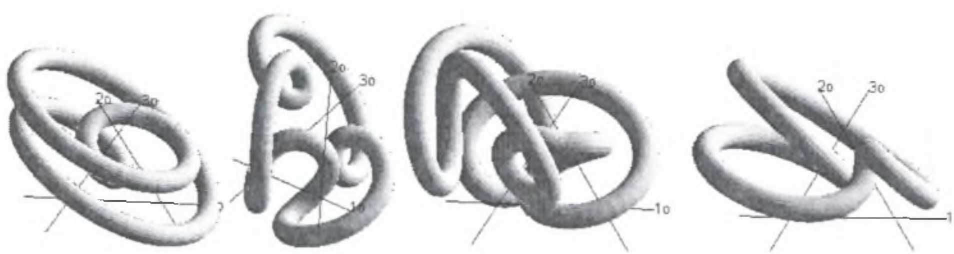


Fig. 9

In Fig. 10 there is displayed cyclical surface with axes in superposition I. 1, where parallel axes of revolutions are situated in one plane yz and parameters m_i are (1,3,1), surface with axes in superposition I. 2, where parallel axes of revolutions are not situated in one plane and their intersects with plane xy create equiangulag triangel, parameters m_i are (1,4,1), $q_3 = -1$ and surface with axes in superposition I. 3 where parallel axes of revolutions are situated in one plane, but ${}^2o = {}^4o$, so the surface is created by four revolutions with m_i (1,1,3,1).

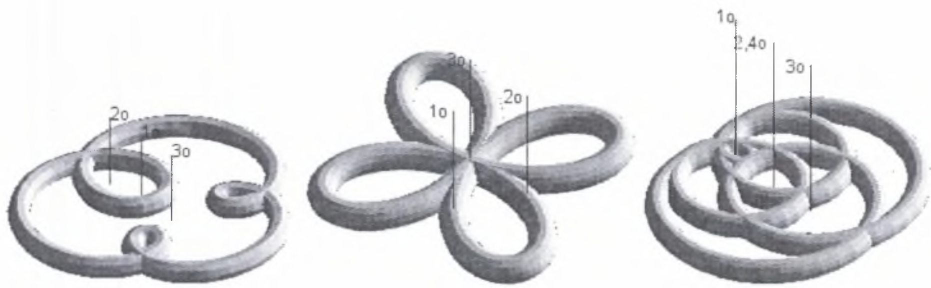


Fig. 10

All preceding surfaces have all axes of revolutions parallel, or intersect or skew to each other. Surfaces with axes in superpositions IV. have different relativ superpositions of axes of revolutions, as we can see in Tab. 1. All four combinations of relativ superpositions of axes are applird on the surfaces displayed in Fig. 11. Parameters of these surfaces are (1,–4,1), (1,4,1), (1,1,3), (1,6,1).

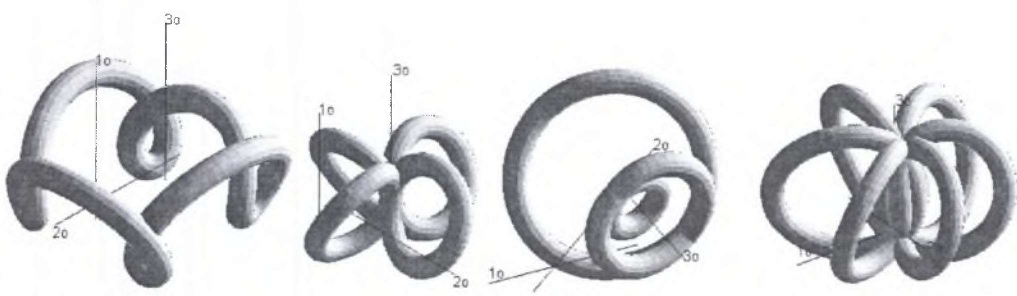


Fig. 11

5. Conclusions

There exist infinitely many different forms of three-axial cyclical surfaces of revolution and their forms change in dependence on superposition of axes of revolutions, in dependence on position of point A with respect to these axes of revolutions and also in dependence on parameters m_i, q_i. It was illustrate in previous text. It will be possible to create various interesting and beautiful forms of these surfaces which can be applied in design practice as constructive or ornamental structural components.

Bibliography

1. Velichová D.: Multy-Axial Revolutions of Euler Type. In: Proceedings of 5th International Conference on Applied Mathematics APLIMAT 2006, SjF STU, Bratislava 2006, p. 191-197.

2. Olejníková T.: Composed Cyclical Surfaces. In: Transactions of the Universities of Košice, No. 3, 2007, p. 54-60.

3. Olejníková T.: Cyclical Surfaces Created by Conical Helix. In: KoG, Scientific and Professional Journal of Croatian Society for Geometry and Graphics, No. 11, 2007, p. 33-38.

Omówienie

Artykuł przedstawia specjalne klasy powierzchni – trzyosiowe, cykliczne powierzchnie obrotowe. Cztery specjalne podklasy powierzchni są określone z uwagi na superpozycje trzech osi obrotu. Specjalne pozycje osi są wybrane w osiach współrzędnych lub liniach równoległych do osi współrzędnych, lub w pozycji ogólnej. Dla każdej transformacji tworzone są macierze obrotu. Niektóre trajektorie ruchu punktu, a także powierzchnie cykliczne tworzone przez przełożenie okręgu wzdłuż tych trajektorii są wizualizowane z ich równań wektorów.