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CLASSIFICATION OF SUBGROUPS AND FACTOR GROUPS OF THE FREE NILPOTENT GROUP OF CLASS 2 WITH TWO GENERATORS

Summary. In the paper a description of all subgroups of a normal discrete Heisenberg's group N_2 is given. Moreover centres of factor groups are described.

Introduction: The purpose of this paper is to determine subgroups and factor groups of the free nilpotent group N_2 of class 2 with two generators x and y. The basic notions are contained in the monography [2].

In section I classification of subgroups of N_2 is presented as well as classification of normal subgroups.

In section II presentation of N_2 is assigned and nest taking advantage of the theorem 2.1 [2], presentations of all factor groups of the group N_2 and theirs centers are given.

Definition

The free nilpotent group N_2 of class 2 with two generators x and y is the set

$$\left\{ x^{\alpha} y^{\beta} z^{\gamma} : \alpha, \beta, \gamma, \in \mathbb{Z}, \ z = [x, y] \right\}$$

in which the group operation is defined in the following way:

$$x^{\alpha} y^{\beta} z^{\gamma} \cdot x^{\alpha'} y^{\beta'} z^{\gamma'} := x^{\alpha+\alpha'} y^{\beta+\beta'} z^{\gamma+\gamma'+\alpha'\beta}$$

Here $[x, y] = x^{-1}y^{-1}xy$.

I. CLASSIFICATION OF SUBGROUPS OF THE GROUP N_2

From lemma 1.9 of [1] is follows that each subgroup of N_2 is finitely generated.

Lemma 1

Each subgroup of N_2 is generated by at most three elements:

$$z^{\gamma_0}, \quad x^{\alpha_1} z^{\gamma_1}, \quad x^{\alpha_2} y^{\beta_2}, \quad \gamma_0 \neq 0, \quad \alpha_1 \neq 0, \quad \beta_2 \neq 0.$$

Proof: There is a central sequence (see theorem 1.8 [1])

$$N_2 = G_0 \geq G_1 \geq G_2 \geq G_3 = 1$$

such that G_i/G_i are cyclic and $G_1 = \{\Gamma_2, x\}$, $G_2 = \Gamma_2$, $\Gamma_2 = [N_2, N_2]$.
Let $H \leq N_2$, than the sequence

$$H = H_0 \geq H_1 \geq H_2 \geq H_3 = 1$$

is a central sequence of the subgroup H where $H_i = H \cap G_i$, $i = 0,1,2,3$. The factor groups H_{i-1}/H_i are also cyclic, we can choose an element $x_i \in H_{i-1}$ such that $H_{i-1} = \{H_i, x_i\}$. Since $H_1 = H \cap G_1$, $G_1 = \{x^{\alpha_1} z^{\gamma_1} : \alpha_1, \gamma_1 \in Z\}$ and $H_2 = H \cap \Gamma_2$, $\Gamma_2 = \{z^{\gamma} : \gamma \in Z\}$, and so there exists an element $z^{\gamma_0} \in H_2$, $\gamma_0 \neq 0$, such that $H_2 = \{H_3, z^{\gamma_0}\}$. There also exists an element $x^{\alpha_1} z^{\gamma_1} \in H_1$, $\alpha_1 \neq 0$, such that $H_1 = \{H_2, x^{\alpha_1} z^{\gamma_1}\} = \{H_3, z^{\gamma_0}, x^{\alpha_1} z^{\gamma_1}\}$ as well as an element $x^{\alpha_2} y^{\beta_2} z^{\gamma_2} \in H$, $\beta_2 \neq 0$, such that $H = \{H_1, x^{\alpha_2} y^{\beta_2} z^{\gamma_2}\}$. Finally $H = gp\{z^{\gamma_0}, x^{\alpha_1} z^{\gamma_1}, x^{\alpha_2} y^{\beta_2} z^{\gamma_2}\}$ and lemma 1 follows.

Among all subgroups of the group N_2 we can distinguish seven separate classes $\mathcal{H}_i$, $1 \leq i \leq 7$.

1.1. Class $\mathcal{H}_1$

Class $\mathcal{H}_1$ consists of subgroups of the form $H = gp(z^{\gamma_0})$, $\gamma_0 \in Z$, $\gamma_0 \neq 0$. Let us notice, that subgroups belonging to class $\mathcal{H}_1$ are subgroups of the centre of N_2 and therefore they are normal subgroups of N_2.

1.2. Class $\mathcal{H}_2$

Class $\mathcal{H}_2$ consists of subgroups $H = gp(x^{\alpha_1} z^{\gamma_1})$, where $\alpha_1, \gamma_1 \in Z$, $\alpha_1 \neq 0$. All elements of $\mathcal{H}_2$ are cyclic. It can be easily noticed that none of the groups of class $\mathcal{H}_2$ is normal. Indeed, since

$$y^{-1} \circ x^{\alpha_1} z^{\gamma_1} \circ y = x^{\alpha_1} z^{\gamma_1 - \alpha_1} \in H = gp(x^{\alpha_1} z^{\gamma_1})$$

if and only if $\gamma_1 - \alpha_1 = \gamma_1$, we see that $\alpha_1 = 0$, contrary to our assumption on $\mathcal{H}_2$.

1.3. Class $\mathcal{H}_3$

The class $\mathcal{H}_3$ is defined as a set of all groups H of the form $H = gp(x^{\alpha_1} y^{\beta_1} z^{\gamma_1})$, where $\alpha_1, \beta_1, \gamma_1 \in \mathbb{Z}$, $\beta_1 \neq 0$.

Let us consider the following product

$$x^{-1} \circ x^{\alpha_1} y^{\beta_1} z^{\gamma_1} \circ x = x^{\alpha_1} y^{\beta_1} z^{\gamma_1 + \beta_1},$$

it belongs to the group $H = gp(x^{\alpha_1} y^{\beta_1} z^{\gamma_1})$ if and only if $\gamma_1 + \beta_1 = \gamma_1$. It follows that $\beta_1 = 0$, which is impossible because of the definition of the class $\mathcal{H}_3$, and thus none of element of $\mathcal{H}_3$ is normal.

1.4. Class $\mathcal{H}_4$

Class $\mathcal{H}_4$ consists groups of the form $H = gp(z^{\gamma_0}, x^{\alpha_1} z^{\gamma_1})$, $\gamma_0 \neq 0$, $\alpha_1 \neq 0$. Each of the element of class $\mathcal{H}_4$ is determined by the triple $(\gamma_0, \alpha_1, \gamma_1)$. It turned out that this correspondence is not 1-1. We have

Lemma 1.4.1

Let the groups $H = gp(z^{\gamma_0}, x^{\alpha_1} z^{\gamma_1})$, $H_1 = gp(z^{\bar{\gamma}_0}, x^{\bar{\alpha}_1} z^{\bar{\gamma}_1})$ belong to class $\mathcal{H}_4$. Then $H = H_1$ if and only if one of the following conditions is satisfied:

(I) $\bar{\gamma}_0 = \gamma_0$, $\bar{\alpha}_1 = \alpha_1$, $\bar{\gamma}_1 = \gamma_1 \pmod{\gamma_0}$,

(II) $\bar{\gamma}_0 = \gamma_0$, $\bar{\alpha}_1 = -\alpha_1$, $\bar{\gamma}_1 = (-\gamma_1) \pmod{\gamma_0}$,

(III) $\bar{\gamma}_0 = -\gamma_0$, $\bar{\alpha}_1 = \alpha_1$, $\bar{\gamma}_1 = \gamma_1 \pmod{\gamma_0}$,

(IV) $\bar{\gamma}_0 = -\gamma_0$, $\bar{\alpha}_1 = -\alpha_1$, $\bar{\gamma}_1 = (-\gamma_1) \pmod{\gamma_0}$.

Proof. If $H = H_1$, then that generators of one of them can be expressed by generators from the other, and conversely. Therefore there exist integers n, m such that

$$z^{\gamma_0} = x^{n\bar{\alpha}_1} z^{n\bar{\gamma}_1 + m\bar{\gamma}_0} \tag{1.4.1}$$

and integers n_1, m_1 such that

$$x^{\alpha_1} z^{\gamma_1} = x^{n_1\bar{\alpha}_1} z^{n_1\bar{\gamma}_1 + m_1\bar{\gamma}_0}. \tag{1.4.2}$$

Hence $n = 0$ and

$$\begin{aligned} \gamma_0 &= m\bar{\gamma}_0 \\ \alpha_1 &= n_1\bar{\alpha}_1 \\ \gamma_1 &= n_1\bar{\gamma}_1 + m_1\bar{\gamma}_0. \end{aligned} \tag{1.4.3}$$

On the other hand there exist numbers $m_2, n_3, m_3 \in \mathbf{Z}$ where

$$\begin{aligned} \bar{\gamma}_0 &= m_2\gamma_0 \\ \bar{\alpha}_1 &= n_3\alpha_1 \\ \bar{\gamma}_1 &= n_3\gamma_1 + m_3\gamma_0. \end{aligned} \tag{1.4.4}$$

It follows from (1.4.3), (1.4.4) that

$$\gamma_0 = mm_2\gamma_0$$

$$1 = m\, m_2$$

hence $m = m_2 = 1$ or $m = m_2 = -1$.
From the second equations of the simultaneous equations following that

$$\alpha_1 = n_1\, n_3\, \alpha_1$$

$$1 = n_1 n_3$$

hence $n_1 = n_3 = 1$ or $n_1 = n_3 = -1$.

We infer from our previous considerations that one of the following four cases must be satisfied:

Case I ($m = m_2 = 1$, $n_1 = n_3 = 1$).

In this case we get

$$\bar{\gamma}_0 = \gamma_0, \; \bar{\alpha}_1 = \alpha_1$$

and

$$\begin{aligned} \gamma_1 &= \bar{\gamma}_1 + m_1\gamma_0 \\ \bar{\gamma}_1 &= \gamma_1 + m_3\gamma_0. \end{aligned} \tag{1.4.5}$$

Hence $\bar{\gamma}_1 = \gamma_1 \pmod{\gamma_0}$.

Case II ($m = m_2 = 1$, $n_1 = n_3 = -1$).

It follows from the equations (1.4.4) that $\bar{\gamma}_0 = \gamma_0$, $\alpha_1 = -\bar{\alpha}_1$. Hence

$$\begin{aligned} \gamma_1 &= -\bar{\gamma}_1 + m_1\gamma_0 \\ \bar{\gamma}_1 &= -\gamma_1 + m_3\gamma_0, \end{aligned} \tag{1.4.6}$$

which is equivalent to the relation $\bar{\gamma}_1 = (-\gamma_1) \pmod{\gamma_0}$.

Case III ($m = m_2 = -1$, $n_1 = n_3 = 1$).

For this case $\bar{\gamma}_0 = -\gamma_0$, $\bar{\alpha}_1 = \alpha_1$ and

$$\begin{aligned} \gamma_1 &= \bar{\gamma}_1 - m_1\gamma_0 \\ \bar{\gamma}_1 &= \gamma_1 + m_3\gamma_0, \end{aligned} \tag{1.4.7}$$

which is equivalent to the relation $\bar{\gamma}_1 = (\gamma_1) \pmod{\gamma_0}$.

Case IV ($m = m_2 = -1$, $n_1 = n_3 = -1$).

In the last case we obtain $\bar{\gamma}_0 = -\gamma_0$, $\bar{\alpha}_1 = -\alpha_1$ and

$$\begin{aligned} \gamma_1 &= -\bar{\gamma}_1 - m_1\gamma_0 \\ \bar{\gamma}_1 &= -\gamma_1 + m_3\gamma_0, \end{aligned} \tag{1.4.8}$$

which is equivalent to the relation $\bar{\gamma}_1 = (-\gamma_1) \pmod{\gamma_0}$.

On the other hand if the triples $(\gamma_0, \alpha_1, \gamma_1)$, $(\bar{\gamma}_0, \bar{\alpha}_1, \bar{\gamma}_1)$ satisfy one of the conditions from (I) to (IV), then the groups H and H_1 are identical. Indeed, we have, for example, if $\bar{\gamma}_0 = \gamma_0$, $\bar{\alpha}_1 = \alpha_1$ and $\bar{\gamma}_1 = \gamma_1 \pmod{\gamma_0}$, then

$$z^{\gamma_0} = x^{0\bar{\alpha}_1} z^{0\bar{\gamma}_1 + \bar{\gamma}_0}$$

$$x^{\alpha_1} z^{\gamma_1} = x^{\bar{\alpha}_1} z^{\bar{\gamma}_1 - m_1\bar{\gamma}_0}$$

and

$$z^{\gamma_0} = x^{0\alpha_1} z^{0\gamma_1 + \gamma_0}$$

$$x^{\bar{\alpha}_1} z^{\bar{\gamma}_1} = x^{\alpha_1} z^{\gamma_1 + m_1\gamma_0}$$

and this means that $H = gp(z^{\gamma_0}, x^{\alpha_1} z^{\gamma_1}) = gp(z^{\bar{\gamma}_0}, x^{\bar{\alpha}_1} z^{\bar{\gamma}_1}) = H_1$. Other cases are analogous.

Theorem 1.4.1

The subgroup $H = gp(z^{\gamma_0}, x^{\alpha_1} z^{\gamma_1}) \in \mathcal{H}_4$ is a normal if and only if $\gamma_0 | \alpha_1$.

Proof. It is clear that H is normal if and only if

$$y^{-1} \circ x^{\alpha_1} z^{\gamma_1} \circ y \in H$$

i.e.

$$x^{\alpha_1} z^{\gamma_1 - \alpha_1} = (x^{\alpha_1} z^{\gamma_1})^n \cdot (z^{\gamma_0})^m = x^{n\alpha_1} z^{n\gamma_1 + m\gamma_0}$$

for some integers n, m, and theorem 1.4.1 follows.

Corollary 1.4.1

All normal subgroups of N_2 to from the class $\mathcal{H}_4$ have the form $H = gp(z^{\gamma_0}, x^{l\gamma_0} z^r)$, $l, \gamma_0 \in \mathbf{N}$, $0 \leqslant r \leqslant \gamma_0 - 1$.

1.5. Class $\mathcal{H}_5$

The class $\mathcal{H}_5$ consists, by definition, of all groups

$$H = gp(z^{\gamma_0}, x^{\alpha_1} y^{\beta_1} z^{\gamma_1}) \quad \text{with} \quad \gamma_0 \neq 0, \quad \beta_1 \neq 0.$$

Lemma 1.5.1

Let groups $H = gp(z^{\gamma_0}, x^{\alpha_1} y^{\beta_1} z^{\gamma_1})$, $H_1 = gp(z^{\bar{\gamma}_0}, x^{\bar{\alpha}_1} y^{\bar{\beta}_1} z^{\bar{\gamma}_1}) \in \mathcal{H}_5$, then $H = H_1$ if and only if one of the following conditions is satisfied:

(I) $\bar{\gamma}_0 = \gamma_0$, $\bar{\alpha}_1 = \alpha_1, \bar{\beta}_1 = \beta_1$, $\bar{\gamma}_1 = \gamma_1 \pmod{\gamma_0}$

(II) $\bar{\gamma}_0 = \gamma_0$, $\bar{\alpha}_1 = -\alpha_1, \bar{\beta}_1 = -\beta_1, \bar{\gamma}_1 = (-\gamma_1 + \alpha_1\beta_1) \pmod{\gamma_0}$

(III) $\bar{\gamma}_0 = -\gamma_0$, $\bar{\alpha}_1 = \alpha_1, \bar{\beta}_1 = \beta_1, \bar{\gamma}_1 = \gamma_1 \pmod{\gamma_0}$

(IV) $\bar{\gamma}_0 = -\gamma_0$, $\bar{\alpha}_1 = -\alpha_1, \bar{\beta}_1 = -\beta_1, \bar{\gamma}_1 = (-\gamma_1 + \alpha_1\beta_1) \pmod{\gamma_0}$.

Proof. Suppose that $H = H_1$. Then generators of the group H can be expressed as a word in generators of the group H_1, and conversely. There exist numbers $m, p \in \mathbf{N}$; $n, q \in \mathbf{Z}$ such that

$$z^{\gamma_0} = x^{m\bar{\alpha}_1} y^{m\bar{\beta}_1} z^{n\bar{\gamma}_0 + m\bar{\gamma}_1 + \binom{m}{1}\alpha_1\bar{\beta}_1} \tag{1.5.1}$$

or

$$z^{\gamma_0} = x^{-p\bar{\alpha}_1} y^{-p\bar{\beta}_1} z^{q\bar{\gamma}_0 - p\bar{\gamma}_1 + \binom{p+1}{2}\bar{\alpha}_1\bar{\beta}_1} \tag{1.5.2}$$

and numbers $k, s \in \mathbf{Z}$; $l, t \in \mathbf{N}$ such that

$$x^{\alpha_1} y^{\beta_1} z^{\gamma_1} = x^{l\bar{\alpha}_1} y^{l\bar{\beta}_1} z^{k\bar{\gamma}_0 + l\bar{\gamma}_1 + \binom{l}{2}\bar{\alpha}_1\bar{\beta}_1} \tag{1.5.3}$$

or

$$x^{\alpha_1} y^{\beta_1} z^{\gamma_1} = x^{-t\bar{\alpha}_1} y^{-t\bar{\beta}_1} z^{s\bar{\gamma}_0 - t\bar{\gamma}_1 + \binom{t+1}{2}\bar{\alpha}_1\bar{\beta}_1}. \tag{1.5.4}$$

It follows from the equalities (1.5.1) and (1.5.2) that $m = p = 0$ and $n = q$. Taking into account the relations (1.5.3), (1.5.4) we get the following equalities

$$\begin{aligned} \alpha_1 &= l\bar{\alpha}_1 \\ \beta_1 &= l\bar{\beta}_1 \\ \gamma_1 &= k\bar{\gamma}_0 + l\bar{\gamma}_1 + \binom{l}{2}\bar{\alpha}_1\bar{\beta}_1 \end{aligned} \tag{1.5.5}$$

or

$$\begin{aligned} \alpha_1 &= -t\bar{\alpha}_1 \\ \beta_1 &= -t\bar{\beta}_1 \\ \gamma_1 &= s\bar{\gamma}_0 - t\bar{\gamma}_1 + \binom{t+1}{2}\bar{\alpha}_1\bar{\beta}_1. \end{aligned} \tag{1.5.6}$$

Similarly there exist numbers $m_1, p_1, l_1, t_1 \in \mathbf{N}$; $n_1, q_1, k_1, s_1 \in \mathbf{Z}$ such that

$$z^{\gamma_0} = x^{m_1\alpha_1} y^{m_1\beta_1} z^{n_1\gamma_0 + m_1\gamma_1 + \binom{m_1}{2}\alpha_1\beta_1} \tag{1.5.7}$$

or

$$z^{\bar{\gamma}_0} = x^{-p_1\alpha_1} y^{-p_1\beta_1} z^{q_1\gamma_0 - p_1\gamma_1 + \binom{p_1+1}{2}\alpha_1\beta_1} \tag{1.5.8}$$

and

$$x^{\bar{\alpha}_1} y^{\bar{\beta}_1} z^{\bar{\gamma}_1} = x^{l_1\alpha_1} y^{l_1\beta_1} z^{k_1\gamma_0 + l_1\gamma_1 + \binom{l_1}{2}\alpha_1\beta_1} \tag{1.5.9}$$

or

$$x^{\bar{\alpha}_1} y^{\bar{\beta}_1} z^{\bar{\gamma}_1} = x^{-t_1\alpha_1} y^{-t_1\beta_1} z^{s\gamma_0 - t_1\gamma_1 + \binom{t_1+1}{2}\alpha_1\beta_1}. \tag{1.5.10}$$

Hence we get the following equalities

$$m_1 = p_1 = 0, \quad n_1 = q_1 \tag{1.5.11}$$

and

$$\begin{aligned} \bar{\alpha}_1 &= l_1\alpha_1 \\ \bar{\beta}_1 &= l_1\beta_1 \\ \bar{\gamma}_1 &= k_1\gamma_0 + l_1\gamma_1 + \binom{l_1}{2}\alpha_1\beta_1 \end{aligned} \tag{1.5.12}$$

or

$$\begin{aligned} \bar{\alpha}_1 &= -t_1\alpha_1 \\ \bar{\beta}_1 &= -t_1\beta_1 \\ \bar{\gamma}_1 &= s_1\gamma_0 - t_1\gamma_1 + \binom{t_1+1}{2}\alpha_1\beta_1 \end{aligned} \tag{1.5.13}$$

and thus it follows from the equalities $z^{\gamma_0} = z^{n\bar{\gamma}_0}$ and $z^{\bar{\gamma}_0} = z^{n\gamma_0}$ that $nn_1 = 1$, $n = n_1 = 1$ or $n = n_1 = -1$.

Let us consider two cases:

Case I. The equalities (1.5.5), (1.5.12) are satisfied and from these equalities we infer that $l = l_1 = 1$ and $\gamma_1 = k\bar{\gamma}_0 + \bar{\gamma}_1$ and $\bar{\gamma}_1 = k_1\gamma_0 + \gamma_1$. If $n = n_1 = 1$, then $\gamma_0 = \bar{\gamma}_0$ and $\bar{\gamma}_1 = \gamma_1 \pmod{\gamma_0}$, if $n = n_1 = -1$, then $\bar{\gamma}_0 = -\gamma_0$ and $\bar{\gamma}_1 = \gamma_1 \pmod{\gamma_0}$. Hence we can state the condition (I) or (III) is fulfilled.

Case II. The equalities (1.5.6), (1.5.13) are satisfied, it follows that $t = t_1 = 1$ and $\gamma_1 = s\bar{\gamma}_0 - \bar{\gamma}_1 + \alpha_1\beta_1$ and $\bar{\gamma}_1 = s_1\gamma_0 - \gamma_1 + \alpha_1\beta_1$. If $n = n_1 = 1$, then $\bar{\gamma}_0 = \gamma_0$ and $\bar{\gamma}_1 = (-\gamma_1 + \alpha_1\beta_1) \pmod{\gamma_0}$. As may be easily seen that $s = s_1$. In case when $n = n_1 = -1$, then $\bar{\gamma}_0 = -\gamma_0$ and $\bar{\gamma}_1 = (-\gamma_1 + \alpha_1\beta_1) \pmod{\gamma_0}$. Hence we see that the condition (II) or (IV) are satisfied.

Other two cases, as may be easily verified, lead to a contradiction. It is clear that if one of condition (I) - (IV) is satisfied, then $H_1 = H$ and lemma 1.5.1 follows.

Theorem 1.5.1

The subgroup $H = gp(z^{\gamma_0}, x^{\alpha_1} y^{\beta_1} z^{\gamma_1}) \in \mathcal{H}_5$ is normal in N_2 if and only if $\gamma_0 \mid \alpha_1$ and $\gamma_0 \mid \beta_1$.

Proof. Let us observe that $H = gp(z^{\gamma_0}, x^{\alpha_1} y^{\beta_1} z^{\gamma_1})$ is normal in N_2 if and only if $x^{-1} \circ x^{\alpha_1} y^{\beta_1} z^{\gamma_1} \circ x \in H$ and $y^{-1} \circ x^{\alpha_1} y^{\beta_1} z^{\gamma_1} \circ y \in H$. This is equivalent that there exist integers n, m such that

$$x^{-1} \circ x^{\alpha_1} y^{\beta_1} z^{\gamma_1} \circ x = x^{\alpha_1} y^{\beta_1} z^{n\gamma_0 + \gamma_1}$$

and

$$y^{-1} \circ x^{\alpha_1} y^{\beta_1} z^{\gamma_1} \circ y = x^{\alpha_1} y^{\beta_1} z^{m\gamma_0 + \gamma_1}$$

which, in turn, is equivalent to the statement $\gamma_1 + \beta_1 = n\gamma_0 + \gamma_1$ and $\gamma_1 - \alpha_1 = m\gamma_0 + \gamma_1$. Thus $\beta_1 = n\gamma_0$ and $\alpha_1 = -m\gamma_0$ and theorem 1.5.1 follows.

Corollary 1.5.1

The group $H = gp(z^{\gamma_0}, x^{\alpha_1} y^{\beta_1} z^{r}) \in \mathcal{H}_5$ is normal in N_2 if and only if $\alpha_1 = l\gamma_0$, $\beta_1 = k\gamma_0$; $l \in \mathbb{N}$, $k \in \mathbb{Z}$, $0 \leqslant r \leqslant \gamma_0 - 1$, $\gamma_0 \in \mathbb{N}$.

1.6. Class $\mathcal{H}_6$

Let us define the class $\mathcal{H}_6$ as the set of all groups

$$H = gp(x^{\alpha_1} z^{\gamma_1}, x^{\alpha_2} y^{\beta_2} z^{\gamma_2}) \quad \text{where } \alpha_1 \neq 0,\ \beta_2 \neq 0.$$

Lemma 1.6.1

The group $H = gp(x^{\alpha_1} z^{\vartheta_1}, x^{\alpha_2} y^{\beta_2} z^{\vartheta_2}) \in \mathcal{H}_6$ is equal to the group N_2 if and only if $\alpha_1 = \pm 1$ and $\beta_2 = \pm 1$.

Proof. Let $H = gp(x^{\alpha_1} z^{\vartheta_1}, x^{\alpha_2} y^{\beta_2} z^{\vartheta_2}) \in \mathcal{H}_6$, $H = N_2$. There exist numbers $n, k \in \mathbf{Z}$, $m \in \mathbf{N}$ such that

$$x = x^{n\alpha_1 + m\alpha_2} y^{m\beta_2} z^{m\vartheta_2 + \binom{m}{2}\alpha_2\beta_2 + n\vartheta_1 - k\alpha_1\beta_2}$$

or

$$x = x^{n\alpha_1 - m\alpha_2} y^{-m\beta_2} z^{-m\vartheta_2 + \binom{m+1}{2}\alpha_2\beta_2 + n\vartheta_1 - k\alpha_1\beta_2}.$$

Hence if follows that in both cases $m = 0$ and $n\alpha_1 = 1$. Thus $n = \alpha_1 = 1$ or $n = \alpha_1 = -1$. Similarly, there also exist numbers $n_1, k_1 \in \mathbf{Z}$, $m_1 \in \mathbf{N}$ such that

$$y = x^{n_1\alpha_1 + m_1\alpha_2} y^{m_1\beta_2} z^{n_1\vartheta_1 + m_1\vartheta_2 + \binom{m_1}{2}\alpha_2\beta_2 - k_1\alpha_1\beta_2}$$

or

$$y = x^{n_1\alpha_1 - m_1\alpha_2} y^{-m_1\beta_2} z^{n_1\vartheta_1 - m_1\vartheta_2 + \binom{m_1+1}{2}\alpha_2\beta_2 - k_1\alpha_1\beta_2}.$$

Hence $m_1\beta_2 = 1$ or $-m_1\beta_2 = 1$ and consequently $m_1 = \beta_2 = 1$ or $m_1 = \beta_2 = -1$ or $m_1 = 1$ and $\beta_2 = -1$ or $m_1 = -1$ and $\beta_2 = 1$.

Let us consider the following cases:

Case I ($\alpha_1 = \beta_2 = 1$). Then $H = gp(x z^{\vartheta_1}, x^{\alpha_2} y z^{\vartheta_2})$ and the following equalities are true

$$x = x z^{\vartheta_1} \circ z^{-\vartheta_1}$$

$$y = (x z^{\vartheta_1})^{-\alpha_2} \quad x^{\alpha_2} y z^{\vartheta_2} \circ z^{\alpha_2\vartheta_1 - \vartheta_2}.$$

Case II ($\alpha_1 = \beta_2 = -1$). Then $H = gp(x^{-1} z^{\vartheta_1}, x^{\alpha_2} y^{-1} z^{\vartheta_2})$ and the following equalities are true

$$x = (x^{-1} z^{\vartheta_1})^{-1} \quad z^{\vartheta_1}$$

$$y = (x^{-1} z^{\gamma_1})^{-\alpha_2} \circ (x^{\alpha_2} y^{-1} z^{\gamma_2})^{-1} \circ z^{\alpha_2 \gamma_1 + \gamma_2 + \alpha_2}.$$

Case III ($\alpha_1 = 1$, $\beta_2 = -1$). Then $H = gp(x z^{\gamma_1}, x^{\alpha_2} y^{-1} z^{\gamma_2})$ and the following equalities are true

$$x = x z^{\gamma_1} \quad z^{-\gamma_1}$$

$$y = (x z^{\gamma_1})^{\alpha_2} \circ (x^{\alpha_2} y^{-1} z^{\gamma_2})^{-1} \circ z^{-\alpha_2 \gamma_1 + \gamma_2 + \alpha_2}.$$

Case IV $\alpha_1 = -1$, $\beta_2 = 1$. Then $H = gp\ x^{-1} z, x\ \ y z$ and the following equalities are true

$$x = (x^{-1} z^{\gamma_1})^{-1} \circ z^{\gamma_1}$$

$$y = (x^{-1} z^{\gamma_1})^{\alpha_2} . x^{\alpha_2} y z^{\gamma_2} \quad z^{-\alpha_2 \gamma_1 - \gamma_2}.$$

In each of the above cases $N_2 \subset H$, and this means that $H = N_2$.

<u>Theorem 1.6.1</u>

Class $\mathcal{H}_6$ does not include proper normal subgroups of the group N_2.

Proof. Suppose that there exists a normal, proper subgroup H in the class $\mathcal{H}_6$. Thus $H = gp(x^{\alpha_1} z^{\gamma_1}, x^{\alpha_2} y^{\beta_2} z^{\gamma_2})$ and at the same time the equalities are not with satisfied $\alpha_1 \neq \pm 1$ or $\beta_2 \neq \pm 1$. We have

$$x^{-1} \circ x^{\alpha_2} y^{\beta_2} z^{\gamma_2} \circ x = x^{\alpha_2} y^{\beta_2} z^{\gamma_2 + \beta_2} =$$

$$= x^{n\alpha_1 + m\alpha_2} y^{m\beta_2} z^{n\gamma_1 + m\gamma_2 + \binom{m}{2}\alpha_2\beta_2 - k\alpha_1\beta_2}$$

for some $n, k \in \mathbb{Z}$, $m \in \mathbb{N}$. Hence

$$\alpha_2 = n\alpha_1 + m\alpha_2$$

$$\beta_2 = m\beta_2$$

$$\gamma_2 + \beta_2 = n\gamma_1 + m\gamma_2 + \binom{m}{2}\alpha_2\beta_2 - k\alpha_1\beta_2.$$

Solving this system of equations we get $m = 1$, $n = 0$ and $\alpha_1 = 1$ or $\alpha_1 = -1$. In the case when $x^{-1} \circ x^{\alpha_2} y^{\beta_2} z^{\gamma_2} \circ x = x^{n\alpha_1 - m\alpha_2} y^{-m\beta_2} z^{n\gamma_1 - m\gamma_2 + \binom{m+1}{2}\alpha_2\beta_2 - k\alpha_1\beta_2}$ we obtain the same result.

Let us consider the product $y^{-1} \; x^{\alpha_1} \; z^{\gamma_1} \; y \in H$, there exist numbers $n, k \in \mathbf{Z}$, $m \in \mathbf{N}$ such that

$$x^{\alpha_1} z^{\gamma_1 - \alpha_1} = x^{n\alpha_1 + m\alpha_2} y^{m\beta_2} z^{n\gamma_1 + m\gamma_2 + \binom{m}{2}\alpha_2\beta_2 - k\alpha_1\beta_2}.$$

As we can see in both cases the coefficient $m = 0$. Similarly

$$x^{\alpha_1} z^{\gamma_1 - \alpha_1} = x^{n\alpha_1} z^{n\gamma_1 - k\alpha_1\beta_2}.$$

Hence $n = 1$, and $\beta_2 = 1$ or $\beta_2 = -1$, which is a contradiction.

1.7. Class $\mathcal{H}_7$

The class $\mathcal{H}_7$ comprises all subgroups of the N_2 of the following form: $H = gp(z^{\gamma_0}, x^{\alpha_1} z^{\gamma_1}, x^{\alpha_2} y^{\beta_2} z^{\gamma_2})$, $\gamma_0 \neq 0$, $\alpha_1 \neq 0$, $\beta_2 = 0$. Each element $h \in H$ can be presented in the form

$$h = (z^{\gamma_0})^n \circ (x^{\alpha_1} z^{\gamma_1})^m \circ (x^{\alpha_2} y^{\beta_2} z^{\gamma_2})^{\varepsilon k} \circ (z^{-\alpha_1\beta_2})^t$$

$n, m, t \in \mathbf{Z}$, $k \in \mathbf{N}$, $\varepsilon = \pm 1$, $z^{-\alpha_1\beta_2} = \left[x^{\alpha_1} z^{\gamma_1}, x^{\alpha_2} y^{\beta_2} z^{\gamma_2}\right]$. Let us denote by $d = $ g.c.d. $(\gamma_0, \alpha_1\beta_2)$, then there exist numbers $p, q \in \mathbf{Z}$ such that $p\gamma_0 + q\alpha_1\beta_2 = d$. Hence if follows that each element $h \in H$ is of the form

$$h = (z^d)^n \circ (x^{\alpha_1} z^{\gamma_1})^m \circ (x^{\alpha_2} y^{\beta_2} z^{\gamma_2})^{\varepsilon k}.$$

Let $H_1 = gp(z^{\bar{\gamma}_0}, x^{\bar{\alpha}_1} z^{\bar{\gamma}_1}, x^{\bar{\alpha}_2} y^{\bar{\beta}_2} z^{\bar{\gamma}_2}) \in \mathcal{H}_7$, by d_1 we denote g.c.d. $(\bar{\gamma}_0, \bar{\alpha}_1\bar{\beta}_2)$.

Lemma 1.7.1

Two subgroups $H, H_1 \in \mathcal{H}_7$ are equal if and only if one of the following conditions is fulfiled:

(I) $d_1 = d$, $\bar{\alpha}_1 = \alpha_1$, $\bar{\gamma}_1 = \gamma_1 \pmod d$, $\bar{\alpha}_2 = \alpha_2 + q\alpha_1$,

$\bar{\beta}_2 = \beta_2$, $\bar{\gamma}_2 = (\gamma_2 + q\gamma_1) \pmod d$,

(II) $d_1 = d$, $\bar{\alpha}_1 = \alpha_1$, $\bar{\gamma}_1 = \gamma_1 \pmod d$, $\bar{\alpha}_2 = -\alpha_2 + q\alpha_1$,

$\bar{\beta}_2 = -\beta_2$, $\bar{\gamma}_2 = (-\gamma_2 + q\gamma_1 + \alpha_2\beta_2) \pmod d$,

(III) $d_1 = d$, $\bar{\alpha}_1 = -\alpha_1$, $\bar{\gamma}_1 = (-\gamma_1) \pmod d$, $\bar{\alpha}_2 = \alpha_2 + q\alpha_1$,

$\bar{\beta}_2 = \beta_2$, $\bar{\gamma}_2 = (\gamma_2 + q\gamma_1) \pmod d$,

(IV) $d_1 = d$, $\bar{\alpha}_1 = -\alpha_1$, $\bar{\gamma}_1 = (-\gamma_1) \pmod d$, $\bar{\alpha}_2 = -\alpha_2 + q\alpha_1$,

$\bar{\beta}_2 = -\beta_2$, $\bar{\gamma}_2 = (-\gamma_2 + q\gamma_1 + \alpha_2\beta_2) \pmod d$,

$q \in Z$.

Proof. Let H, $H_1 \in \mathcal{H}_7$ and $H = H_1$, then there exist numbers n, t, p, q, $r \in \mathbf{Z}$, $w \in N$ such that

$$z^{d_1} = z^{hd} \tag{1.7.1}$$

$$x^{\bar{\alpha}_1} z^{\bar{\gamma}_1} = z^{pd} x^{t\alpha_1} z^{t\gamma_1} \tag{1.7.2}$$

(1.7.3)

(a) $$x^{\bar{\alpha}_2} y^{\bar{\beta}_2} z^{\bar{\gamma}_2} = z^{rd} \cdot x^{q\alpha_1} z^{q\gamma_1} \cdot x^{w\alpha_2} y^{w\beta_2} z^{w\gamma_2 + \binom{w}{2}\alpha_2\beta_2}$$

or

(b) $$x^{\bar{\alpha}_2} y^{\bar{\beta}_2} z^{\bar{\gamma}_2} = z^{rd} \cdot x^{q\alpha_1} z^{q\gamma_1} \cdot x^{-w\alpha_2} y^{-w\beta_2} z^{-w\gamma_2 + \binom{w+1}{2}\alpha_2\beta_2}$$

and there exist numbers n_1, t_1, p_1, q_1, $r_1 \in \mathbf{Z}$, $w_1 \in \mathbf{N}$ such that

$$z^d = z^{n_1 d_1} \tag{1.7.4}$$

$$x^{\alpha_1} z^{\gamma_1} = z^{p_1 d_1} \cdot x^{t_1\bar{\alpha}_1} z^{t_1\bar{\gamma}_1} \tag{1.7.5}$$

$$\begin{cases} (a') \quad x^{\alpha_2} y^{\beta_2} z^{\gamma_2} = z^{r_1 d_1} \cdot x^{q_1\bar{\alpha}_1} z^{q_1 \cdot \bar{\gamma}_1} \cdot x^{w_1\bar{\alpha}_2} y^{w_1\bar{\beta}_2} z^{w_1\bar{\gamma}_2 + \binom{w_1}{2}\bar{\alpha}_2\bar{\beta}_2} \\ \quad \text{or} \\ (b') \quad x^{\alpha_2} y^{\beta_2} z^{\gamma_2} = z^{r_1 d_1} \cdot x^{q_1\bar{\alpha}_1} z^{q_1\bar{\gamma}_1} \cdot x^{-w_1\bar{\alpha}_2} y^{-w_1\bar{\beta}_2} z^{-w_1\bar{\gamma}_2 + \binom{w_1+1}{2}\bar{\alpha}_2\bar{\beta}_2} \end{cases} \tag{1.7.6}$$

From the conditions (1.7.1) and (1.7.4) if follows that $nn_1 = 1$, and from the conditions (1.7.2) and (1.7.5) we infer that $t \cdot t_1 = 1$. Let us point out the fact that if the condition (1.7.3) (a) is satisfied then the condition (1.7.6) (a') is satisfied, too; if the condition (1.7.3) (b) is fulfilled then the condition (1.7.6) (b') is fulfilled, too and conversely. The conditions (a) and (b'), (b) and (a') cannot be satisfied at the same time, because is such cases we get disagreement.

So let $n = n_1 = 1$ and $t = t_1 = 1$, then taking into consideration the conditions (1.7.1), (1.7.2), (1.7.4), (1.7.5) we obtain $d_1 = d$, $\bar{\alpha}_1 = \alpha_1$, $\bar{\gamma}_1 = \gamma_1 \pmod d$. If the conditions (1.7.3) (a) and (1.7.6) (a') are fulfielled, then $\bar{\alpha}_2 = \alpha_2 + q\alpha_1$, $\bar{\beta}_2 = \beta_2$ and $\bar{\gamma}_2 = (\gamma_2 + q\gamma_1) \pmod d$, when the donditions (1.7.3) (b) and (1.7.6) (b') are satisfied, then $\bar{\alpha}_2 = -\alpha_2 + q\alpha_1$, $\bar{\beta}_2 = -\beta_2$ and $\bar{\gamma}_2 = (-\gamma_2 + q\gamma_1 + \alpha_2\beta_2) \pmod d$. In case, when $t = t_1 = -1$ from the condition (1.7.2) if follows $\bar{\alpha}_1 = -\alpha_1$ and $\bar{\gamma}_1 = (-\gamma_1) \pmod d$ and from the condition (1.7.3) we can see that $\alpha_2 = \alpha_2 + q\alpha_1$, $\bar{\beta}_2 = \beta_2$ and $\bar{\gamma}_2 = (\gamma_2 + q\gamma_1) \pmod d$ or $\bar{\alpha}_2 = -\alpha_2 + q\alpha_1$, $\bar{\beta}_2 = -\beta_2$, $\bar{\gamma}_2 = (-\gamma_2 + q\gamma_1 + \alpha_2\beta_2) \pmod d$.

Suppose, that on of the conditions (I) to (IV) is satisfied. Let say the condition (I). Then $d_1 = d$, $\bar{\alpha}_1 = \alpha_1$, $\bar{\gamma}_1 = \gamma_1 + pd$, $\bar{\alpha}_2 = \alpha_2 + q\alpha_1$, $\bar{\beta}_2 = \beta_2$, $\bar{\gamma}_2 = \gamma_2 + q\gamma_1 + rd$; where $p, q, r \in \mathbb{Z}$. Hence it immediately follows that

$$z^{d_1} = z^d,$$

$$x^{\bar{\alpha}_1} z^{\bar{\gamma}_1} = x^{\alpha_1} z^{\gamma_1} \cdot z^{pd},$$

$$x^{\bar{\alpha}_2} y^{\bar{\beta}_2} z^{\bar{\gamma}_2} = z^{rd} \cdot x^{q\alpha_1} z^{q\gamma_1} \cdot x^{\alpha_2} y^{\beta_2} \cdot z^{\gamma_2}.$$

On the other hand $\gamma_1 = \bar{\gamma}_1 - pd_1$, $\alpha_2 = \bar{\alpha}_2 - q\bar{\alpha}_1$, $\gamma_2 = \bar{\gamma}_2 + (qp - r)d_1 - q\bar{\gamma}_1$, hence

$$z^d = z^{d_1},$$

$$x^{\alpha_1} z^{\gamma_1} = x^{\bar{\alpha}_1} z^{\bar{\gamma}_1} z^{-pd_1},$$

$$x^{\alpha_2} y^{\beta_2} z^{\gamma_2} = z^{(qp-r)d_1} \cdot x^{-q\bar{\alpha}_1} z^{-q\bar{\gamma}_1} \cdot x^{\bar{\alpha}_2} y^{\bar{\beta}_2} z^{\bar{\gamma}_2}.$$

Therefore the groups $H = gp(z^{\vartheta_0}, x^{\alpha_1} z^{\vartheta_1}, x^{\alpha_2} y^{\beta_2} z^{\vartheta_2})$ and $H_1 = gp(z^{\overline{\vartheta}_0}, x^{\overline{\alpha}_1} z^{\overline{\vartheta}_1}, x^{\overline{\alpha}_2} y^{\overline{\beta}_2} z^{\overline{\vartheta}_2})$ are equal. We consider other cases in a similar way.

Colollary 1.7.1

Let $H = gp(z^{\vartheta_0}, x^{\alpha_1} z^{\vartheta_1}, x^{\alpha_2} y^{\beta_2} z^{\vartheta_2}) \in \mathcal{H}_7$, then $H = N_2$ if and only if $\alpha_1 = \pm 1$ and $\beta_2 = \pm 1$.

Proof. Let $H = N_2 = gp(x, y)$, then $d_1 = g.c.d.\ (0, 1)$, $\overline{\alpha}_1 = 1$ and $\overline{\beta}_2 = 1$. If follows from the lemma 1.7.1 that $\alpha_1 = 1$ or $\alpha_1 = -1$ and $\beta_2 = 1$ or $\beta_2 = -1$ and $d = g.c.d.\ (\vartheta_0, \alpha_1\beta_2) = 1$.

Let $H = gp(z^{\vartheta_0}, x^{\alpha_1} z^{\vartheta_1}, x^{\alpha_2} y^{\beta_2} z^{\vartheta_2}) \in \mathcal{H}_7$ such that $\alpha_1 = \pm 1$ and $\beta_2 = \pm 1$. Let us notice that $d = g.c.d.\ (\vartheta_0, \alpha_1\beta_2) = 1$, then each element $h \in H$ is the form $h = z^n . x^m z^{m\vartheta_1} (x^{\alpha_2} y z^{\vartheta_2})^{\varepsilon k}$, $n, m \in \mathbf{Z}$, $k \in \mathbf{N}$, $\varepsilon = \pm 1$. It follows that $x = z^{-\vartheta_1} . x z^{\vartheta_1}$, $y = z^{\alpha_2\vartheta_1 - \vartheta_2} . x^{-\alpha_2} z^{-\alpha_2\vartheta_1} x^{\alpha_2} y z^{\vartheta_2}$. Hence $N_2 \leqslant H$. Similarly $H = N_2$.

Theorem 1.7.1

The subgroup $H = gp(z^{\vartheta_0}, x^{\alpha_1} z^{\vartheta_1}, x^{\alpha_2} y^{\beta_2} z^{\vartheta_2}) \in \mathcal{H}_7$, $H < N_2$ is a normal subgroup of the group N_2 if and only if $d | \alpha_1$, $d | \alpha_2$, $d | \beta_2$; where $d = g.c.d.\ (\vartheta_0, \alpha_1\beta_2)$.

Proof. Suppose that $H < N_2$, $H \lhd N_2$. Hence $x^{-1} . x^{\alpha_2} y^{\beta_2} z^{\vartheta_2} . x \in H$, $y^{-1} . x^{\alpha_1} z^{\vartheta_1} . y \in H$, $y^{-1} . x^{\alpha_2} y^{\beta_2} z^{\vartheta_2} . y \in H$, this means there exist numbers n, $m \in \mathbf{Z}$, $t \in \mathbf{N}$ such that

$$x^{\alpha_2} y^{\beta_2} z^{\vartheta_2 + \beta_2} = x^{m\alpha_1} z^{m\vartheta_1 + nd} . (x^{\alpha_2} y^{\beta_2} z^{\vartheta_2})^{\varepsilon t} \qquad (1.7.7)$$

and numbers m_i, $n_i \in \mathbf{Z}$, $t_2 \in \mathbf{N}$, $i = 1,2$, such that

$$x^{\alpha_1} z^{\vartheta_1 - \alpha_1} = x^{m_1\alpha_1} z^{m_1\vartheta_1 + n_1 d} \qquad (1.7.8)$$

$$x^{\alpha_2} y^{\beta_2} z^{\vartheta_2 - \alpha_2} = x^{m_2\alpha_1} z^{m_2\vartheta_1 + n_2 d} (x^{\alpha_2} y^{\beta_2} z^{\vartheta_2})^{\varepsilon t_2}, \qquad (1.7.9)$$

$\varepsilon = \pm 1$. Hence we obtain the following systems equations:

$$\begin{cases} \alpha_2 = m\alpha_1 + t\alpha_2 \\ \beta_2 = t\beta_2 \\ \gamma_2 + \beta_2 = nd + m\gamma_1 + t\gamma_2 + \binom{t}{2}\alpha_2\beta_2 \end{cases} \tag{1.7.10}$$

or

$$\begin{cases} \alpha_2 = m\alpha_1 - t\alpha_2 \\ \beta_2 = -t\beta_2 \\ \gamma_2 + \beta_2 = nd + m\gamma_1 - t\gamma_2 + \binom{t+1}{2}\alpha_2\beta_2. \end{cases} \tag{1.7.10)'$$

$$\begin{cases} \alpha_1 = m_1\alpha_1 \\ \gamma_1 - \alpha_1 = n_1 d + m_1\gamma_1. \end{cases} \tag{1.7.11}$$

$$\begin{cases} \alpha_2 = m_2\alpha_1 + t_2\alpha_2 \\ \beta_2 = t_2\beta_2 \\ \gamma_2 - \alpha_2 = n_2 d + m_2\gamma_1 + t_2\gamma_2 + \binom{t_2}{2}\alpha_2\beta_2 \end{cases} \tag{1.7.12}$$

or

$$\begin{cases} \alpha_2 = m_2\alpha_1 - t_2\alpha_2 \\ \beta_2 = -t_2\beta_2 \\ \gamma_2 - \alpha_2 = n_2 d + m_2 d + m_2\gamma_1 - t_2\gamma_2 + \binom{t_2+1}{2}\alpha_2\beta_2. \end{cases} \tag{1.7.12)'$$

Let us note that it is enough to consider the systems (1.7.10), (1.7.11) and (1.7.12). The system (1.7.10)' is reduced to the system (1.7.10) because $t = -1$. Analogously the system (1.7.12)' is reduced to the system (1.7.12). From the system (1.7.10) it follows that $t = 1$, $m = 0$ and $\gamma_2 + \beta_2 = nd + \gamma_2$. We have $\beta_2 = nd$, hence $d \mid \beta_2$. From the system (1.7.11) it follows that $m_1 = 1$ and $\gamma_1 - \alpha_1 = n_1 d + \gamma_1$. Hence $\alpha_1 = -n_1 d$, thus $d \mid \alpha_1$. From the system (1.7.12) it follows that $t_2 = 1$, $m_2 = 0$ and $\gamma_2 - \alpha_2 = n_2 d + \gamma_2$. Hence $\alpha_2 = -n_2 d$, thus $d \mid \alpha_2$.

Now let us assume that $d | \alpha_1$, $d | \alpha_2$, $d | \beta_2$ i.e. $\alpha_1 = l_1 d$, $\alpha_2 = l_2 d$, $\beta_2 = l_3 d$ for some $l_1, l_2, l_3 \in \mathbf{Z}$. Then each element of the group H can be presented as follows

$$(z^d)^n \cdot (x^{l_1 d} z^{\vartheta_1})^m \cdot (x^{l_2 d} y^{l_3 d} z^{\vartheta_2})^{\varepsilon t}$$

n, $m \in \mathbf{Z}$, $t \in \mathbf{N}$, $\varepsilon = \pm 1$. Observe that

$$x^{-1} \cdot x^{l_2 d} y^{l_3 d} z^{\vartheta_2} \cdot x = x^{l_2 d} y^{l_3 d} z^{\vartheta_2} \cdot z^{l_3 d}$$

$$y^{-1} \cdot x^{l_1 d} z^{\vartheta_1} \cdot y = x^{l_1 d} z^{\vartheta_1} \cdot z^{-l_1 d}$$

$$y^{-1} \cdot x^{l_2 d} y^{l_1 d} z^{\vartheta_2} \cdot y = x^{l_2 d} y^{l_3 d} z^{\vartheta_2} \cdot z^{-l_2 d}.$$

Thus H is normal in N_2 and theorem 1.7.1 follows.

Corollary 1.7.2

The subgroup $H \in \mathcal{H}_7$ is a normal subgroup of the group N_2 if and only if $H = gp(zd,\ x^{rd} z^{\vartheta_1},\ x^{kd} y^{sd} z^{\vartheta_2})$, $d, r, k, s \in \mathbf{N}$, $0 \leq \vartheta_i \leq d-1$, $i = 1,2$.

II. Factor groups of N_2.

2.1. Presentations of factor groups of the group N_2.

We start with the very known theorem (cf. [2]).

Theorem 2.1

Let G has a presentation $G = \langle a, b, c, \ldots;\ P, Q, R, \ldots \rangle$ and let H is a normal subgroup of G generated by the words $S(a,b,c,\ldots)$, $T(a,b,c,\ldots)$,.... Then the factor group G/H has the presentation $\langle a,b,c,\ldots;\ P, Q, R, \ldots, S, T, \ldots \rangle$.

It is clear that N_2 has a presentation

$$N_2 = \langle a,b,c;\ caba^{-1}b^{-1},\ a^{-1}c^{-1}ac,\ b^{-1}c^{-1}bc \rangle$$

under the mapping

$$x \to a, \quad y \to b, \quad z \to c$$

and therefore we have

Corollary 2.1.1

If $H \lhd N_2$, $H \in \mathcal{H}_1$, then the group N_2/H has the presentation

$$<a,\ b,\ c;\ caba^{-1}b^{-1},\ a^{-1}c^{-1}ac,\ b^{-1}c^{-1}bc,\ c^{\vartheta_0}> \qquad (2.1.1)$$

$\vartheta_0 \in \mathbb{Z}$, (under the mapping $xH \to a$, $yH \to b$, $zH \to c$; of course).

Corollary 2.1.2

If $H \lhd N_2$, $H \in \mathcal{H}_4$, then the group N_2/H has the presentation

$$<a,\ b,\ c;\ caba^{-1}b^{-1},\ a^{-1}c^{-1}ac,\ b^{-1}c^{-1}bc,\ c^{\vartheta_0},\ a^{l\vartheta_0}c^{r}>, \qquad (2.1.2)$$

$l, \vartheta_0 \in \mathbb{N}$, $0 \leq r \leq \vartheta_0 - 1$.

Corollary 2.1.3

If $H \lhd N_2$, $H \in \mathcal{H}_5$, then the group N_2/H has the presentation

$$<a,\ b,\ c;\ caba^{-1}b^{-1},\ a^{-1}c^{-1}ac,\ b^{-1}c^{-1}bc,\ c^{\vartheta_0},\ a^{l\vartheta_0}b^{k\vartheta_0}c^{r}> \qquad (2.1.3)$$

$l, \vartheta_0 \in \mathbb{N}$, $k \in \mathbb{Z}$, $0 \leq r \leq \vartheta_0 - 1$.

Corollary 2.1.4

If $H \lhd N_2$, $H \in \mathcal{H}_7$, then the group N_2/H has the presentation

$$<a,b,c;\ caba^{-1}b^{-1},\ a^{-1}c^{-1}ac,\ b^{-1}c^{-1}bc,\ c^{d},\ a^{rd}c^{\vartheta_1},\ a^{kd}b^{sd}c^{\vartheta_2}> \qquad (2.1.4)$$

$d, r, k, s \in \mathbb{N}$, $0 \leq \vartheta_i \leq d - 1$, $i = 1,2$.

This together with part I of our paper give complete description of factor groups of N_2.

2.2. The centers of factor groups of the group N_2

Let a group G have the following presentation

$$<a,\ b,\ c;\ caba^{-1}b^{-1},\ a^{-1}c^{-1}ac,\ b^{-1}c^{-1}bc,\ P,\ Q,\ R,\ldots>$$

We denote by $S(P,Q,R,\ldots)$ the group generated by the words $P,Q,R,\ldots$ and by $S(c)$ the group generated by the word c. Let $d = \min\{l > 0;\ c^{l} \in S(P,Q,R,\ldots)\}$ and $d = 0$ if $S(P,Q,R,\ldots) \cap S(c) = \emptyset$. Let us point out that if $S(P,Q,R,\ldots) \cap S(c) \neq \emptyset$, then we can replace relators $P,Q,R,\ldots$ by a set of relators $c^{d}, P', Q', R', \ldots$ such that among the relators $P', Q', R', \ldots$ there are no relators of the form c^p, $p \in \mathbb{N}$. Then the group G has the presentation

$$\langle a,b,c;\ caba^{-1}b^{-1},\ a^{-1}c^{-1}ac,\ b^{-1}c^{-1}bc,\ c^d, P', Q', R', \ldots \rangle \quad (2.2.1)$$

The following lemma ist easy to check.

Lemma 2.2.1

Let a group G have the presentation (2.2.1). The element of G assigned by the word $a^{\alpha}b^{\beta}c^{\gamma}$ belongs to the center $C(G)$ of G if and only if $\alpha = kd$, $\beta = ld$, $l,k \in \mathbb{Z}$.

Theorem 2.2.1

Let H be a normal subgroup of N_2.

a) If $H \in \mathcal{H}_1$, $H = gp(z^{\gamma_0})$ and $\gamma_0 \in \mathbb{N}$, then we have

$$C(N_2/H) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}(\gamma_0).$$

b) If $H \in \mathcal{H}_4$, $H = gp(z^{\gamma_0}, x^{l\gamma_0} z^r)$, $\gamma_0 \in \mathbb{N}$ and $\gamma_0 > 1$, $l \in \mathbb{N}$, $0 < r < \gamma_0 - 1$ and $l\gamma_0 = p_1^{\mu_1} \cdot \ldots \cdot p_k^{\mu_k}$, $p_i \neq p_j$ for $i \neq j$, then we have

$$C(N_2/H) \cong \mathbb{Z} \times \mathbb{Z}(p_1^{\mu_1}) \times \ldots \times \mathbb{Z}(p_k^{\mu_k}).$$

c) If $H \in \mathcal{H}_5$, $H = gp(z^{\gamma_0}, x^{l\gamma_0} y^{k\gamma_0} z^r)$, γ_0, $l \in \mathbb{N}$, $k \in \mathbb{Z}$, $0 \leq r \leq \gamma_0 - 1$ and $q = g.c.d.(l,k,r,\gamma_0)$, then for some $p \in \mathbb{N}$ we have

$$C(N_2/H) \cong \mathbb{Z}(q) \times \mathbb{Z}(pq) \times \mathbb{Z}$$

provided $q \neq 1$, and if $q = 1$, then the rank of the group $C(N_2/H)$ equals two and exists number $t \in \mathbb{N}$ such that

$$C(N_2/H) \cong \mathbb{Z}(t\gamma_0) \times \mathbb{Z}.$$

d) If $H \in \mathcal{K}_7$, $H = gp(z^d, x^{rd} z^{\gamma_1}, x^{kd} y^{sd} z^{\gamma_2})$, $d, r, k, s \in \mathbf{N}$, $0 \leqslant \gamma_i \leqslant d-1$, $i = 1,2$, and $q_1 = g.c.d\,(d,r,k,s,\gamma_1,\gamma_2) \neq 1$, then the rank of the group $C(N_2/H)$ equals three and there exists numbers $p_i \in N$, $i = 1,2$, such that

$$C(N_2/H) \cong \mathbf{Z}(q_1) \times \mathbf{Z}(p_1 q_1) \times \mathbf{Z}(p_2 p_1 q_1).$$

Proof. a) The elements of the group N_2/H belonging to $C(N_2/H)$ are determined by the words of the form $a^{n\gamma_d} b^{m\gamma_0} c^{\gamma}$, $n, m \in \mathbf{Z}$, $0 \leqslant \gamma \leqslant \gamma_0 - 1$. Thus $C(N_2/H)$ is the direct product of the groups generated by the words a^{γ_0}, b^{γ_0} and c, and consequently $C(N_2/H) \cong \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}(\gamma_0)$.

b) Every element of $C(N_2/H)$ is uniquely determined by the words $a^{n\gamma_0} b^{m\gamma_0} c^{\gamma}$ with $0 \leqslant n \leqslant l-1$, $m \in \mathbf{Z}$, $0 \leqslant \gamma \leqslant \gamma_0 - 1$. Let us notice that $C(N_2/H)$ is the direct product of the groups $H_1 = gp(b)$ and $H_2 = gp(a^{\gamma_0}, c)$. The group H_2 is a finite torsion abelian group of order $l\gamma_0$, therefore it is the direct product. Thus $C(N_2/H) \cong \mathbf{Z} \times \mathbf{Z}(p_1^{\mu_1}) \times \ldots \times \mathbf{Z}(p_k^{\mu_k})$. Let us point out that if $\gamma_0 = 1$, then we have $C(N_2/H) \cong \mathbf{Z} \times \mathbf{Z}(1)$. If $\gamma_0 > 1$ and $r = 0$, then for every $l \in \mathbf{N}$ we have $C(N_2/H) \cong \mathbf{Z} \times \mathbf{Z}(1) \times \mathbf{Z}(\gamma_0)$.

c) The group N_2/H has the presentation (2.1.3). The centre of the group N_2/H is an abelian group in which the elements are determined by the words $a^{n\gamma_0} b^{m\gamma_0} c^{\gamma}$, $n, m \in \mathbf{Z}$, $0 \leqslant \gamma \leqslant \gamma_0 - 1$.

We denote by A_3 a free abelian group of the rank three, then there exists a subgroup $\overline{H}$ of the group A_3 such that $C(N_2/H) \cong A_3/\overline{H}$. Let $\overline{H} \leqslant A_3$ be a subgroup generated by the words $x_1^l x_2^k x_3^r$, $x_3^{\gamma_0}$, then the group $A_3/\overline{H}$ has the following presentation

$$< a_1, a_2, a_3;\ a_1^l a_2^k a_3^r,\ a_3^{\gamma_0},\ x_1 x_2 x_1^{-1} x_2^{-x} > . \qquad (2.2.2)$$

It can be easily noticed that $C(N_2/H)$ is isomorphic to the factor group $A_3/\overline{H}$ and thus it has the presentation (2.2.2). The following matrix relation is related to the presentation (2.2.2) of the group $A_3/\overline{H}$

$$M = \begin{pmatrix} l & 0 & 0 \\ k & 0 & 0 \\ r & \gamma_0 & 0 \end{pmatrix}$$

The k-th invariant factors of the matrix M are equal respectivelly:

$$\varepsilon_1(M) = \text{g.c.d.} \ (1,k,r,\gamma_0), \quad \varepsilon_2(M) = \text{g.c.d.} \ (k\gamma_0, -l\gamma_0) \quad \varepsilon_3(M) = 0.$$

It should be noticed (see theorem 3.6 of [2]) that the torsion numbers τ_1, τ_2 are equal respectivelly q and pq, where $p \in \mathbf{N}$, and Betti number ρ equals one. Thus $C(N_2/H) \cong \mathbf{Z}(q) \times \mathbf{Z}(pq) \times \mathbf{Z}$. It can be easily noticed that if $q = \varepsilon_1(M) = 1$ and $\tilde{q} = \varepsilon_2(M) \neq 1$, then there exists a number $t \in \mathbf{N}$ such that $\tilde{q} = t\gamma_0$. Thus the rank of group $C(N_2/H)$ equals two and $C(N_2/H) \cong \mathbf{Z}(t\gamma_0) \times \mathbf{Z}$.

d) The group N_2/H has the presentation (2.1.4). Let $\tilde{\tilde{H}}$ be the subgroup generated by the words x_3^d, $x_1^r x_3^{\gamma_1}$, $x_1^k x_2^s x_3^{\gamma_2}$ in the free abelian group A_3, then the group $A_3/\tilde{\tilde{H}}$ has the presentation

$$< a_1, a_2, a_3; \quad a_3^d, a_1^r a_3^{\gamma_1}, a_1^k a_2^s a_3^{\gamma_2}, X_1X_2X_1^{-1}X_2^{-1} > . \qquad (2.2.3)$$

It is easy to see that $C(N_2/H) \cong A_3/\tilde{\tilde{H}}$ and thus $C(N_2/H)$ has the presentation (2.2.3). The matrix relation of the presentation (2.2.3) is the following

$$M_1 = \begin{pmatrix} 0 & r & k \\ 0 & 0 & s \\ d & \gamma_1 & \gamma_2 \end{pmatrix}$$

and the k-th invariant factors of the matrix M_1 equal:

$$\varepsilon_1(M_1) = \text{g.c.d.} \ (d,r,k,s,\gamma_1,\gamma_2),$$

$$\varepsilon_2(M_1) = \text{g.c.d.} \ (-s\gamma_1, sd, -(r\gamma_2 - k\gamma_1), -kd, rd, rs),$$

$$\varepsilon_3(M_1) = drs.$$

Let us notice that the equalities $d = r = s = 1$ are imposible. The torsion numbers τ_1, τ_2, τ_3 are equal $\tau_1 = q_1 = \varepsilon_1(M_1)$, $\tau_2 = p_1q_1$, $\tau_3 = p_2p_1q_1$, where $p_1, p_2 \in \mathbf{N}$, and Betti number ρ equals zero. Therefore $C(N_2/H) \cong \mathbf{Z}(q_1) \times \mathbf{Z}(p_1q_1) \times \mathbf{Z}(p_2p_1q_1)$.

If $q_1 = 1$ and $q_2 = \varepsilon_2(M_1) \neq 1$, then the rank of the group $C(N_2/H)$ equals two and exists a number $t \in \mathbf{N}$ such that $C(N_2/H) \cong \mathbf{Z}(q_1) \times \mathbf{Z}(tq_1)$.

If $q_1 = q_2 = 1$, then the rank of the group $C(N_2/H)$ equals one and $C(N_2/H) \cong \mathbf{Z}(drs)$. Theorem 2.2.1 is thus proved.

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КЛАССИФИКАЦИЯ ПОДГРУПП И ФАСТОР ГРУПП СВОБОДНОЙ НИЛЬПОТЕНТНОЙ ГРУППЫ КЛАССА 2 С ДВУМЯ ГЕНЕРАТОРАМИ

Р е з ю м е

Работа состоит из описания всех нормальных подгрупп дискретной группы Гейзенберга N_2 а также из описания центров фастор груп.

KLASYFIKACJA PODGRUP I GRUP ILORAZOWYCH WOLNEJ NILPOTENTNEJ GRUPY KLASY 2 Z DWOMA GENERATORAMI

S t r e s z c z e n i e

W pracy tej zawarty jest opis wszystkich podgrup normalnych dyskretnej grupy Heisenberga N_2 oraz opis centrów grup ilorazowych.